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C7.6: General Relativity 2

University of Oxford: Part C

Chris Couzens

Office S1.46

Mathematical Institute, University of Oxford

Andrew Wiles Building, Radcliffe Observatory Quarter

Woodstock Road

Oxford, OX2 6GG

Disclaimer: There are almost certainly typos in the notes, if something does not look correct or needs further explanation please let me know.

Please send any comments/corrections to Christopher.Couzens@maths.ox.ac.uk

Recommended books and resources

There are a large variety of good textbooks and lecture notes on general relativity. This course borrows from a number of them, in various different places, chiefly among them is the book by Sean Carroll, [1] and the book by Wald [2].

For the background material one can read the GR1 lecture notes. This should cover all the necessary prerequisites that one would need to know about general relativity.

Some useful lecture notes are by Harvey Reall and by Fay Dowker.

Contents

Conventions

- We will use the god-given signature convention of mostly plus $(-,+,+,+)$. This may differ with the convention you have used in other courses, especially field theory courses. This convention is preferable when thinking about geometry as it gives positive spatial distances. For quantum field theory the other convention is preferable since it ensures that energies and frequencies are positive. You may map between the two conventions through *Wick rotation*, essentially allowing the coordinates to become complex.

- Spacetime indices will be taken to be greek letters from the middle of the alphabet: $\mu, \nu, \rho, \ldots$ and run over $0, 1, 2, 3$. Latin indices $i, j, k, ..$ run over the spatial directions and take values $1, 2, 3$.

- We employ Einstein summation convention, repeated indices are summed over, unless otherwise stated.

- We work in units where the speed of light c is set to 1. Occasionally it is instructive to reintroduce c which can be done by dimensional analysis.

- The Minkowski metric will be denoted by $\eta_{\mu\nu} = \text{diagonal}(-1, 1, 1, 1)_{\mu\nu}$.

- After introducing curvature we will take the metric to be $g_{\mu\nu}$ and the determinant will be $\det(g_{\mu\nu}) \equiv g$.

- The set of all vector fields on a manifold M is $\mathcal{X}(M)$.

Useful formulae

- The Lagrangian for the geodesic equation of a massive test particle is

$$\mathcal{L}\Big(\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, x^\mu\Big) = \sqrt{-g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,,$$

  with λ an arbitrary parameter along the worldline.

- The geodesic equation for a massive particle is

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0\,, \qquad g_{\mu\nu}(x)\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = -1\,,$$

  where τ is the proper time. For light, the first equation takes the same form just replacing τ with an affine parameter. The second is modified by $-1 \to 0$.

- The Christoffel symbols (Levi–Civita connection) are

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \Big(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho} \Big) \,.$$

- The Riemann tensor is

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\rho\lambda} \Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\lambda} \Gamma^{\lambda}{}_{\nu\rho} \,.$$

  - Symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \,,$$
$$R_{\mu\nu\rho\sigma} = R_{\sigma\rho\mu\nu} \,.$$

  - Bianchi identity 1

$$R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho} = 0 \,.$$

  - Bianchi Identity 2

$$\nabla_\mu R^{\sigma}{}_{\lambda\nu\rho} + \nabla_\nu R^{\sigma}{}_{\lambda\rho\mu} + \nabla_\rho R^{\sigma}{}_{\lambda\mu\nu} = 0 \,.$$

- Ricci tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$$

- Ricci scalar

$$R = R_{\mu\nu} g^{\mu\nu} \,.$$

- Einstein tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \,.$$

- Einstein–Hilbert action plus cosmological constant,

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \Big(R + \Lambda \Big) \,.$$

- Under a variation $g_{\mu\nu} \to g_{\mu\nu} \delta g_{\mu\nu}$ we have

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma} \,,$$
$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \,,$$
$$\delta R_{\mu\nu} = \nabla_\rho \delta \Gamma^{\rho}{}_{\mu\nu} - \nabla_\mu \delta \Gamma^{\rho}{}_{\rho\nu} \,.$$

1 Introduction

These are lecture notes for the Part C course General Relativity 2 at Oxford university. They are an extension of the course General Relativity 1 and we assume that the reader is familiar with the material covered there.

To keep our conventions in order we will briefly review the essential material from GR1. For those who have done a GR course but not studied manifolds I recommend consulting the GR1 notes as manifolds will appear at times in the lectures.

1.1 Manifolds

The underlying structure of General relativity is differential geometry. This is the study of manifolds.

Definition Let X be any set and $\mathcal{T} = \{U_i | i \in I\}$ denote a certain collection of subsets of X. The pair $(X, \mathcal{T})$ is called a *topological space* if $\mathcal{T}$ satisfies

1. Both the set X and the empty set $\emptyset$ are open subsets: $M \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2. If $\mathcal{T}$ is any, possibly infinite, subcollection of I, then the family $\{U_j | j \in J\}$ satisfies $\cup_{j\in J} U_j \in \mathcal{T}$.
3. If K is any finite subcollection of I then the set $\{U_k | k \in K\}$ satisfies $\cap_{k\in K} U_k \in \mathcal{T}$.

Definition M is an n-dimensional *differentiable manifold* if satisfies:

1. M is a Hausdorff topological space,
2. M is provided with a family of pairs $\{(U_i, \varphi_i)\}$;
3. $\{U_i\}$ is a family of open sets which covers M: $\cup_i U_i = M$.
4. φ_i is a homeomorphism from U_i onto an open subset U'_i of $\mathbb{R}^n$,
5. Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, then the map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cup U_j)$ is infinitely differentiable. ψ_{ij} is known as a *transition function.*

Differentiable maps Let $f : M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. A point $p \in M$ is mapped to a point $f(p) \in N$. We may take a chart (U, φ) on M and a chart (V, ψ) in N where for all $p \in U$, $f(p) \in V$. Then f has the following coordinate presentation:

$$\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \,. \tag{1.1}$$

If we write $\varphi(p) = \{x^\mu\}$ and $\psi\big(f(p)\big) = \{y^\alpha\}$ then, $\psi \circ f \circ \varphi^{-1}$ is just the usual vector-valued function $y = \psi \circ f \circ \varphi^{-1}(x)$ of m variables. Sometimes it is useful to abuse notation and write $y = f(x)$ or $y^\alpha = f^\alpha(x^\mu)$ when we know the coordinate systems on M and N that are in use.

