Exercise sheet 4. W1 of Hilary Term. All lectures.

Q1. Suppose in this exercise that $\mathrm{char}(k) = 0$. Find the singularities of the following curves C in k^2. For each singular point $P \in C$ compute the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space. Here $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{C,P}$.

(1) $\mathrm{Z}(x^6 + y^6 - xy)$

(2) $\mathrm{Z}(y^2 + x^4 + y^4 - x^3)$

You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

Solution. (1) Note that $\dim(C) = 1$ by Krull's theorem and by Theorem 8.7. Thus we need to find the points of C where the gradient of $x^6 + y^6 - xy$ vanishes. The gradient of $x^6 + y^6 - xy$ is $\langle 6x^5 - y, 6y^5 - x\rangle$. Hence we need to solve the equations $x^6 + y^6 - xy = 6x^5 - y = 6y^5 - x = 0$. We have

$$(x/6)(6x^5 - y) - (y/6)(6y^5 - x) + 2y^6 - xy = x^6 + y^6 - xy$$

and thus these equations are equivalent to

$$2y(y^5 - x) = 6x^5 - y = 6y^5 - x = 0.$$

Now if $y = 0$ then $x = 0$. If $y \neq 0$ then $y^5 = x = x/6$ so $y = 0$, which is a contradiction. So we must have $x = y = 0$. So $\langle 0, 0\rangle$ is the only singular point of C.

For $P = \langle 0, 0\rangle$ the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space cannot be 1, since otherwise the ring $\mathcal{O}_{C,P}$ would be regular (apply Proposition 13.3). Since $\mathfrak{m}_P$ is generated as a $k[x, y]$-module by the elements x and y, we see that $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension at most 2. Hence $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension 2.

(2) The reasoning is similar. Solve $y^2 + x^4 + y^4 - x^3 = 4x^3 - 3x^2 = 2y + 4y^3 = 0$. Combining, we obtain

$$4(y^2 + x^4 + y^4 - x^3) + (1/4 - x)(4x^3 - 3x^2) - y(2y + 4y^3) = (-3/4)x^2 + 2y^2 = 0.$$

Now if $x \neq 0$ then $x = 3/4$ since $4x^3 - 3x^2 = 0$ and so $y^2 = 27/128$. But then $y(2y+4y^3) = 6503409/67108864$ which is a contradiction. So we have $x = 0$ and also $y = 0$. We conclude again that the origin is the only singular point of C. By the same reasoning as above, we see that $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension 2.

Q2. (blowing up the origin in affine space) Let $n \geqslant 1$. Let $x_1, \dots, x_n$ be variables for k^n and let $y_1, \dots, y_n$ be homogenous variables for $\mathbb{P}^{n-1}(k)$. Note that contrary to what is customary, the index of the homogenous variables runs between 1 and n here (not 0 and $n - 1$).

(1) Let Z be the subset of $k^n \times \mathbb{P}^{n-1}(k)$ defined by the equations $\{x_i y_j - x_j y_i = 0\}_{i,j\in\{1,\dots,n\}}$ (note that this makes sense because the polynomials are homogenous in the y-variables). Show that Z is a closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$. The variety Z is called the *blow-up* of k^n at the origin of k^n. Let $\phi : Z \to k^n$ the map obtained by restricting the projection $k^n \times \mathbb{P}^{n-1}(k) \to k^n$ to Z.

(2) Show that $\phi^{-1}(\{0\})$ is canonically isomorphic to $\mathbb{P}^{n-1}(k)$. Show that the points of $\phi^{-1}(0)$ are in one-to-one correspondence with the lines going through the origin of k^n.

(3) Show that the restriction of ϕ to the open subvariety $\phi^{-1}(k^n\backslash\{0\})$ of Z induces an isomorphism $\phi^{-1}(k^n\backslash\{0\}) \simeq k^n\backslash\{0\}$.

Solution. (1) On the open affine subset $k^n \times U^{n-1}_{j_0}$, Z is given by the equations

$$\{x_i y_j - x_j y_i = 0, x_i - x_{j_0} y_i = 0\}_{i\in\{1,\dots,n\}, j\in\{1,\dots,j_0-1,j_0+1,\dots,n\}}.$$

The set $Z \cap k^n \times U_{j_0}^{n-1}$ is thus closed in $k^n \times U_{j_0}^{n-1}$. Since the $k^n \times U_j^{n-1}$ cover $k^n \times \mathbb{P}^{n-1}(k)$, we see that Z is closed.

(2) It follows from the definitions that $\phi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1}(k)$.

