C4.3 Functional Analytic Methods for PDEs
Lectures 15-16

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In the last lectures

- Linear elliptic equations of second order.
- Classical and weak solutions.
- Energy estimates.
- First existence theorem: Riesz representation theorem.
- First existence theorem: Direct method of the calculus of variation.
- Second existence theorem: Fredholm alternative.
- Third existence theorem: Spectral theory.
- H^2 regularity of weak solutions to linear elliptic equations.

This lecture

- H^2 regularity of weak solutions to linear elliptic equations.
- Continuity of weak solutions to linear elliptic equations.
- A priori L^∞ estimates.
- Guided reading groups' presentation.

A priori H^2 estimates in the general case

- We prove for $a_{ij} = \delta_{ij}$ that if $u \in H^1(\mathbb{R}^n)$ satisfies $-\partial_i(a_{ij}\partial_j u) = f$ on $\mathbb{R}^n$ with $f \in L^2(\mathbb{R}^n)$, then $u \in H^2(\mathbb{R}^n)$.
- We now turn to the case where a is variable. To better convey central ideas, we will focus in the rest of this course to a priori estimates: We assume that the solution has the right regularity and will be concerned with establishing quantitative estimates.
- More precisely, we suppose that u belongs to $\underline{H^2(\mathbb{R}^n)}$ and is a weak solution to $Lu = f$ in $\mathbb{R}^n$, and would like to bound $\|u\|_{H^2(\mathbb{R}^n)}$ in terms of the bounds for the coefficients of L, $\|f\|_{L^2(\mathbb{R}^n)}$ and $\|u\|_{H^1(\mathbb{R}^n)}$.
- For simplicity, we will assume that $b \equiv 0$ and $c \equiv 0$. You should check that the methods we use work in the general case.

Method of freezing coefficients

Theorem

Suppose $a \in C^1(\mathbb{R}^n)$, $\nabla a \in L^\infty(\mathbb{R}^n)$ *and* $L = -\partial_i(a_{ij}\partial_j)$.
There exist $0 < \delta_0 \ll 1$ *and* $C > 0$ *such that if* $\|a_{ij} - \delta_{ij}\|_{L^\infty(\mathbb{R}^n)} \leq \delta_0$ *and if* $u \in H^2(\mathbb{R}^n)$ *and satisfies* $Lu = f$ *in* $\mathbb{R}^n$ *in the weak sense, then*

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^1(\mathbb{R}^n)}).$$

Proof

- Claim: u satisfies

$$-\Delta u = f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u =: \tilde{f},$$

that is, for all $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v\,dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u\Big] v\,dx.$$

Method of freezing coefficients

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i \partial_j u + \partial_i a_{ij} \partial_j u \Big] v \, dx.$$

  - ⋆ We note that $(a_{ij} - \delta_{ij})v \in C_c^1(\mathbb{R}^n)$. Hence, by definition of weak derivatives,

$$\begin{aligned}
\int_{\mathbb{R}^n} (a_{ij} - \delta_{ij})\partial_i \partial_j u v \, dx &= -\int_{\mathbb{R}^n} \partial_j u \partial_i [(a_{ij} - \delta_{ij})v] \, dx \\
&= -\int_{\mathbb{R}^n} \partial_j u [(a_{ij} - \delta_{ij})\partial_i v + \partial_i a_{ij} v] \, dx \\
&= \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx - \int_{\mathbb{R}^n} a_{ij} \partial_j u \partial_i v \, dx \\
&\quad - \int_{\mathbb{R}^n} \partial_i a_{ij} v \, dx.
\end{aligned}$$

Method of freezing coefficients

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v\, dx = \int_{\mathbb{R}^n} \Big[f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u\Big] v\, dx.$$

  - ⋆ $$\int_{\mathbb{R}^n} (a_{ij} - \delta_{ij})\partial_i\partial_j u v\, dx = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v\, dx - \int_{\mathbb{R}^n} a_{ij}\partial_j u \partial_i v\, dx - \int_{\mathbb{R}^n} \partial_i a_{ij} v\, dx.$$
  - ⋆ As $Lu = f$, we have

$$\int_{\mathbb{R}^n} a_{ij}\partial_j u \partial_i v\, dx = \int_{\mathbb{R}^n} f\, v\, dx.$$

  - ⋆ Putting the two identities together, we obtain the claim.