Definition We say that a function $f : M \to \mathbb{R}$ is *smooth* if the map $f \circ \varphi^{-1} : U \to \mathbb{R}$ is smooth for all charts. We let the set of all small functions on M be denoted by $\mathcal{F}(M)$.

Definition We say that a map $f : M \to N$ between two manifolds is smooth if the map $\psi \circ f \circ \varphi^{-1} : U \to V$ is smooth for all charts $\varphi : M \to \mathbb{R}^m$ and $\psi : N \to \mathbb{R}^n$. If $y = \psi \circ f \circ \varphi^{-1}(x)$ is C^∞ then we say that f is *differentiable* at p. This is actually independent of the coordinate system.

Definition Let $f : M \to N$ be a homeomorphism and ψ and φ coordinate functions. If $\psi \circ f \circ \varphi^{-1}$ is invertible, f is called a *diffeomorphism* and M is said to be *diffeomorphic* to N and vice-versa. This is denoted by $M \equiv N$.

Since the map is invertible it follows that if $M \equiv N$ then $\dim M = \dim N$. Homeomorphisms classify spaces according to whether it is possible to deform one space into another *continuously*. *Diffeomorphisms* classify spaces into equivalence classes according to whether it is possible to deform one space into the other *smoothly*. As such a diffeomorphism is stronger than a homeomorphism, it requires that both the map and its inverse are smooth. Two diffeomorphic manifolds are viewed as the same manifold.

Tangent vectors We can define curves on our manifold, $\gamma : (a, b) \to M$ and the tangent to such a curve. If we collect all curves passing through the point p and find all tangent vectors to the point p, this defines the *tangent space* at p: $T_p(M)$ which is a vector space. A basis of the tangent space is given by

$$\{e_\mu\} = \left\{ \frac{\partial}{\partial x^\mu} \right\}, \tag{1.2}$$

and any vector field X may be expanded in terms of this basis as

$$X = X^\mu \frac{\partial}{\partial x^\mu}\,. \tag{1.3}$$

When we are looking at vector fields in $T_p(M)$ the X^μ are just numbers, however we can equally consider the tangent bundle which is the union of all tangent spaces in M. Then a vector field in the tangent bundle has X^μ which are functions on M.

Let U_i, j be two coordinate patches with coordinates $x = \varphi_i(p)$ and $y = \varphi_j(p)$ respectively and let $p \in U_i \cup U_j$. Then we can give the vector field X in both sets of coordinates and we have that

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}\,, \tag{1.4}$$

and therefore the components of the vector field X transform as

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu} \quad \Rightarrow \quad \tilde{X}^\mu = X^\nu \frac{y^\mu}{x^\nu} \,. \tag{1.5}$$

One-forms Since $T_p(M)$ is a vector space there exists a dual vector space whose element is a linear function $T_p(M) \to \mathbb{R}$. The dual space is called the *cotangent space* at p, and denoted $T_p^*(M)$. An element $\omega \in T_p^*(M)$ is a linear map $T_p(M) \to \mathbb{R}$ and is called a *cotangent vector, dual vector* or *one-form*.

The natural basis of the cotangent space is given by the differential of the coordinates: $\{\mathrm{d}x^\mu\}$. Using the bilinear map arising from the tangent and cotangent spaces being dual vector spaces, one takes

$$\left\langle \mathrm{d}x^\mu, \frac{\partial}{\partial x^\nu} \right\rangle = \delta^\mu_\nu \,. \tag{1.6}$$

An arbitrary one-form can then be expanded out in this basis as $\omega = \omega_\mu \mathrm{d}x^\mu$. Let us take $p \in U_i \cup U_j$ as before, then for $\omega \in T_p^*(M)$ we have

$$\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\mu \mathrm{d}y^\mu \quad \Rightarrow \quad \tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu} \,. \tag{1.7}$$

Tensors We can now define tensors of type (q,r) to be a multilinear object which maps q elements of $T_p^*(M)$ and r elements of $T_p(M)$ to $\mathbb{R}$. We denote the set of (q,r) tensors at p to be $\mathcal{T}_p^{(q,r)}(M)$. An element of $\mathcal{T}^{(q,r)}(M)$ can be written in terms of the bases described above as

$$T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \cdots \frac{\partial}{\partial x^{\mu_q}} \mathrm{d}x^{\nu_1} \dots \mathrm{d}x^{\nu_r} \,. \tag{1.8}$$

T is a linear function

$$T : \otimes^q T_p^*(M) \otimes^r T_p(M) \to \mathbb{R} \,. \tag{1.9}$$

Let $V_i = V_i^\mu \frac{\partial}{\partial x^\mu}$ with $1 \leq i \leq r$ and $\omega_j = \omega_{j\mu} \mathrm{d}x^\mu$ with $1 \leq j \leq q$ then the action of T is

$$T(\omega_1, ..., \omega_q; V_1, ...V_r) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \, \omega_{1\mu_1} ....\omega_{q\mu_q} V_1^{\mu_1} ....V_r^{\mu_r} \,. \tag{1.10}$$

Tensor fields So far we have defined vectors, one-forms and tensors at a particular point $p \in M$. We want to be able to smoothly assign such an object to every point of M. For a vector we call such an object a *vector field*. In other words if V is a vector field then for every $f \in \mathcal{F}(M)$ then $V[f] \in \mathcal{F}(M)$. We will denote the set of all vector fields on M as $\mathcal{X}(M)$. A vector field X at $p \in M$ is denoted by $X|_p$ which is an element of $T_p(M)$. Similarly we may define a *tensor field* of type (q,r) by a smooth assignment of an element of $\mathcal{T}^q_{r,p}(M)$ at each point $p \in M$. The set of tensor fields of type (q,r) on M is denoted by $\mathcal{T}^q_r(M)$.