(3) Suppose that $\langle X_1, \dots, X_n\rangle \neq 0$. Then there is an i_0 such that $X_{i_0} \neq 0$. The equations for Z then give $Y_j = X_j(Y_{i_0}/X_{i_0})$ for all j. Up to multiplication of all the Y_j by a non zero scalar factor, the only solution to this set of equations is $\langle X_1, \dots, X_n\rangle$. In particular, we have

$$\phi^{-1}(\langle X_1, \dots, X_n\rangle) = \{\langle X_1, \dots, X_n\rangle\} \times \{[X_1, \dots, X_n]\}.$$

This shows that the morphism $\phi^{-1}(k^n\backslash\{0\}) \to k^n\backslash\{0\}$ is a bijection. To show that it is an isomorphism, we shall provide an inverse morphism. For this, consider the morphism $q : k^n\backslash\{0\} \to \mathbb{P}^{n-1}(k)$ introduced in Q6 of Sheet 3. We define a map $k^n\backslash\{0\} \to Z$ by the formula $g := \mathrm{Id}_{k^n\backslash\{0\}} \prod q$. By construction, this gives an inverse of the morphism $\phi^{-1}(k^n\backslash\{0\}) \to k^n\backslash\{0\}$.

Q3. (blowing up a point of an affine variety) Let $X \subseteq k^n$ be a closed subvariety (ie an algebraic set). Let $\bar{v} := \langle v_1, \dots, v_n\rangle \in X$ and suppose that $\{\bar{v}\}$ is not an irreducible component of X. Let $\tau_{\bar{v}} : k^n \to k^n$ be the map such that $\tau_{\bar{v}}(\langle w_1, \dots, w_n\rangle) = \langle w_1 + v_1, \dots, w_n + v_n\rangle$ for all $\bar{w} = \langle w_1, \dots, w_n\rangle \in k^n$ (note that this is an automorphism of the variety k^n). Let $Y := \tau_{-\bar{v}}(X)$. Note that by construction we have $0 \in Y$. Let $\phi : Z \to k^n$ be the morphism defined in Q2.

We define the *blow-up* $\mathrm{Bl}(X, \bar{v})$ of X at $\bar{v}$ to be the closure of $\phi^{-1}(Y\backslash\{0\})$ in Z.

(1) Show that $\phi(\mathrm{Bl}(X, \bar{v})) = Y$.

Let $b : \mathrm{Bl}(X, \bar{v}) \to X$ be the morphism $\tau_{\bar{v}} \circ \phi|_{\mathrm{Bl}(X,\bar{v})}$.

(2) Suppose that X is irreducible. Show that $\mathrm{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y) \subseteq k^n \times \mathbb{P}^{n-1}(k)$. Show that b is a birational morphism. If $X \neq k^n$, show that the irreducible components of $\phi^{-1}(Y)$ are $\mathrm{Bl}(X, \bar{v})$ and $\{0\} \times \mathbb{P}^{n-1}(k)$.

The closed set $b^{-1}(\{v\}) = \mathrm{Bl}(X, \bar{v}) \cap (\{0\} \times \mathbb{P}^{n-1}(k))$ is called the *exceptional divisor* of $\mathrm{Bl}(X, \bar{v})$.

Solution. (1) Note first that $\bar{v}$ lies in the closure of $X\backslash\{\bar{v}\}$. To see this, let C be the irreducible component of X containing $\bar{v}$. Then $C\backslash\{\bar{v}\}$ is non-empty (by assumption) and it is open in C (since $\{\bar{v}\}$ is closed). Furthermore, $C\backslash\{\bar{v}\}$ is not closed in C, for otherwise C would be disconnected and hence reducible. Thus $\bar{v}$ lies in the closure of $C\backslash\{0\}$ in C (which must be C) and hence $\bar{v}$ lies in the closure of $X\backslash\{\bar{v}\}$ in X.

Now since $\mathbb{P}^{n-1}(k)$ is complete (see Theorem 12.9), we know that $\phi(\mathrm{Bl}(X, \bar{v}))$ is closed. By (3) of Q2, we now that $\phi(\mathrm{Bl}(X, \bar{v}))\backslash\{\bar{v}\} = X\backslash\{\bar{v}\}$ and thus by the reasoning in the last paragraph, we see that $\bar{v} \in \phi(\mathrm{Bl}(X, \bar{v}))$. In particular, $\phi(\mathrm{Bl}(X, \bar{v})) = Y$.