Method of freezing coefficients

Proof

- We have proved the claim that $-\Delta u = \tilde{f} = f + (a_{ij} - \delta_{ij})\partial_i\partial_j u + \partial_i a_{ij}\partial_j u$.
- By the lemma on the H^2 regularity for $-\Delta$, we have a constant C such that

$$\begin{aligned}\|\nabla^2 u\|_{L^2} &\leq C\|\tilde{f}\|_{L^2} \\ &\leq C\Big[\|f\|_{L^2} + \|a_{ij} - \delta_{ij}\|_{L^\infty}\|\nabla^2 u\|_{L^2(\Omega)} \\ &\qquad + \|\partial_i a_{ij}\|_{L^\infty}\|\nabla u\|_{L^2}\Big].\end{aligned}$$

- It is readily seen that if $C\|a_{ij} - \delta_{ij}\|_{L^\infty} < 1$, then the second term on the right hand side can be absorbed back to the left hand side, giving the conclusion:

$$\|\nabla^2 u\|_{L^2} \leq C'\Big[\|f\|_{L^2} + \|\nabla u\|_{L^2}\Big].$$

Method of differentiating the equation

Theorem

Suppose $a \in C^1(\mathbb{R}^n)$, $\nabla a \in L^\infty(\mathbb{R}^n)$ and $L = -\partial_i(a_{ij}\partial_j)$.
There exists $C > 0$ such that if $u \in H^2(\mathbb{R}^n)$ and satisfies $Lu = f$ in $\mathbb{R}^n$ in the weak sense, then

$$\|u\|_{H^2(\mathbb{R}^n)} \leq C(\|f\|_{L^2(\mathbb{R}^n)} + \|u\|_{H^1(\mathbb{R}^n)}).$$

Proof

- Let $w = \partial_k u \in H^1(\mathbb{R}^n)$. We would like to bound $\|w\|_{H^1}$.
- Claim: w satisfies

$$Lw = \partial_i h_i \text{ where } h_i = \partial_k a_{ij}\partial_j u + f\,\delta_{ik},$$

that is, for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\,dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\,dx.$$

Method of differentiating the equation

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v \, dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v \, dx.$$

  ⋆ Note that $a_{ij}\partial_i v \in C_c^1(\mathbb{R}^n)$. Hence, by definition of weak derivatives,

$$\begin{aligned}\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v \, dx &= \int_{\mathbb{R}^n} \partial_k \partial_j u \, (a_{ij}\partial_i v) \, dx = -\int_{\mathbb{R}^n} \partial_j u \, \partial_k (a_{ij}\partial_i v) \, dx \\ &= -\int_{\mathbb{R}^n} a_{ij}\partial_j u \, \partial_k \partial_i v \, dx - \int_{\mathbb{R}^n} \partial_j u \, \partial_k a_{ij} \partial_i v \, dx\end{aligned}$$

Method of differentiating the equation

Proof

- Claim: for $v \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\, dx = -\int_{\mathbb{R}^n} [\partial_k a_{ij}\partial_j u + f\,\delta_{ik}]\partial_i v\, dx.$$

  ⋆ $\displaystyle\int_{\mathbb{R}^n} a_{ij}\partial_j w \partial_i v\, dx = -\int_{\mathbb{R}^n} a_{ij}\partial_j u\, \partial_k \partial_i v\, dx - \int_{\mathbb{R}^n} \partial_j u\, \partial_k a_{ij} \partial_i v\, dx.$

  ⋆ On the other hand, using $\partial_k v$ as a test function for $Lu = f$, we have

$$\int_{\mathbb{R}^n} a_{ij}\partial_j u\, \partial_i \partial_k v\, dx = \int_{\mathbb{R}^n} f\, \partial_k v\, dx.$$

  ⋆ Putting the two identities together we get the claim.