Differential forms A *differential form* of order r, or more succinctly an r-form, is a totally anti-symmetric tensor of type $(0, r)$.

The *Wedge product* $\wedge$ of r one-forms is defined to be the totally anti-symmetric tensor product of the one-forms

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge ...\mathrm{d}x^{\mu_r} \equiv \sum_{P \in S_r} \mathrm{sgn}(P) \mathrm{d}x^{\mu_{P(1)}} \otimes \mathrm{d}x^{\mu_{P(2)}} \otimes .... \otimes \mathrm{d}x^{\mu_{P(r)}} . \tag{1.11}$$

Thus

$$\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} - \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\mu} . \tag{1.12}$$

The wedge product satisfies the following conditions

- $\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} = 0$ if some index is repeated.
- $\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} = \mathrm{sgn}(P) \mathrm{d}x^{\mu_{P(1)}} \wedge ... \wedge \mathrm{d}x^{\mu_{P(r)}}$.
- $\mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r}$ is linear in each $\mathrm{d}x^{\mu}$.

We will denote the vector space of r-forms at the point $p \in M$ by $\Omega^r_p(M)$, a basis is provided by the set of all wedge products in (1.11). We can then expand an element of $\Omega^r_p(M)$ as

$$\omega = \frac{1}{r!} \omega_{\mu_1 ... \mu_r} \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} , \tag{1.13}$$

where $\omega_{\mu_1 ... \mu_r}$ are taken to be totally anti-symmetric.

We may define the *exterior product* to be the map $\wedge : \Omega^q_p(M) \times \Omega^r_p(M) \to \Omega^{q+r}_p(M)$. Its action follows by trivial extension of the wedge product defined above. Let $\omega \in \Omega^q_p(M)$ and $\xi \in \Omega^r_p(M)$ be an q-form and and r-form respectively. The action of the $(q + r)$-form $\omega \wedge \xi$ on $q + r$ vectors V_i is

$$(\omega \wedge \xi)(V_1, ..., V_{q+r}) = \frac{1}{q!r!} \sum_{P \in S_{q+r}} \mathrm{sgn}(P) \omega\big(V_{P(1)}, ..., V_{P(q)}\big) \xi\big(V_{P(q+1)}, ..., V_{P(q+r)}\big) . \tag{1.14}$$

The *exterior deriavtive* d_r is a map $\Omega^r(M) \to \Omega^{r+1}(M)$, whose action on an r-form

$$\omega = \frac{1}{r!} \omega_{\mu_1 ... \mu_r} \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} , \tag{1.15}$$

is

$$\mathrm{d}_r \omega = \frac{1}{r!} \left(\frac{\partial}{\partial x^{\nu}} \omega_{\mu_1 ... \mu_r} \right) \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} . \tag{1.16}$$

It is common to drop the r subscript and simply write d. The wedge product automatically anti-symmetrises the coefficient so it is indeed a $(r+1)$-form that we obtain. It follows that for $\xi \in \Omega_p^q(M)$, $\eta \in \Omega_p^r(M)$ we have

$$\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-1)^q \xi \wedge \mathrm{d}\eta\,. \tag{1.17}$$

The exterior derivative satisfies $\mathrm{d}^2 = 0$.

Let X be a vector field and $\omega \in \Omega^r(M)$ then the *interior product* of the r-form ω with respect to the vector X is

$$\mathrm{i}_X \omega(X_1, ..., X_{r-1}) \equiv \omega(X, X_1, ..., X_{r-1})\,. \tag{1.18}$$

If we introduce coordinates: $X = X^\mu \frac{\partial}{\partial x^\mu}$ then

$$\mathrm{i}_X \omega = \frac{1}{(r-1)!} X^\nu \omega_{\nu\mu_1 \ldots \mu_{r-1}} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{r-1}}\,. \tag{1.19}$$

1.2 Riemannian geometry

Definition: Let M be a differentiable manifold. A *Riemannian metric* g on M is a type $(0,2)$ tensor field on M which at each point $p \in M$ satisfies

- Symmetric: $g_p(X,Y) = g_p(Y,X)$,
- $g_p(X,X) \geq 0$ with equality iff $X = 0$

with $X, Y \in T_p(M)$. A tensor field g of type $(0,2)$ is a *pseudo-Riemannian metric* if it satisfies the first condition and

- Non-degenerate. If for any $p \in M$ $g_p(X,Y) = 0$ for all $Y \in T_p(M)$ then $X_p = 0$,

We may extend the tensor g_p over the full manifold. With a choice of coordinates we can write the metric as

$$g = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\,. \tag{1.20}$$

We will often write this as the line elements $\mathrm{d}s^2$,

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu\,. \tag{1.21}$$

We may view $g_{\mu\nu}$ as a matrix, which by the symmetry property above is symmetric. This implies that the matrix is diagonalisable, with real eigenvalues. If there are i positive eigenvalues and j negative eigenvalues the pair (i,j) is called the *index* of the metric. If $j = 1$ the metric is called a *Lorentz metric*, for $j = 0$ we have a *Euclidean* metric. The number of negative entries is called the *signature* and by Sylvester's law of inertia[1], this is independent of the choice of basis.

[1]This has nothing to do with inertia, Sylvester just wanted a law of inertia like Newton.