(2) From Q2 (3) we know that the natural morphism $\phi^{-1}(Y\backslash\{0\}) \to Y\backslash\{0\}$ is an isomorphism. Now if X is irreducible, so is Y and so is $Y\backslash\{0\}$. Hence $\mathrm{Bl}(X, \bar{v})$ is irreducible by Q4 (1) of Sheet 2. On the other hand, $\mathrm{Bl}(X, \bar{v}) \subseteq \phi^{-1}(Y)$ since $\phi^{-1}(Y)$ is closed in Z. Since $\mathrm{Bl}(X, \bar{v})$ contains the non empty open subset set $\phi^{-1}(Y\backslash\{0\})$ of $\phi^{-1}(Y)$, we see that $\mathrm{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y)$. Since $\phi^{-1}(Y\backslash\{0\}) \to Y\backslash\{0\}$ is an isomorphism, the morphism b is birational.

On the other hand, we have by construction $\phi^{-1}(Y) = \mathrm{Bl}(X, \bar{v}) \cup (\{0\} \times \mathbb{P}^{n-1}(k))$. Now suppose that $X \neq k^n$. We then have $\{0\} \times \mathbb{P}^{n-1}(k) \not\subseteq \mathrm{Bl}(X, \bar{v})$ because

$$\dim(\{0\} \times \mathbb{P}^{n-1}(k)) = n - 1 \geqslant \dim(\mathrm{Bl}(X, \bar{v})) = \dim(X) \leqslant n - 1$$

(use Proposition 9.2, Q6 of Sheet 6 and Theorem 8.7). Since $\{0\} \times \mathbb{P}^{n-1}(k)$ is irreducible (since it is isomorphic to $\mathbb{P}^{n-1}(k)$) we see that the irreducible components of $\phi^{-1}(Y)$ are $\mathrm{Bl}(X, \bar{v})$ and $\{0\} \times \mathbb{P}^{n-1}(k)$.

Q4. Let C be the plane curve considered in (1) of Q1. Consider the blow-up B of C at each of its singular points in turn. How many irreducible components does the exceptional divisor of B have? Is B non-singular?

Solution. Consider the curve $\mathrm{Z}(x_1x_2 - x_1^6 - x_2^6) \subseteq k^2$ of (1) of Q1. Use the terminology of Q2 and Q3, letting $n = 2$ and $X = \mathrm{Z}(x_1x_2 - x_1^6 - x_2^6) = Y$ (note that the point to blow-up is the origin by the solution Q1 (1) so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = \mathrm{Z}(x_1y_2 - x_2y_1, x_1x_2 - x_1^6 - x_2^6)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subseteq \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} &\phi^{-1}(X) \cap (k^2 \times U_1) = \mathrm{Z}(x_1y_2 - x_2, x_1x_2 - x_1^6 - x_2^6) = \mathrm{Z}(x_1y_2 - x_2, x_1^2y_2 - x_1^6 - x_1^6y_2^6) \\ = \;& \mathrm{Z}(x_1y_2 - x_2, x_1^2(y_2 - x_1^4 - x_1^4y_2^6)) = \mathrm{Z}(x_1y_2 - x_2, x_1) \cup \mathrm{Z}(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6)) \\ = \;& \{0\} \times U_1 \cup \mathrm{Z}(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2) \end{aligned}$$

Now $\mathrm{Z}(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2)$ does not contain $\{0\} \times U_1$ (since setting $x_1 = x_2 = 0$ implies that $y_2 = 0$) so we have $\mathrm{Bl}(X, 0) \cap (k^2 \times U_1) = \mathrm{Z}(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2)$ by Q3 (2). Finally, note that $\mathrm{Z}(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2) \cap (\{0\} \times U_1)$ contains only the point $\{0\} \times \{[1, 0]\}$. In other words, the intersection of the exceptional divisor of $\mathrm{Bl}(X, 0)$ with $\{0\} \times U_1$ is the point $\{0\} \times \{[1, 0]\}$.

Let now $U_2 := \{[Y_1, 1] \mid Y_1 \in k\} \subseteq \mathbb{P}^1(k)$. We compute as before

$$\begin{aligned} &\phi^{-1}(X) \cap (k^2 \times U_1) = \mathrm{Z}(x_1 - x_2y_1, x_1x_2 - x_1^6 - x_2^6) = \mathrm{Z}(x_1 - x_2y_1, y_1x_2^2 - x_2^6y_1^6 - x_2^6) \\ = \;& \mathrm{Z}(x_1 - x_2y_1, x_2) \cup \mathrm{Z}(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) = \{0\} \times U_2 \cup \mathrm{Z}(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) \end{aligned}$$

We conclude as before that

$$\mathrm{Bl}(X, 0) \cap (k^2 \times U_2) = \mathrm{Z}(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4)$$

We compute $\mathrm{Z}(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) \cap (\{0\} \times U_2) = \{0\} \times \{[0, 1]\}$. So the intersection of the exceptional divisor of $\mathrm{Bl}(X, 0)$ with $\{0\} \times U_2$ is the point $\{0\} \times [0, 1]$.

