Exercise sheet 4. W1 of Hilary Term. All lectures.

Q1. Suppose in this exercise that $\mathrm{char}(k) = 0$. Find the singularities of the following curves C in k^2. For each singular point $P \in C$ compute the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space. Here $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{C,P}$.

(1) $\mathrm{Z}(x^6 + y^6 - xy)$

(2) $\mathrm{Z}(y^2 + x^4 + y^4 - x^3)$

You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

Q2. (blowing up the origin in affine space) Let $n \geqslant 1$. Let $x_1, \dots, x_n$ be variables for k^n and let $y_1, \dots, y_n$ be homogenous variables for $\mathbb{P}^{n-1}(k)$. Note that contrary to what is customary, the index of the homogenous variables runs between 1 and n here (not 0 and $n-1$).

(1) Let Z be the subset of $k^n \times \mathbb{P}^{n-1}(k)$ defined by the equations $\{x_i y_j - x_j y_i = 0\}_{i,j \in \{1,\dots,n\}}$ (note that this makes sense because the polynomials are homogenous in the y-variables). Show that Z is a closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$. The variety Z is called the *blow-up* of k^n at the origin of k^n. Let $\phi : Z \to k^n$ the map obtained by restricting the projection $k^n \times \mathbb{P}^{n-1}(k) \to k^n$ to Z.

(2) Show that $\phi^{-1}(\{0\})$ is canonically isomorphic to $\mathbb{P}^{n-1}(k)$. Show that the points of $\phi^{-1}(0)$ are in one-to-one correspondence with the lines going through the origin of k^n.

(3) Show that the restriction of ϕ to the open subvariety $\phi^{-1}(k^n \backslash \{0\})$ of Z induces an isomorphism $\phi^{-1}(k^n \backslash \{0\}) \simeq k^n \backslash \{0\}$.

Q3. (blowing up a point of an affine variety) Let $X \subseteq k^n$ be a closed subvariety (ie an algebraic set). Let $\bar{v} := \langle v_1, \dots, v_n \rangle \in X$ and suppose that $\{\bar{v}\}$ is not an irreducible component of X. Let $\tau_{\bar{v}} : k^n \to k^n$ be the map such that $\tau_{\bar{v}}(\langle w_1, \dots, w_n \rangle) = \langle w_1 + v_1, \dots, w_n + v_n \rangle$ for all $\bar{w} = \langle w_1, \dots, w_n \rangle \in k^n$ (note that this is an automorphism of the variety k^n). Let $Y := \tau_{-\bar{v}}(X)$. Note that by construction we have $0 \in Y$. Let $\phi : Z \to k^n$ be the morphism defined in Q2.

We define the *blow-up* $\mathrm{Bl}(X, \bar{v})$ of X at $\bar{v}$ to be the closure of $\phi^{-1}(Y \backslash \{0\})$ in Z.

(1) Show that $\phi(\mathrm{Bl}(X, \bar{v})) = Y$.

Let $b : \mathrm{Bl}(X, \bar{v}) \to X$ be the morphism $\tau_{\bar{v}} \circ \phi|_{\mathrm{Bl}(X,\bar{v})}$.

(2) Suppose that X is irreducible. Show that $\mathrm{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y) \subseteq k^n \times \mathbb{P}^{n-1}(k)$. Show that b is a birational morphism. If $X \neq k^n$, show that the irreducible components of $\phi^{-1}(Y)$ are $\mathrm{Bl}(X, \bar{v})$ and $\{0\} \times \mathbb{P}^{n-1}(k)$.

The closed set $b^{-1}(\{v\}) = \mathrm{Bl}(X, \bar{v}) \cap (\{0\} \times \mathbb{P}^{n-1}(k))$ is called the *exceptional divisor* of $\mathrm{Bl}(X, \bar{v})$.

Q4. Let C be the plane curve considered in (1) of Q1. Consider the blow-up B of C at each of its singular points in turn. How many irreducible components does the exceptional divisor of B have? Is B non-singular?

Q5. Let C be the curve $y^2 = x^3$ in k^2. Let $b : \mathrm{Bl}(C, 0) \to C$ of C be the blow-up of C at the origin.

(1) Show that $\mathrm{Bl}(C, 0) \simeq k$.

(2) Show that the map b is a homeomorphism but is not an isomorphism.

Q6. Let $V \subseteq k^2$ be the algebraic set defined by the equation $x_1 x_2 = 0$. Show that $\mathrm{Bl}(V, 0)$ has two disjoint irreducible components and that each of these components is isomorphic to k.

Q7. (1) Let $f : X \to Y$ be a dominant morphism of varieties. Suppose that Y is irreducible. Show that $\dim(X) \geqslant \dim(Y)$.

(2) Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Suppose that the field extension $\kappa(X)|\kappa(Y)$ is algebraic. Show that there are affine open subvarieties $U \subseteq X$ and $W \subseteq Y$ such that $f(U) = W$ and such that the map of rings $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is injective and finite.

(3) Let $f : X \to Y$ be a dominant morphism of irreducible quasi-projective varieties. Show that there is a $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$. [*Hint. Reduce to the situation where Y is affine and apply Noether's normalisation lemma to show that you may assume wlog that $Y = k^n$ for some n. Now use the existence of transcendence bases and (2) to show that there is an open subvariety $U \subseteq X$ and an open subvariety W of $k^{\dim(X)-\dim(Y)} \times k^n$ such that $f|_U$ factors as a finite and surjective morphism $U \to W$, followed by the projection to k^n. Now deduce the result from (1) and a computation of the dimension of the fibres of the projection $k^{\dim(X)-\dim(Y)} \times k^n \to k^n$.*]

(4) Deduce that in the situation of (3), the set of $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$ is dense in Y.

Q8. (1) Show that all the morphisms from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use Q7 and the projective dimension theorem.]*

(2) Deduce from (1) that for any $n \geqslant 2$ the morphisms from $\mathbb{P}^n(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use (1) and Q7 of Sheet 2.]*