Method of differentiating the equation

Proof

- We have thus shown that $Lw = \partial_i h_i$ with $h_i = \partial_k a_{ij} \partial_j u + f\,\delta_{ik}$.
- Using w as a test function for this equation, we get

$$\int_{\mathbb{R}^n} a_{ij} \partial_j w \partial_i w \, dx = - \int_{\mathbb{R}^n} h_i \partial_i w \, dx.$$

- Using ellipticity on the left side and Cauchy-Schwarz' inequality on the right side we get

$$\lambda \|\nabla w\|_{L^2}^2 \leq \|h\|_{L^2} \|\nabla w\|_{L^2} \leq \frac{\lambda}{2} \|\nabla w\|_{L^2}^2 + \frac{1}{2\lambda} \|h\|_{L^2}^2.$$

- We thus have

$$\|\nabla w\|_{L^2} \leq C \|h\|_{L^2} \leq C \Big[\|f\|_{L^2} + \|\nabla u\|_{L^2} \Big].$$

Recalling that $w = \partial_k u$, we're done.

Example

- Recall the example of the equation $-(au')' = f$ in $(-1,1)$ with $a = \chi_{(-1,0)} + 2\chi_{(0,1)}$.
- If $f \in L^q$, then $au' \in W^{1,q}$ and so u' is presumably discontinuous.
- Nevertheless as u' exists by assumption, u is continuous.
- In higher dimension, the existence of ∇u (in L^2) doesn't ensure continuity of u. Nevertheless, a major result due to De Giorgi, Moser and Nash around late 50s asserts that u is indeed continuous!

De Giorgi-Moser-Nash's theorem

Theorem (De Giorgi-Moser-Nash's theorem)

Suppose that $a, b, c \in L^\infty(\Omega)$, *a is uniformly elliptic, and* $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. *If* $u \in H^1(\Omega)$ *satisfies* $Lu = f$ *in* Ω *in the weak sense for some* $f \in L^q(\Omega)$ *with* $q > \frac{n}{2}$, *then u is locally Hölder continuous , and for any open* ω *such that* $\bar{\omega} \subset \Omega$ *we have*

$$\|u\|_{C^{0,\alpha}(\omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{H^1(\Omega)})$$

where the constant C depends only on $n, \Omega, \omega, a, b, c$, *and the Hölder exponent* α *depends only on* n, Ω, ω, a.

A digression

We make some observations:

- In De Giorgi-Moser-Nash's theorem, no continuity is assumed on the coefficients a_{ij}.
- If a_{ij} is continuous, one can imagine using the method of freezing coefficients to reduce to the case a_{ij} is constant. Hence the model equation is $-\Delta u = f$.
- In $1d$, we have $-u'' = f$. If $f \in L^q$, we then have that $u \in W^{2,q}_{loc}$.
- It turns out that, in any dimension, if $-\Delta u = f$ and $f \in L^q$, then $u \in W^{2,q}_{loc}$.
  In particular, when $n/2 < q < n$, by the embedding $W^{2,q}_{loc} \hookrightarrow W^{1,\frac{qn}{n-q}}_{loc} \hookrightarrow C^{0,2-\frac{n}{q}}_{loc}$, we have u is Hölder continuous.

Global a priori L^∞ estimate

To illustrate the method, we will assume for simplicity that $b \equiv 0$ and $c \equiv 0$. We will focus on

- a priori L^∞ estimates: we assume that the solution $u \in L^\infty$ and try to establish estimates for $\|u\|_{L^\infty}$.
- We assume in addition for now a boundary condition: $u = 0$ on ∂B_1.

Theorem (Global a priori L^∞ estimates)

Suppose that $a \in L^\infty(B_1)$, a is uniformly elliptic, $b \equiv 0$, $c \equiv 0$ and $L = -\partial_i(a_{ij}\partial_j)$. If $u \in H_0^1(B_1) \cap L^\infty(B_1)$ satisfies $Lu = f$ in B_1 in the weak sense and $f \in L^q(B_1)$ with $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)})$$

where the constant C depends only on n, q, a.

Truncations and powers of H^1 functions

Lemma

Suppose that $u \in H^1_0(B_1) \cap L^\infty(B_1)$. Then, for $p \geq 1$ and $k \geq 0$, one has $(u_+ + k)^p - k^p \in H^1_0(B_1)$.