Putting everything together, we see that the exceptional divisor of $\mathrm{Bl}(X, 0)$ consists of the points $\{0\} \times \{[1, 0]\}$ and $\{0\} \times \{[0, 1]\}$. In particular, the exceptional divisor of $\mathrm{Bl}(X, 0)$ has two irreducible components.

We now check non-singularity. We only have to check the non-singularity of $\mathrm{Bl}(X, 0)$ at $\{0\} \times \{[1, 0]\}$ and $\{0\} \times \{[0, 1]\}$ since $\mathrm{Bl}(X, 0) \backslash \{\{0\} \times \{[1, 0]\} \cup \{0\} \times \{[0, 1]\}\}$ is isomorphic to $X \backslash \{0\}$ and $X \backslash \{0\}$ is non-singular by the solution of Q1(1).

We first check non-singularity at $\{0\} \times \{[1, 0]\}$. Let $Q_1 := x_1y_2 - x_2$ and $Q_2 := y_2 - x_1^4 - x_2^4y_2^2$. We have

$$\begin{pmatrix} \frac{\partial}{\partial x_1}Q_1 & \frac{\partial}{\partial x_2}Q_1 & \frac{\partial}{\partial y_2}Q_1 \\ \frac{\partial}{\partial x_1}Q_2 & \frac{\partial}{\partial x_2}Q_2 & \frac{\partial}{\partial y_2}Q_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -4x_1^3 & -4x_2^3y_2^2 & 1 - 2x_2^4y_2 \end{pmatrix}$$

and evaluating at 0 we get the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has rank 2. Using Lemma 13.5 we see that $\mathrm{Bl}(X,0)$ is non-singular at $\{0\}\times\{[1,0]\}$.

We now check non-singularity at $\{0\}\times\{[0,1]\}$. Let $Q_1 := x_1 - x_2y_1$ and $Q_2 := y_1 - x_2^4y_1^6 - x_2^4$. We have

$$\begin{pmatrix} \frac{\partial}{\partial x_1}Q_1 & \frac{\partial}{\partial x_2}Q_1 & \frac{\partial}{\partial y_2}Q_1 \\ \frac{\partial}{\partial x_1}Q_2 & \frac{\partial}{\partial x_2}Q_2 & \frac{\partial}{\partial y_2}Q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4x_2^3 - 4x_2^3y_1^6 & 1-6x_2^4y_1^5 \end{pmatrix}$$

and evaluating at 0 we get the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which again has rank 2. Again using Lemma 13.5 we see that $\mathrm{Bl}(X,0)$ is non-singular at $\{0\}\times\{[0,1]\}$.

So all in all $\mathrm{Bl}(X,0)$ is non-singular and its exceptional divisor has two irreducible components (which are points).

Q5. Let C be the curve $y^2 = x^3$ in k^2. Let $b : \mathrm{Bl}(C,0) \to C$ of C be the blow-up of C at the origin.

(1) Show that $\mathrm{Bl}(C,0) \simeq k$.

(2) Show that the map b is a homeomorphism but is not an isomorphism.

Solution. Use the terminology of Q2 and Q3, letting $n = 2$ and $X = \mathrm{Z}(x_2^2 - x_1^3) = Y$ (note that the point to blow-up is the origin so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = \mathrm{Z}(x_1y_2 - x_2y_1, x_2^2 - x_1^3)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subset \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} &\phi^{-1}(X) \cap (k^2 \times U_1) = \mathrm{Z}(x_1y_2 - x_2, x_2^2 - x_1^3) = \mathrm{Z}(x_1y_2 - x_2, x_1^2y_2^2 - x_1^3) \\ = \quad &\mathrm{Z}(x_1y_2 - x_2, x_1) \cup \mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1) = (\{0\} \times U_1) \cup \mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1) \end{aligned}$$

The closed set $\mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1)$ does not contain $\{0\} \times U_1$. Also $\phi^{-1}(X) \cap (k^2 \times U_1)$ has at most two irreducible components by Q2 (2) so we conclude that $\mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1) = \mathrm{Bl}(X,0) \cap (k^2 \times U_1)$. On the other hand, $\mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1) \cap (\{0\} \times U_1) = \{0\} \times \{[1,0]\}$.