Lorentzian manifolds For our purposes Riemannian manifolds are not what we want to consider, instead we want to consider Lorentzian manifolds. The simplest example is Minkowski space. This is $\mathbb{R}^{1,m-1}$ equipped with the metric

$$\eta = -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1 + ... + \mathrm{d}x^{m-1} \otimes \mathrm{d}x^{m-1} \,, \tag{1.22}$$

which has components $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$. Note that on a Lorentzian manifold we take the index to run over $0, 1, .., m-1$.

At any point p on a general Lorentzian manifold it is always possible to find an orthonormal basis $\{e_\mu\}$ of $T_p(M)$ such that locally the metric looks like the Minkowski metric

$$g_{\mu\nu}|_p = \eta_{\mu\nu} \,. \tag{1.23}$$

This is closely related to the equivalence principle (see later).

The fact that locally the metric looks locally like Minkowski space allows us to import some of the ideas of special relativity, namely we can classify the elements of $T_p(M)$ into three classes

- $g(X, X) > 0 \longrightarrow X$ is *spacelike* ,
- $g(X, X) = 0 \longrightarrow X$ is *lightlike* or *null* ,
- $g(X, X) < 0 \longrightarrow X$ is *timelike* .

At each point on M we can then draw light cones which are the null tangent vectors at that point. The novelty is that the directions of these light cones can vary smoothly as we move around the manifold. This specifies the causal structure of spacetime which determines which regions of spacetime can interact together.

We can use the metric to determine the length of curves. The nature of a curve is inherited from the nature of its tangent vector. A curve is called *timelike* if its tangent vector is everywhere timelike. We then measure the proper time

$$\tau = \int_a^b \mathrm{d}t \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \,. \tag{1.24}$$

The existence of a metric comes with a large number of benefits.

The metric as an isomorphism The metric gives a natural isomorphism between vectors and covectors, $g : T_p(M) \to T_p^*(M)$ for each p. In a coordinate basis we can write $X = X^\mu \partial_\mu$, and map it to a one-form $X = X_\mu \mathrm{d}x^\mu$, as

$$X_\mu = g_{\mu\nu} X^\nu \,. \tag{1.25}$$

We will usually say that we use the metric to lower (or raise) an index. What we really mean is that the metric provides and isomorphism between a vector space and its dual. Since g is non-degenerate and is thus invertible we also have the inverse map. We take the inverse of $g_{\mu\nu}$ to be $g^{\mu\nu}$ so that $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$. This can then be thought of as the components of a symmetric $(2,0)$ tensor

$$\hat{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu \,. \tag{1.26}$$

Then

$$X^\mu = g^{\mu\nu}X_\nu \,. \tag{1.27}$$

The Volume form The metric also gives a natural volume form on the manifold M. On a Riemannian manifold we take the volume form to be

$$\mathrm{vol}(M) = \sqrt{\det(g_{\mu\nu})}\mathrm{d}x^1 \wedge ...\mathrm{d}x^m \,, \tag{1.28}$$

and we use the shorthand $\sqrt{\det(g_{\mu\nu})} = \sqrt{g}$. On a Lorentzian manifold the determinant is negative and therefore we take the volume form to be

$$\mathrm{vol}(M) = \sqrt{-g}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^{n-1} \,. \tag{1.29}$$

As it is written it looks coordinate dependent however it is not.

Hodge dual On an oriented manifold M we can use the totally anti-symmetric tensor density to define a map which takes a p-form $\omega \in \Omega^p(M)$ to a $(m-p)$-form $\star\omega \in \Omega^{m-p}(M)$. We define this map to be

$$(\star\omega)_{\mu_1...\mu_{m-p}} = \frac{1}{p!}\sqrt{|g|}\epsilon_{\mu_1...\mu_{m-p}\nu_1...\nu_p}\omega^{\nu_1..\nu_p} \,, \tag{1.30}$$

where $\epsilon_{\mu_1...\mu_m}$ is the totally anti-symmetric tensor, with $\epsilon_{123...m} = 1$ and for even permutations, -1 for odd permutations and 0 otherwise.

This is called the *Hodge dual* and is independent of coordinates. One can see that it satisfies

$$\star(\star\omega) = \pm(-1)^{p(m-p)}\omega \,, \tag{1.31}$$

with $+$ for a Riemannian metric and $-$ for a Lorentzian.

Connections An *affine connections* ∇ is a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, $(X,Y) \mapsto \nabla_X Y$ which satisfies

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \,, \tag{1.32}$$

$$\nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z\,, \tag{1.33}$$
$$\nabla_X(fY) = X[f]Y + f\nabla_X Y\,, \tag{1.34}$$

for vector fields $X, Y, Z \in \mathcal{X}(M)$ and functions $f, g \in \mathcal{F}(M)$.

We may introduce connection coefficients so that the connection acts on an arbitrary tensor of rank (q, r) as

$$\begin{aligned}\nabla_\mu T^{\nu_1\ldots\nu_q}{}_{\rho_1\ldots\rho_r} = & \frac{\partial}{\partial x^\mu} T^{\nu_1\ldots\nu_q}{}_{\rho_1\ldots\rho_r} + \Gamma^{\nu_1}{}_{\mu\sigma} T^{\sigma\ldots\nu_q}{}_{\rho_1\ldots\rho_r} + \ldots. + \Gamma^{\nu_q}{}_{\mu\sigma} T^{\nu_1\ldots\nu_{q-1}\sigma}{}_{\rho_1\ldots\rho_r} \\ & - \Gamma^\sigma{}_{\mu\rho_1} T^{\nu_1\ldots\nu_q}{}_{\sigma\ldots\rho_r} - \ldots - \Gamma^\sigma{}_{\mu\rho_r} T^{\nu_1\ldots\nu_q}{}_{\rho_1\ldots\rho_{r-1}\sigma}\,.\end{aligned} \tag{1.35}$$

In words, you first differentiate the tensor and then for each upper index you add in a $+\Gamma T$ and for every down index a $-\Gamma T$. The connection takes tensors to tensors, the (q, r) tensor gets mapped to a $(q, r + 1)$ tensor.