Proof

- As $u \in L^\infty(B_1)$, we can suppose $|u| \leq M$ a.e. in B_1.
- By Sheet 3, $u_+ \in H^1(B_1)$.
- Select a function $g \in C^1(\mathbb{R})$ such that $g(t) = (t_+ + k)^p - k^p$ for $t \leq M$, and $g(t) = (M + k + 1)^p - k^p$ for $t \geq M + 1$. Note that $(u_+ + k)^p - k^p = g(u)$.
- Then $|g(t)| + |g'(t)| \leq C$ on $\mathbb{R}$.
- By the chain rule (Sheet 2), $g(u)$ has weak derivatives $\nabla g(u) = g'(u)\nabla u \in L^2(B_1)$. Hence $g(u) \in H^1(B_1)$.

Truncations and powers of H^1 functions

Proof

- $g(u) \in H^1(B_1)$.
- We next show that $g(u) \in H^1_0(B_1)$.
  Approximate u by $(u_m) \in C_c^\infty(B_1)$. The argument above shows that $g(u_m) \in H^1(B_1)$.
  As $g(u_m)$ is continuous, we have that the its trace on ∂B_1 is zero, hence $g(u_m) \in H^1_0(B_1)$.
- We have, by Lebesgue's dominated convergence theorem

$$\int_{B_1} |g(u_m) - g(u)|^2 \, dx \to 0.$$

  So $g(u_m) \to g(u)$ in L^2.

Truncations and powers of H^1 functions

Proof

- Next, we have

$$\int_B |\nabla g(u_m) - \nabla g(u)|^2\,dx = \int_B |g'(u_m)\nabla u_m - g'(u)\nabla u|^2\,dx$$
$$\leq \int_B |g'(u_m) - g'(u)|^2 |\nabla u|^2\,dx$$
$$+ \int_B |g'(u_m)|^2 |\nabla u_m - \nabla u|^2\,dx \to 0,$$

  where we use Lebesgue's dominated convergence theorem to treat the first integral and the convergence of ∇u_m to ∇u in L^2 to treat the second integral.
  Hence $\nabla g(u_m) \to \nabla g(u)$ in L^2.
- We have thus shown that $g(u_m) \in H_0^1(B)$ and $g(u_m) \to g(u)$ in $H^1(B)$. The conclusion follows.

Global a priori L^∞ estimates

We now prove the statement that if $u \in H_0^1(B_1) \cap L^\infty(B_1)$ is such that $Lu = f$ in B_1 with $f \in L^q(B_1)$ for some $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)}).$$

- We use Moser iteration method. We write $B = B_1$ and fix some $k > 0$, $p \geq 1$.
- Let $w = u_+ + k$ and we use $v = w^p - k^p$ as test function. This is possible because we just proved that $v \in H_0^1(B_1)$.
  We have

$$\begin{aligned} \int_B f\, v\, dx &= \int_B a_{ij}\partial_j u \partial_i v\, dx \\ &= \int_B p w^{p-1} a_{ij} \partial_j u \partial_i u_+\, dx \\ &\overset{ellipticity}{\geq} \lambda p \int_B w^{p-1} |\nabla u_+|^2\, dx. \end{aligned}$$

Global a priori L^∞ estimate

Proof

- We thus have
$$\int_B |\nabla w^{\frac{p+1}{2}}|^2 \, dx \leq Cp \int_B |f| \, |v| \, dx \leq Cp \int_B |f| w^p \, dx.$$
- By Friedrichs' inequality, this gives
$$\|w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}}\|_{H^1}^2 \leq Cp \int_B |f| w^p \, dx.$$
- By Gagliardo-Nirenberg-Sobolev's inequality, this implies that
$$\|w^{\frac{p+1}{2}} - k^{\frac{p+1}{2}}\|_{L^{\frac{2n}{n-2}}}^2 \leq Cp \int_B |f| \, w^p \, dx.$$
- We thus have
$$\|w^{\frac{p+1}{2}}\|_{L^{\frac{2n}{n-2}}}^2 \leq Cp \int_B \left(\frac{|f|}{k} + 1\right) w^{p+1} \, dx.$$

Global a priori L^∞ estimate

Proof

- $\|w^{\frac{p+1}{2}}\|^2_{L^{\frac{2n}{n-2}}} \leq Cp \int_B (\frac{|f|}{k} + 1)\, w^{p+1}\, dx.$
- Using Hölder's inequality, we then arrive at

$$\|w^{p+1}\|_{L^{\frac{n}{n-2}}} \leq Cp(\|\frac{|f|}{k}\|_{L^q} + 1)\|w^{p+1}\|_{L^{q'}}.$$