We now repeat the above reasoning for $U_2 := \{[Y_1, 1] \mid Y_1 \in k\} \subseteq \mathbb{P}^1(k)$ instead of U_1. We have

$$\begin{aligned} &\phi^{-1}(X) \cap (k^2 \times U_2) = \mathrm{Z}(x_1 - x_2y_1, x_2^2 - x_1^3) = \mathrm{Z}(x_1 - x_2y_1, x_2^2 - x_2^3y_1^3) \\ = \quad &\mathrm{Z}(x_1 - x_2y_1, x_2) \cup \mathrm{Z}(x_1 - x_2y_1, 1 - x_2y_2^3) = (\{0\} \times U_2) \cup \mathrm{Z}(x_1 - x_2y_1, 1 - x_2y_2^3) \end{aligned}$$

As before, we have $\mathrm{Z}(x_1 - x_2y_1, 1 - x_2y_2^3)) \cap (k^2 \times U_2) = \mathrm{Bl}(X,0) \cap (k^2 \times U_2)$. On the other hand, a simple calculation shows that $\mathrm{Z}(x_1 - x_2y_1, 1 - x_2y_2^3) \cap (\{0\} \times U_2) = \emptyset$.

So we conclude that the exceptional divisor of $\mathrm{Bl}(X,0)$ consist of only the point $\{0\} \times \{[1,0]\}$. In particular, the map $b : \mathrm{Bl}(X,0) \to X$ is bijective. Since $\mathrm{Bl}(X,0)$ is complete, the morphism b sends closed sets to closed sets (see Theorem 12.9 and Corollary 12.10) and thus (since b is bijective), b sends open sets to open sets. Hence b is a homeomorphism. This answers part of (2). On the other hand

$$\phi^{-1}(X) \cap k^2 \times (\mathbb{P}^1 \backslash U_1) = \mathrm{Z}(x_1y_2 - x_2y_1, x_2^2 - x_1^3, y_1) = \mathrm{Z}(x_1, y_1, x_2) = \{0\} \times \{[0,1]\}$$

and this set is not in $\mathrm{Bl}(X,0)$ by the above. Hence

$$\mathrm{Bl}(X,0) = \mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1) \subseteq \{0\} \times U_1 \subseteq k^3$$

We claim that the map $A(t) = \langle t^2, t^3, t\rangle$ gives an isomorphism between k and $\mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1)$. Indeed this map has an inverse, which is the restriction to $\mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1)$ of the map $B : k^3 \to k$ given by the formula $B(X_1, X_2, Y_2) = Y_2$. To verify this, note first that we clearly have $A(t) \in \mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1)$ and $B(A(t)) = t$. Secondly, for $\langle X_1, X_2, Y_2\rangle \in \mathrm{Z}(x_1y_2 - x_2, y_2^2 - x_1)$ we have

$$A(B(X_1, X_2, Y_2)) = (Y_2^2, Y_2^3, Y_2)$$

and we have $Y_2^2 = X_1$, $Y_2^3 = X_1Y_2 = X_2$. We conclude that $\mathrm{Bl}(X, 0) \simeq k$.

Q6. Let $V \subseteq k^2$ be the algebraic set defined by the equation $x_1x_2 = 0$. Show that $\mathrm{Bl}(V, 0)$ has two disjoint irreducible components and that each of these components is isomorphic to k.

Solution. Use the terminology of Q2 and Q3, letting $n = 2$ and $X = \mathrm{Z}(x_1x_2) = Y$ (note that the point to blow-up is the origin so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = \mathrm{Z}(x_1y_2 - x_2y_1, x_1x_2)$$

Let $U_1 := \{[1, Y_2] \,|\, Y_2 \in k\} \subseteq \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} &\phi^{-1}(X) \cap (k^2 \times U_1) = \mathrm{Z}(x_1y_2 - x_2, x_1x_2) = \mathrm{Z}(x_1y_2 - x_2, x_1) \cup \mathrm{Z}(x_1y_2 - x_2, x_2) \\ = \;& \{0\} \times U_1 \cup \mathrm{Z}(x_1y_2, x_2) = \{0\} \times U_1 \cup Z(x_1, x_2) \cup \mathrm{Z}(y_2, x_2) = \{0\} \times U_1 \cup \mathrm{Z}(y_2, x_2)\end{aligned}$$