The connection coefficients are not tensors themselves, but transform as

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = (\Lambda^{-1})^\mu{}_\kappa \Lambda^\sigma{}_\rho \Lambda^\tau{}_\nu \Gamma^\kappa{}_{\sigma\tau} + (\Lambda^{-1})^\mu{}_\kappa \Lambda^\sigma{}_\rho \partial_\sigma \Lambda^\kappa{}_\nu\,, \qquad \text{with} \qquad \Lambda^\mu{}_\nu = \frac{\partial y^\mu}{\partial x^\nu}\,. \tag{1.36}$$

The difference

$$T^\kappa{}_{\sigma\tau} = \Gamma^\kappa{}_{\sigma\tau} - \Gamma^\kappa{}_{\tau\sigma}\,, \tag{1.37}$$

is called the torsion tensor, and is indeed a tensor. If the torsion tensor vanishes we say that the connection is *torsion free.*

Levi–Civita connection Given a metric there we have:

Theorem There exists a unique, torsion free, connection that is compatible with the metric g:

$$\nabla_X g = 0\,, \tag{1.38}$$

for all vector fields X.

The connection compatible with the metric is called the *Levi–Civita connection* while the components of the Levi–Civita connection are called the *Christoffel symbols* and are given by:

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\big(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} - \partial_\rho g_{\mu\nu}\big)\,. \tag{1.39}$$

Given a vector field X which is tangent to the curve γ with coordinates x^μ, we say that a tensor field T is parallel transported along γ if

$$\nabla_X T = 0\,. \tag{1.40}$$

Let γ connect two points $p, q \in M$. The condition (1.40) provides a map from the vector space defined at p to the vector space defined at q. Consider a second vector field Y. In components (1.40) reads

$$X^\nu\big(\partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\rho} Y^\rho\big) = 0\,. \tag{1.41}$$

If we evaluate it on the curve γ, we can write $Y^\mu = Y^\mu(x(\lambda))$ and therefore the condition is

$$\frac{\mathrm{d}Y^\mu}{\mathrm{d}\lambda} + X^\nu \Gamma^\mu{}_{\nu\rho} Y^\rho\,. \tag{1.42}$$

A *geodesic* is a curve tangent to a vector field X that obeys

$$\nabla_X X = 0\,. \tag{1.43}$$

Along the curve γ with coordinates x^μ and tangent vector X this implies

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0\,. \tag{1.44}$$

This is the same geodesic equation one obtains by varying the action

$$S = \int \mathrm{d}\lambda \sqrt{-g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,, \tag{1.45}$$

and picking an affine parameter.

Using the Levi–Civita connection we can define the curvature and torsion tensors. In components the Riemann tensor is

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\lambda{}_{\nu\rho}\Gamma^\sigma{}_{\mu\lambda} - \Gamma^\lambda{}_{\mu\rho}\Gamma^\sigma{}_{\nu\lambda}\,. \tag{1.46}$$

It has the following symmetries and properties

$$R^\sigma{}_{\rho\mu\nu} = -R^\sigma{}_{\rho\nu\mu}\,, \tag{1.47}$$

$$R_{\mu\nu\rho\sigma} = R_{\sigma\rho\mu\nu}\,, \tag{1.48}$$

$$R_{\mu[\nu\rho\sigma]} = 0\,, \tag{1.49}$$

$$\nabla_{[\mu} R_{\sigma\rho]\tau\nu} = 0\,. \tag{1.50}$$

Given a rank $(1,3)$ tensor we can construct a rank $(0,2)$ tensor by contraction, for the Riemann tensor the resultant $(0,2)$-rank tensor is called the *Ricci* tensor and is defined by

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}\,. \tag{1.51}$$

It inherits symmetry in its indices from the properties of the Riemann tensor

$$R_{\mu\nu} = R_{\nu\mu}\,. \tag{1.52}$$

We can create a scalar by contracting the indices again

$$R = g^{\mu\nu} R_{\mu\nu}\,. \tag{1.53}$$

1.3 Einsteins equations

The Einstein–Hilbert action is

$$S_{\rm EH} = \int \mathrm{d}^4x\sqrt{-g}R\,. \tag{1.54}$$

Variation with respect to the metric gives Einstein's field equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0\,. \tag{1.55}$$

A cosmological constant term may be added to the action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}(R-2\Lambda)\,. \tag{1.56}$$

Varying the action as before yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu}\,. \tag{1.57}$$

Coupling to matter We can couple gravity to matter. We do this via minimal coupling. We replace covariant derivatives with the connection, add in the correct volume measure and insert a metric for summed space-time indices.

We need to consider the combined action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}(R-2\Lambda) + S_{\rm Matter}\,, \tag{1.58}$$

where $S_{\rm Matter}$ is the action for any matter fields in the theory minimally coupled to gravity. The *Energy-Momentum tensor* is defined to be

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm Matter}}{\delta g^{\mu\nu}}\,. \tag{1.59}$$

1.4 Schwarzschild solution

The Scwarzschild solution is

$$\mathrm{d}s^2 = -\left(1-\frac{2G_NM}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2G_NM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\,. \tag{1.60}$$

This solves Einstein's equations in a vacuum, $R_{\mu\nu}=0$.

Birkhoff's theorem The Schwarzschild solution is the unique spherically symmetric asymptotically flat solution to the vacuum Einstein equations.

New coordinates The Schwarzschild solution in Schwarzschild coordinates has a coordinate singularity at $r = R_s = 2G_N M$. This surface is called the *event horizon*. In GR no signals can come out from within the event-horizon, once you fall past the event horizon you are lost to the outside world.