- We now choose k to be any number larger than $\|f\|_{L^q}$ and obtain from the above that

$$\|w\|^{p+1}_{L^{\frac{n(p+1)}{n-2}}} \leq Cp\|w\|^{p+1}_{L^{q'(p+1)}}.$$

Recalling that $q > n/2$, we have $q' < \frac{n}{n-2}$. Thus the above inequality is self-improving: If w has a bound in $L^{q'(p+1)}$, then it has a bound in $L^{\frac{n(p+1)}{n-2}}$.

Global a priori L^∞ estimate

Proof

- $\|w\|^{p+1}_{L^{\frac{n(p+1)}{n-2}}} \leq C(p+1)\|w\|^{p+1}_{L^{q'(p+1)}}$.
- Now let $\chi = \frac{n}{(n-2)q'} > 1$ and $t_m = \gamma\chi^m$ for some $\gamma > 2q'$, then the above gives

$$\begin{aligned}\|w\|_{L^{t_{m+1}}} &\leq (Ct_m)^{\frac{q'}{t_m}}\|w\|_{L^{t_m}}\\ &= (C\gamma)^{q'\gamma^{-1}\chi^{-m}}\chi^{q'\gamma^{-1}m\chi^{-m}}\|w\|_{L^{t_m}}.\end{aligned}$$

  Hence by induction,

$$\|w\|_{L^{t_{m+1}}} \leq (C\gamma)^{q'\gamma^{-1}\sum_m \chi^{-m}}\chi^{q'\gamma^{-1}\sum_m m\chi^{-m}}\|w\|_{L^\gamma} \leq C\|w\|_{L^\gamma}.$$

- Sending $m \to \infty$, we obtain

$$\|w\|_{L^\infty} \leq C\|w\|_{L^\gamma} \text{ provided } \gamma > 2q'.$$

Global a priori L^∞ estimate

Proof

- $\|w\|_{L^\infty} \leq C\|w\|_{L^\gamma}$ when $\gamma > 2q'$.
- We now reduce from L^γ to L^2:

$$\|w\|_{L^\infty} \leq C\Big\{\int_B |w|^\gamma\,dx\Big\}^{1/\gamma} \leq C\|w\|_{L^\infty}^{1-\frac{2}{\gamma}}\Big\{\int_B |w|^2\,dx\Big\}^{1/\gamma}.$$

  This gives

$$\|w\|_{L^\infty} \leq C\|w\|_{L^2}.$$

- Recalling that $w = u_+ + k$ and k can be any positive constant larger than $\|f\|_{L^q}$, we have thus shown that

$$\|u_+\|_{L^\infty} \leq C(\|u\|_{L^2} + \|f\|_{L^q})$$

- Applying the same argument to u_-, we get the corresponding bound for u_- and conclude the proof.

Global a priori L^∞ estimate

Theorem (Global a priori L^∞ estimates)

Suppose that $a, b, c \in L^\infty(B_1)$, a is uniformly elliptic, $b \equiv 0$, $c \equiv 0$ and $L = -\partial_i(a_{ij}\partial_j)$. If $u \in H_0^1(B_1) \cap L^\infty(B_1)$ satisfies $Lu = f$ in B_1 in the weak sense and $f \in L^q(B_1)$ with $q > n/2$, then

$$\|u\|_{L^\infty(B_1)} \leq C(\|f\|_{L^q(B_1)} + \|u\|_{L^2(B_1)})$$

where the constant C depends only on n, q, a, b, c.

Remark

When L is injective, the term $\|u\|_{L^2(B_1)}$ on the right hand side can be dropped yielding the estimate:

$$\|u\|_{L^\infty(B_1)} \leq C\|f\|_{L^q(B_1)}.$$

Energy estimate with L^q right hand side

The remark is a consequence of:

Theorem

Suppose that $a, b, c \in L^\infty(B_1)$, a is uniformly elliptic, and $L = -\partial_i(a_{ij}\partial_j) + b_i\partial_i + c$. Suppose that the only solution in $H_0^1(B_1)$ to $Lu = 0$ is the trivial solution. Then, for every $u \in H_0^1(B_1)$ and $f \in L^q(B_1)$ with $q \geq \frac{2n}{n+2}$ satisfying $Lu = f$ in B_1, there holds

$$\|u\|_{H^1(B_1)} \leq C\|f\|_{L^q(B_1)}$$

where the constant C depends only on n, q, a, b, c.