Now note that by definition $\mathrm{Bl}(X, 0)$ is the closure of $\phi^{-1}(X\backslash 0)$. In particular, $\mathrm{Bl}(X, 0)$ is the union of the closures of $\phi^{-1}(\mathrm{Z}(x_1)\backslash 0)$ and $\phi^{-1}(\mathrm{Z}(x_1)\backslash 0)$, ie the blow-ups of $\mathrm{Z}(x_1)$ and of $\mathrm{Z}(x_2)$, respectively. Now note that $\phi^{-1}(\mathrm{Z}(x_1)\backslash 0) \cap (k^2 \times U_1) = \emptyset$ (see the solution to Q2 (3)). Noting also that $\mathrm{Z}(y_2, x_2)$ is irreducible, we see that $\mathrm{Bl}(X, 0) \cap (k^2 \times U_1)) = \mathrm{Z}(y_2, x_2)$.

A completely similar reasoning with U_2 in place of U_1 shows that $\mathrm{Bl}(X, 0) \cap (k^2 \times U_2) = \mathrm{Z}(y_1, x_1)$. Hence $\mathrm{Bl}(X, 0) \subseteq \mathrm{Z}(y_2, x_2) \cup \mathrm{Z}(y_1, x_1) \subseteq k^2 \times \mathbb{P}^1(k)$, where we view the polynomials x_1, x_2, y_1, y_2 as homogenous polynomials in the y-variables. On the other hand we have $\mathrm{Z}(y_2, x_2) \cap \mathrm{Z}(y_1, x_1) = \mathrm{Z}(x_1, x_2, y_1, y_2) = \emptyset$ and $\mathrm{Z}(y_2, x_2) \simeq \mathrm{Z}(y_1, x_1) \simeq k$. Since $\mathrm{Bl}(X, 0)$ has two irreducible components of dimension 1 by the above, we thus have $\mathrm{Bl}(X, 0) = \mathrm{Z}(y_2, x_2) \cup \mathrm{Z}(y_1, x_1)$.

Q7. (1) Let $f : X \to Y$ be a dominant morphism of varieties. Suppose that Y is irreducible. Show that $\dim(X) \geqslant \dim(Y)$.

(2) Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Suppose that the field extension $\kappa(X)|\kappa(Y)$ is algebraic. Show that there are affine open subvarieties $U \subseteq X$ and $W \subseteq Y$ such that $f(U) = W$ and such that the map of rings $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is injective and finite.

(3) Let $f : X \to Y$ be a dominant morphism of irreducible quasi-projective varieties. Show that there is a $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$. [*Hint. Reduce to the situation where Y is affine and apply Noether's normalisation lemma to show that you may assume wlog that $Y = k^n$ for some n. Now use the existence of transcendence bases and (2) to show that there is an open subvariety $U \subseteq X$ and an open subvariety W of $k^{\dim(X)-\dim(Y)} \times k^n$ such that $f|_U$ factors as a finite and surjective morphism $U \to W$, followed by the projection to k^n. Now deduce the result from (1) and a computation of the dimension of the fibres of the projection $k^{\dim(X)-\dim(Y)} \times k^n \to k^n$.*]

(4) Deduce that in the situation of (3), the set of $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X)-\dim(Y)$ is dense in Y.

Solution. (1) Let $\{X_i\}$ be the irreducible components of X. Then $f(X_i)$ is irreducible for all i and hence the closure $\overline{f(X_i)}$ is also irreducible for all i (by Q4 (1) of Sheet 2). Hence we must have $\cup_i \overline{f(X_i)} = Y$, otherwise f is not dominant. Now if $\overline{f(X_i)} \neq Y$ for all i then Y is not irreducible, which is impossible. So there is an index i_0 such that $\overline{f(X_{i_0})} = Y$. In that case we have a field extension $\kappa(X_{i_0})|\kappa(Y)$ and thus $\dim(X_{i_0}) \geqslant \dim(Y)$ by Proposition 9.2. In now follows from the definition of dimension that $\dim(X) \geqslant \dim(Y)$.

(2) We first prove the following statement of commutative algebra. Let $\phi : A \to B$ be a homomorphism of finitely generated integral k-algebras. Suppose that $\mathrm{Spm}(\phi)(\mathrm{Spm}(B))$ is dense in $\mathrm{Spm}(A)$ and suppose that the induced map $\mathrm{Frac}(\phi) : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$ is an algebraic extension of fields. Then there is an element $f \in A$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective and finite.