The apparent singularity at $r = R_s$ is only a coordinate singularity and can be removed by a coordinate transformation. First introduce the *tortoise coordinate* r_*

$$r_* = r + 2G_N M \log\left(\frac{r - 2G_N M}{2G_N M}\right), \tag{1.61}$$

then in these coordinates the null radial in-going/out-going geodesics are particularly simple:

$$t = \pm r_* + \text{constant}\,. \tag{1.62}$$

Next introduce a pair of null coordinates further adapted to the null geodesics:

$$v = t + r_*\,, \qquad u = t - r_*\,. \tag{1.63}$$

Ingoing Eddington–Finkelstein coordinates Eliminating t via $t = v - r_*(r)$, known as ingoing Eddington–Finkelstein coordinates, we find

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}s^2(S^2)\,. \tag{1.64}$$

Even though the metric coefficient g_{vv} vanishes at $r = 2G_N M$ there is no real degeneracy there and the metric is well-defined as one can see by computing the determinant.

There is also the complementary *outgoing Eddington–Finkelstein coordinates* where we eliminate t using u above. With Eddington–Finkelstein coordinates we are able to continue the Schwarzschild solution beyond the horizon to $r > 0$. In fact there are two ways to do this with either the ingoing or outgoing Eddington–Finkelstein coordinates. In fact we can do better and write a metric which captures both of these regions simultaneously.

To begin write the Schwarzschild metric using both null (u, v)-coordinates, the metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{r}\right)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}s^2(S^2)\,, \tag{1.65}$$

where r is a function of $u - v$. In these coordinates the metric is again degenerate at $r = R_s$ so we need to perform another change of coordinates. We introduce the *Kruskal-Szekeres coordinates*,

$$U = -\exp\left(-\frac{u}{4G_N M}\right), \qquad V = \exp\left(\frac{v}{4G_N M}\right), \tag{1.66}$$

both are null coordinates. The original Schwarzschild black hole is parametrised by $U < 0$ and $V > 0$. Outside the horizon they satisfy

$$UV = -\exp\left(\frac{r_*}{2G_N M}\right) = \frac{2G_N M - r}{2G_N M}\exp\left(\frac{r}{2G_N M}\right), \tag{1.67}$$

and similarly

$$\frac{U}{V} = -\exp\left(-\frac{t}{2G_N M}\right). \tag{1.68}$$

The metric is then

$$\mathrm{d}s^2 = -\frac{32(G_N M)^3}{r}\mathrm{e}^{-\frac{r}{2G_N M}}\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}s^2(S^2), \tag{1.69}$$

with $r(U, V)$ defined by inverting (1.67). The original Schwarzschild metric covers just $U < 0$ and $V > 0$ however there is no obstruction to extending $U, V \in \mathbb{R}$. Nothing bad happens at $r = 2G_N M$, the metric is smooth and non-degenerate and now we have a metric which covers all regions. The Kruskal spacetime is the maximal extension of the Schwarzschild solution.

2 Killing vectors

Killing vectors play an important role in general relativity and in understanding black holes. In this section we will introduce the notion of a Killing vector and show how they give rise to conserved quantities along geodesics.

2.1 Lie derivative

Let (M, g) be a Lorentzian manifold with metric g. Given a smooth vector-field X on M we define an integral curve $\gamma(\lambda) : \mathbb{R} \to M$ to be a curve whose tangent vector is equal to X at every point $p \in \gamma$. That is we demand

$$X^\mu\Big|_p = \frac{\mathrm{d}}{\mathrm{d}\lambda} x^\mu(\lambda)\Big|_p \,. \tag{2.1}$$

This is equivalent to solving a set of first order ODEs with fixed initial conditions, and therefore there is a unique solution at least locally.

Let $\gamma(\lambda, p)$ be the integral curve of X which passes through the point p when $\lambda = 0$. The map $\gamma : \mathbb{R} \times M \to M$ defines the flow generated by X. The flow defines an abelian group since one can show that $\sigma(\lambda_1, \sigma(\lambda_2, p)) = \sigma(\lambda_1 + \lambda_2, p)$. Let $\sigma_\lambda(p) = \sigma(\lambda, p)$ then

$$\begin{aligned}
&\sigma_\lambda\big(\sigma_\tau(p)\big) = \sigma_{\lambda+\tau}(p)\,,\\
&\sigma_0 = \text{Unit element}\,,\\
&\sigma_{-\lambda} = (\sigma_\lambda)^{-1}\,.
\end{aligned} \tag{2.2}$$

This allows us to move points along the curve, in particular by using the flow we can move tensors from one point on the flow to another, recall that this goes by the name of *push-forward* or *pull back* depending on what object we are acting on.[2] This allows us to define the Lie derivative along the vector field X. For a vector Y we have

$$\mathcal{L}_X Y|_p = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big[\big(\sigma_{-\epsilon}(p)\big)_* Y\big|_{\sigma_\epsilon(p)} - Y\big|_p\Big]\,. \tag{2.3}$$

One can show that

$$\mathcal{L}_X Y = [X, Y]\,, \tag{2.4}$$

with $[\ ,\ ] : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ the Lie bracket

$$[X, Y] = \Big(X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu\Big)\partial_\mu\,. \tag{2.5}$$

[2]Given a smooth function $f : M \to N$ the push forward $f_* : T_p(M) \to T_{f(p)}(N)$ acts on a vector field V as $(f_*V)[g] = V[g \circ f]$. The pullback $f^* : T^*_{f(p)}(N) \to T^*_p(M)$, acts as $\langle f^*\omega, V\rangle = \langle \omega, f_* V\rangle$.