Proof

- When $q = 2$, the result is a consequence of the Fredholm alternative and the inverse mapping theorem.

Energy estimate with L^q right hand side

Proof

- Let us consider first the case that $b \equiv 0$ and $c \equiv 0$.
  - ⋆ In this case, by using u as a test function, we have

$$\lambda \|\nabla u\|_{L^2}^2 \leq \int_{B_1} a_{ij} \partial_j u \partial_i u \, dx = \int_B f u \, dx \leq \|f\|_{L^q} \|u\|_{L^{q'}}.$$

  - ⋆ By Friedrichs' inequality, we have $\|u\|_{H^1} \leq C \|\nabla u\|_{L^2}$. As $q \geq \frac{2n}{n+2}$, $q' \leq \frac{2n}{n-2}$. Hence, by Gagliardo-Nirenberg-Sobolev's inequality, $\|u\|_{L^{q'}} \leq C \|u\|_{H^1}$.
  - ⋆ Therefore

$$\|u\|_{H^1}^2 \leq C \|\nabla u\|_{L^2}^2 \leq C \|f\|_{L^q} \|u\|_{L^{q'}} \leq C \|f\|_{L^q} \|u\|_{H^1},$$

from which we get $\|u\|_{H^1} \leq C \|f\|_{L^q}$, as desired.

Energy estimate with L^q right hand side

Proof

- Let us now consider the general case. By using u as a test function, we have

$$B(u,u) = \int_{B_1} fu\,dx \leq \|f\|_{L^q}\|u\|_{L^{q'}},$$

  where B is the bilinear form associated with L.

- The right hand side is treated as before and is bounded from above by $C\|f\|_{L^q}\|u\|_{H^1}$. For the left hand side, we use Friedrichs' inequality together with energy estimates:

$$B(u,u) + C\|u\|_{L^2}^2 \geq \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 \geq \frac{1}{C}\|u\|_{H^1}^2.$$

  We thus have

$$\|u\|_{H^1}^2 \leq C\|f\|_{L^q}\|u\|_{H^1} + C\|u\|_{L^2}^2.$$

Energy estimate with L^q right hand side

Proof

- $\|u\|_{H^1}^2 \leq C\|f\|_{L^q}\|u\|_{H^1} + C\|u\|_{L^2}^2$.
- By Cauchy-Schwarz' inequality, we then have

$$\|u\|_{H^1}^2 \leq \frac{1}{2}\|u\|_{H^1}^2 + C\|f\|_{L^q}^2 + C\|u\|_{L^2}^2,$$

  and so

$$\|u\|_{H^1}^2 \leq C\|f\|_{L^q}^2 + C\|u\|_{L^2}^2.$$

- In other words,

$$\|u\|_{H^1} \leq C\|f\|_{L^q} + C\|u\|_{L^2}. \tag{*}$$

- To conclude, we show that

$$\|u\|_{L^2} \leq C\|f\|_{L^q}. \tag{**}$$

  More precisely, we show that "(*) + injectivity of L $\Rightarrow$ (**)".

Energy estimate with L^q right hand side

Proof

- Suppose by contradiction that there exists sequence $u_m \in H_0^1(B_1)$, $f_m \in L^q(B_1)$ such that $Lu_m = f_m$ but
$$\|u_m\|_{L^2} > m\|f_m\|_{L^q}.$$
Replacing u_m by $\frac{1}{\|u_m\|_{L^2}} u_m$ if necessary, we can assume that $\|u_m\|_{L^2} = 1$.
- Then $\|u_m\|_{L^2} = 1$, $\|f_m\|_{L^q} < \frac{1}{m}$ and by (*), $\|u_m\|_{H^1} \leq C$.
By the reflexivity of H^1 and Rellich-Kondrachov's theorem, we may assume that $u_m \rightharpoonup u$ in H^1 and $u_m \to u$ in L^2.
Note that $\|u\|_{L^2} = 1$.
- To conclude, we show that $Lu = 0$, which implies $u = 0$ by hypothesis, and amounts to a contradiction with $\|u\|_{L^2} = 1$.