To prove this assertion, note that by Q5 of Sheet 1 we already know that under the given assumptions, ϕ must be injective. Note also that since we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Frac}(A) & \xrightarrow{\mathrm{Frac}(\phi)} & \mathrm{Frac}(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\phi} & B \end{array}$$

all whose maps are injective, the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective for any choice of $f \in A\backslash\{0\}$ (remember that A and B are integral domains). Thus we only have to show that there is $f \in A\backslash\{0\}$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is finite. Now let $b_1, \dots, b_l$ be generators of B as a k-algebra. By assumption, each $b_i/1 \in \mathrm{Frac}(B)$ satisfies a monic polynomial equation with coefficients in $\mathrm{Frac}(A)$. Let $f \in A$ be the product of the denominators of all the coefficients of all these equations. Note that $B[\phi(f)^{-1}]$ is generated as a k-algebra by $1/\phi(f)$ and by the elements $b_i/1$ (use Lemma 5.3 in CA). In particular, $B[\phi(f)^{-1}]$ is generated by the $b_i/1$ as a $A[f^{-1}]$-algebra. On the other hand, by construction, the elements $b_i/1$ all satisfy integral equations over $A[f^{-1}]$. Hence $A[f^{-1}] \to B[\phi(f)^{-1}]$ is a finite map of rings (see section 8 in CA).

Note that the fact that $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective and finite implies that the induced map

$$\mathrm{Spm}(B[\phi(f)^{-1}]) \to \mathrm{Spm}(A[f^{-1}])$$

is surjective (use Th. 8.8 and Cor. 8.10 in CA).

Returning to the problem at hand, note that we may wlog assume that X and Y are affine (take an affine open Y' in Y and an affine open X' in $f^{-1}(Y')$ and replace X by X' (resp. Y by Y'). Applying the result of commutative algebra that we just proved to $A = \mathcal{O}_X(X)$ and $B = \mathcal{O}_Y(Y)$ we obtain the desired result.

(3) Note that Th. 9.1 (Noether's normalisation lemma), Prop. 8.12, Th. 8.8 and Cor. 8.10 in CA imply that for some $n \geqslant 0$ there is a surjective morphism $h : Y \to k^{\dim(Y)}$, such that the fibre $h^{-1}(\bar{v})$ of h over $\bar{v}$ is finite for all $\bar{v} \in k^n$. Since the fibres of the composed morphism $h \circ f$ are finite disjoint unions of fibres of f, we may thus replace f by $h \circ f$ and suppose that $Y = k^n$ for some $n \geqslant 0$.

Now consider the field extension $\kappa(X)|\kappa(Y)$. Choose a transcendence basis $b_1, \dots, b_\delta \in \kappa(X)$ of $\kappa(X)$ over $\kappa(Y)$. Write $\kappa(Y) = \kappa(k^n) = k(x_1, \dots, x_n)$. The set $x_1, \dots, x_n, b_1, \dots, b_\delta$ is then by construction a transcendence basis for $\kappa(X)$ over k. Since we know that $\dim(k^n) = n$ (see Theorem 8.4), we deduce from Proposition 9.2 that $\delta = \dim(X) - \dim(Y)$. Now the subfield $\kappa(Y)(b_1, \dots, b_\delta)$ of $\kappa(X)$ is isomorphic as a k-algebra to $k(x_1, \dots, x_n, y_1, \dots, y_\delta)$, which is the function field of $k^{n+\delta}$. The inclusion $k(x_1, \dots, x_n) \hookrightarrow k(x_1, \dots, x_n, y_1, \dots, y_\delta)$ is induced by the natural projection morphism $\pi : k^{n+\delta} \to k^n$ (unroll the definitions). Hence we have a rational dominant map $a : X \to k^{n+\delta}$ such that the rational dominant map associated with the morphism $f : X \to Y$ is the composition of a with the rational dominant map associated with π (apply Proposition 9.4 and Q3 of Sheet 3). Applying (2) we obtain open affine subvarietes $U \subseteq X$ and $W \subseteq k^{n+\delta}$ and a surjective morphism $g : U \to W$, which represents a. Let now now $f' = \pi \circ g$. Note that by Q3 of Sheet 3 again, we have $f' = f|_U$. Let $y \in \pi(W) = f'(U) = f(U)$. We compute

$$\begin{aligned} & \dim(f^{-1}(y)) \geqslant \dim(f^{-1}(y) \cap U) = \dim((f')^{-1}(y)) \\ = \ & \dim(g^{-1}(\pi^{-1}(y) \cap W)) \geqslant \dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y)) = \delta = \dim(X) - \dim(Y) \end{aligned}$$

Here we used Q6 of Sheet 2 for the first inequality and we used (1) for the inequality

$$\dim(g^{-1}(\pi^{-1}(y) \cap W) \geqslant \dim(\pi^{-1}(y) \cap W)$$

(remember that g is surjective). To justify the equality

$$\dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y)) = \delta$$

note that $\pi^{-1}(y) \simeq k^\delta$. We thus have $\dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y))$ by Proposition 9.2 and we have $\dim(\pi^{-1}(y)) = \delta$ by Theorem 8.4.