The Lie derivative can be extended to any tensor with appropriate generalisation. For tensors one must use a combination of the push-forward and pull-back. Of primary interest to us here is the Lie derivative of the metric. We have

$$\mathcal{L}_X g = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[(\sigma_\epsilon(p))^* g|_{\sigma_\epsilon(p)} - g|_{\sigma_\epsilon(p)}\Big] . \tag{2.6}$$

Note that the pull back uses σ_ϵ rather than $\sigma_{-\epsilon}$, this is not a typo. In coordinates we have

$$(\mathcal{L}_X g)_{\mu\nu} = X^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu X^\sigma + g_{\mu\sigma}\partial_\nu X^\sigma , \tag{2.7}$$

which by using the Levi–Civita connection can be rewritten as

$$(\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu . \tag{2.8}$$

More generally, let T be a tensor of rank (q, r), then the Lie derivative along the vector field X in local coordinates is

$$\begin{aligned}\mathcal{L}_X T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} &= X^\sigma \partial_\sigma T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} - \big(\partial_\sigma X^{\mu_1}\big) T^{\lambda \dots \mu_q}{}_{\nu_1 \dots \nu_r} - \dots - \big(\partial_\sigma X^{\mu_q}\big) T^{\mu_1 \dots \sigma}{}_{\nu_1 \dots \nu_r} \\ &\quad + \big(\partial_{\nu_1} X^\sigma\big) T^{\mu_1 \dots \mu_q}{}_{\sigma \dots \nu_r} + \dots + \big(\partial_{\nu_r} X^\sigma\big) T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \sigma} .\end{aligned} \tag{2.9}$$

To make this more manifestly tensorial one can replace the partial derivatives with *any* torsion free connection[3], not necessarily the Levi–Civita connection. One can show that the Lie derivative satisfies

$$\begin{aligned}\mathcal{L}_X(T+S) &= \mathcal{L}_X T + \mathcal{L}_X S , \\ \mathcal{L}_X(T \otimes S) &= \Big(\mathcal{L}_X T\Big) \otimes S + T \otimes \Big(\mathcal{L}_X S\Big) , \\ \mathcal{L}_{[X,Y]} &= \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X , \\ \mathcal{L}_X f &= X[f] ,\end{aligned} \tag{2.10}$$

where X, Y are vector fields, T and S are arbitrary tensors, and f is a function.

2.2 Killing vectors

The Lie derivative of the metric along a vector field X captures the variation of the metric under the infinitesimal coordinate transformation:

$$x^\mu \to \tilde{x}^\mu = x^\mu + X^\mu . \tag{2.11}$$

Let us transform the metric under the above coordinate transformation

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) \equiv \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x) . \tag{2.12}$$

[3]Recall that a connection is torsion free if the connection coefficients satisfy $\Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}$.

Then

$$\delta g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = \Big(\mathcal{L}_X g\Big)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu\,, \tag{2.13}$$

with ∇ the Levi–Civita connection.

There can be special vectors X where the right-hand-side of (2.8) vanishes, i.e. vector fields X which satisfy

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0\,. \tag{2.14}$$

Such vectors are known as *Killing vectors*. They are vectors which define flows along which the metric does not change. We say that it generates an *isometry* of the spacetime and that the metric has a *symmetry*. We will see later in the course that there are corresponding conserved quantities for these symmetries as one may suspect from Noether's theorem, we will study this in section **??** after introducing some additional technology.

2.3 Maximally symmetric spaces: how many Killing vectors can we have?

It is natural to ask if there an upper limit on the number of Killing vectors a space can have? The answer is yes. Consider Euclidean space in n-dimensions $\mathbb{R}^n$ with the flat metric. What are the symmetries of this space? We know that we have both translations and rotations. Fix a point p in $\mathbb{R}^n$. Translations are the transformations that move the pint: there are n independent axes along which we can move and therefore there are a total of n translations. The rotations, centred at p are those transformations which leave p fixed. We can think of rotations as mapping one of the axes through the point p into one of the others. Since there are n axes and thus $n-1$ axes it can be rotated into. However rotating x into y and y into x are not independent and therefore the total number of rotations is $\frac{n(n+1)}{2}$. This gives a total of

$$\frac{n(n+1)}{2}\,, \tag{2.15}$$

independent symmetries.

This is the maximum number of linearly independent (by constant coefficients) Killing vectors that an n-dimensional space may have.

Definition: *Maximally symmetric space*
An n-dimensional space with the maximum number of Killing vectors, $\frac{n(n+1)}{2}$, is called a *maximally symmetric space*.

Aside: To prove this one needs to use that for a Killing vector K we have

$$\nabla_\mu \nabla_\nu K^\sigma = R^\sigma{}_{\nu\mu\rho} K^\rho\,. \tag{2.16}$$

Then we view the Killing equation (2.14) as a set of first order PDEs for the n functions K^μ. We can now find a solution as a series expansion around some arbitrary point p in the manifold. We would have

$$K^\mu(x) = K^\mu(p) + (x^\nu - p^\nu)\partial_\nu K^\mu\Big|_{x=p} + \frac{1}{2}(x^\nu - p^\nu)(x^\sigma - p^\sigma)\nabla_\nu\nabla_\sigma K^\mu\Big|_{x=p} + ... \quad (2.17)$$

However since (2.16) allows us to express the second derivative of K at p in terms of $K(p)$ and $\partial_\mu K(p)$ it follows that we may eliminate second derivative terms from the expansion. In fact we may go further, whacking (2.16) with another derivative allows us to express the third derivative of K in terms of $K_\nu(p)$ and $\nabla_\mu K_\nu(p)$ too. We can do this infinitely many times to obtain expressions for all higher derivative terms. Therefore the solution is determined uniquely by the initial conditions $K^\mu(p)$ and $\nabla_\mu K^\nu|_{x=p}$. The general solution is then of the form

$$K_\mu(x) = A_\mu{}^\nu(x,p)K_\nu(p) + B_\mu{}^{\nu\rho}(x,p)\nabla_\nu K_\rho\Big|_{x=p}, \quad (2.18)$$

where A and B are complicated functions depending on the initial point p and the metric and its derivatives but independent of the initial data of the Killing vector. Therefore we have shown that every Killing vector can be determined in terms of the initial conditions $K_\mu(p)$ and $\nabla_\mu K_\nu\big|_{x=p}$. There are n-independent components of $K_\mu(p)$ and $\frac{n(n-1)}{2}$ independent components of $\nabla_\mu K_\nu\big|_{x=p}$. The latter comes about because the initial conditions must satisfy the Killing equation, which fixes $\nabla_\mu K_\nu\big|_{x=p}$ to be a $n \times n$ anti-symmetric matrix which has $\frac{n(n-1)}{2}$ independent components. This gives the claimed total of $\frac{n(n-1)}{2}$ Killing vectors.