Energy estimate with L^q right hand side

Proof

- We start with $Lu_m = f_m$ which means

$$\int_{B_1} \Big[a_{ij}\partial_j u_m \partial_i v + b_i \partial_i u_m v + c u_m v\Big]\,dx = \int_{B_1} f_m v\,dx \text{ for all } v \in H_0^1(B_1).$$

  We then send $m \to \infty$ using that $\nabla u_m \rightharpoonup \nabla u$ in L^2, $u_m \to u$ in L^2 and $f_m \to 0$ in L^q to obtain

$$\int_{B_1} \Big[a_{ij}\partial_j u \partial_i v + b_i \partial_i u v + c u v\Big]\,dx = 0 \text{ for all } v \in H_0^1(B_1),$$

  i.e. $Lu = 0$, as desired.

- As $u_m \in H_0^1(B_1)$, we have $u \in H_0^1(B_1)$ and so $u = 0$ by hypothesis. This contradicts the identity $\|u\|_{L^2} = 1$, and finishes the proof.

Non-uniformly elliptic: A case study

Let us now consider an example in $1d$:

$$\begin{cases} -(au')' = f \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

As $k \to 0$, the ellipticity deteriorates. As $k \to \infty$, the boundedness of k deteriorates.

We have proved 2 estimates:

$$\|u\|_{L^\infty(-1,1)} \leq C_1(k)\|f\|_{L^\infty(-1,1)}, \tag{1}$$

$$\|u\|_{L^\infty(-1,1)} \leq C_2(k)(\|f\|_{L^\infty(-1,1)} + \|u\|_{L^2(-1,1)}). \tag{2}$$

We would now like to have a rough appreciation whether (or how) these constants depend on k, as $k \to 0$ or ∞.

Non-uniformly elliptic: A case study

$$\begin{cases} -(au')' = f \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- We empirically take $f = 1$, so that $\|f\|_{L^\infty} = 1$.
- We know that the problem has uniqueness (why?), so it suffices to find a solution.
- The equation gives $-u'' = 1$ in $(-1,0)$ and $-u'' = 1/k$ in $(0,1)$. So u takes the form

$$u(x) = \begin{cases} -\frac{1}{2}(x+1)^2 + \alpha(x+1) & \text{for } x \in (-1,0), \\ -\frac{1}{2k}(x-1)^2 + \beta(x-1) & \text{for } x \in (0,1). \end{cases}$$

Non-uniformly elliptic: A case study

$$\begin{cases} -(au')' = 1 \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- As $u \in H^1(-1,1)$, u is continuous. So

$$-\frac{1}{2} + \alpha = -\frac{1}{2k} - \beta.$$

- As au' is weakly differentiable, it is continuous and so

$$-1 + \alpha = 1 + k\beta.$$

- So we find $\alpha = \frac{k+3}{2(k+1)}$ and $\beta = -\frac{3k+1}{2k(k+1)}$.

Non-uniformly elliptic: A case study

$$\begin{cases} -(au')' = 1 \text{ in } (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \qquad \text{where } a = \chi_{(-1,0)} + k\chi_{(0,1)}.$$

- So we have

$$u(x) = \begin{cases} -\frac{1}{2}(x+1)^2 + \frac{k+3}{2(k+1)}(x+1) & \text{for } x \in (-1,0), \\ -\frac{1}{2k}(x-1)^2 - \frac{3k+1}{2k(k+1)}(x-1) & \text{for } x \in (0,1). \end{cases}$$

- We find $\|u\|_{L^\infty} \sim \frac{1}{k}$ as $k \to 0$, and $\|u\|_{L^\infty} \sim 1$ as $k \to \infty$. Therefore

$$C_1(k) \sim \frac{1}{k} \text{ as } k \to 0, \text{ and } C_1(k) \sim 1 \text{ as } k \to \infty.$$

- Similarly $\|u\|_{L^2} \sim \frac{1}{k}$ as $k \to 0$, and $\|u\|_{L^2} \sim 1$ as $k \to \infty$. Therefore

$$C_2(k) \sim 1 \text{ as } k \to 0, \infty.$$