(4) Let $U \subseteq Y$ be an open subvariety. Applying (3) to the morphism $f^{-1}(U) \to U$, we see that there is a point $y \in U$ such that $\dim(f^{-1}(y)) \geqslant \dim(f^{-1}(U)) - \dim(U) = \dim(X) - \dim(Y)$. Since U was arbitrary, this shows what we want.

Q8. (1) Show that all the morphisms from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use Q7 and the projective dimension theorem.]*

(2) Deduce from (1) that for any $n \geqslant 2$ the morphisms from $\mathbb{P}^n(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use (1) and Q7 of Sheet 2.]*

Solution. (1) Let $f : \mathbb{P}^2(k) \to \mathbb{P}^1(k)$ is a morphism. Suppose for contradiction that f is not constant. By Corollary 12.10, the image $f(\mathbb{P}^2(k))$ is closed, and it is also irreducible, since $\mathbb{P}^2(k)$ is irreducible. Hence $f(\mathbb{P}^2(k)) = \mathbb{P}^1(k)$ (because $\dim(\mathbb{P}^1(k)) = 1$). Now let $y_1, y_2 \in \mathbb{P}^1(k)$ be such that $y_1 \neq y_2$ and $\dim(f^{-1}(y_1)), \dim(f^{-1}(y_1)) \geqslant \dim(\mathbb{P}^2(k)) - \dim(\mathbb{P}^1(k)) = 1$. This exists by Q7. Since $\dim(\mathbb{P}^2(k)) = 2$ we then actually have $\dim(f^{-1}(y_1)) = \dim(f^{-1}(y_1)) = 1$. Let C_1 (resp. C_2) be an irreducible component of $\dim(f^{-1}(y_1))$ (resp. $\dim(f^{-1}(y_2))$) such that $\dim(C_1) = \dim(C_2) = 1$. We have $\dim(C_1) + \dim(C_2) - 2 = 0$ and so by Proposition 11.2 we have $C_1 \cap C_2 \neq \emptyset$. This is a contradiction.

(2) Let $n \geqslant 2$. First note that $\mathbb{P}^2(k)$ is isomorphic to the closed subvariety $\mathrm{Z}(x_3, x_4, \ldots, x_n)$ of $\mathbb{P}^n(k)$. To see this note that the image of the morphism $\iota : \mathbb{P}^2(k) \to \mathbb{P}^n(k)$ given by the formula

$$[X_0, X_1, X_2] \mapsto [X_0, X_1, X_2, 0 \ldots ((n-2)\text{-times}) \ldots, 0]$$

is $\mathrm{Z}(x_3, x_4, \ldots, x_n)$. This morphism is an isomorphism onto $\mathrm{Z}(x_3, x_4, \ldots, x_n)$ because the morphism

$$\mathbb{P}^n(k) \backslash \mathrm{Z}(x_0, x_1, x_2) \to \mathbb{P}^2(k)$$

given by the formula

$$[X_0, X_1, X_2, \ldots, X_n] \mapsto [X_0, X_1, X_2]$$

gives an inverse to ι when restricted to $\mathrm{Z}(x_3, x_4, \ldots, x_n)$.

Let now $f : \mathbb{P}^n(k) \to \mathbb{P}^1(k)$ be a morphism. Suppose for contradiction that f is not constant. Let $\bar{v}_1, \bar{v}_2 \in \mathbb{P}^n(k)$ be two points such that $f(\bar{v}_1) \neq f(\bar{v}_2)$. Let M be an invertible $(n+1) \times (n+1)$-matrix such that $M([1,0,0,\dots,0]) = \bar{v}_1$ and $M([0,1,0,0,\dots,0]) = \bar{v}_2$. Let $\phi_M : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ be the automorphism defined by M (see Q7 of Sheet 2). The morphism $f \circ \phi_M \circ \iota : \mathbb{P}^2(k) \to \mathbb{P}^1(k)$ is then not constant, which is a contradiction by (1).