Examples of maximally symmetric spaces are flat space, spheres, hyperbolic space, Minkowski space and (anti-) de-Sitter space.

If a manifold is maximally symmetric it means that the curvature is the same in all directions. The Riemann tensor can in fact be fixed in terms of the constant Ricci scalar and takes the form

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right). \quad (2.19)$$

This means that locally the space is determined by the Ricci scalar.[4]

2.4 Conserved quantities along geodesics and Killing vectors

We have already seen conserved quantities when we studied geodesics. When we had an ignorable coordinate we found a conserved quantity along the geodesic. This is in fact related

[4]For example both a torus and the Euclidean plane are flat, and hence the Riemann tensor vanishes, however they are very different spaces, one is compact while the other is non-compact. The Ricci scalar therefore does not capture the global difference of the two spaces.

to the presence of Killing vectors.

Consider the action

$$S = \int \mathrm{d}\lambda \sqrt{\left| g_{\mu\nu}\big(x(\lambda)\big) \frac{\mathrm{d}x^\mu(\lambda)}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu(\lambda)}{\mathrm{d}\lambda} \right|} \,. \tag{2.20}$$

For simplicity let us assume that λ is an affine parameter which allows us to consider the action with the square root removed. From GR1 we know that geodesics are the curves which extremise the action, that is geodesics are curves, $x^\mu(\lambda)$, which when deformed by a small amount $\delta x^\mu(\lambda)$, the change in the action vanishes.

Consider deforming the curve as

$$x^\mu \to x^\mu + \epsilon X^\mu \,. \tag{2.21}$$

The change in the action is

$$\begin{aligned}
\delta S &= S(x^\mu + \epsilon X^\mu) - S(x^\mu) \\
&= \epsilon \int \mathrm{d}\lambda \Big[x^\rho \partial_\rho g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu} \big(\dot{X}^\mu \dot{x}^\nu + \dot{x}^\mu \dot{X}^\nu \big) \Big] + \mathcal{O}(\epsilon^2) \\
&= \epsilon \int \mathrm{d}\lambda \; \dot{x}^\rho \dot{x}^\sigma \Big[X^\mu \partial_\mu g_{\rho\sigma} + g_{\nu\sigma} \partial_\rho X^\nu + g_{\nu\rho} \partial_\sigma X^\nu \Big] + \mathcal{O}(\epsilon^2) \\
&= \epsilon \int \mathrm{d}\lambda \; \dot{x}^\rho \dot{x}^\sigma \Big[\nabla_\rho X_\sigma + \nabla_\sigma X_\rho \Big] + \mathcal{O}(\epsilon^2) \,.
\end{aligned} \tag{2.22}$$

We have used that $\dot{X}^\mu = \dot{x}^\rho \partial_\rho X^\mu$. We see that if X is a Killing vector field we have a symmetry of the action. We know from Noether's theorem that there must be a conserved charge.

Conserved charge along a geodesic Given an action S with Lagrangian density $\mathcal{L}$,

$$S = \int \mathrm{d}\lambda \; \mathcal{L}\big(x(\lambda)\big) \,, \tag{2.23}$$

define the conjugate momentum p_μ to be

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \,. \tag{2.24}$$

Then for any Killing vector X,

$$Q = X^\mu p_\mu \,, \tag{2.25}$$

is a conserved quantity along the geodesic.

Proof: Consider a small variation $\delta x^\mu = \epsilon X^\mu$ generated by the Killing vector field X as above. As shown above such variations leave the action invariant: $\delta S = 0$, which is equivalent to

$$\frac{\partial \mathcal{L}}{\partial x^\mu} X^\mu + \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{X}^\mu = 0 \,. \tag{2.26}$$

Along a geodesic the Euler–Lagrange equations are satisfied:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0 \,. \tag{2.27}$$

Therefore along the geodesic (2.26) implies

$$0 = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda} p_\mu \right) X^\mu + p_\mu \frac{\mathrm{d}}{\mathrm{d}\lambda} X^\mu = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(p_\mu X^\mu \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda} Q \,. \tag{2.28}$$

Note that Q is conserved only along the geodesic, for a path which is not a geodesic this is *not* conserved. We can see immediately why this must be the case in the derivation above since we used the Euler–Lagrange equations.

Example: Schwarzschild solution Let us apply this to the Schwarzschild solution. We have used that there are two quantities that are conserved for any geodesic. These were the energy E and the angular momentum j. The Schwarzschild solution has Killing vectors ∂_t and ∂_ϕ. Actually it has a $\mathbb{R}\times\mathrm{SO}(3)$ isometry group, where time translations give the $\mathbb{R}$ factor and ∂_ϕ generates a $\mathrm{U}(1) \subset \mathrm{SO}(3)$.

Then for $X_t = \partial_t$ we have $X_t^\mu = (1, 0, 0, 0)$ and therefore

$$Q_t = X_t^\mu p_\mu = p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left(1 - \frac{2M}{r} \right) \dot{t} \equiv -E \,. \tag{2.29}$$

Similarly for the Killing vector $K_3 = \partial_\phi$ we have $K_3^\mu = (0, 0, 0, 1)$ and therefore

$$Q_3 = K_3^\mu p_\mu = p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \sin^2\theta \dot{\phi} = J \,. \tag{2.30}$$

These are both conserved charges that you are familiar and the intuition about ignorable coordinates giving rise to conserved quantities holds for these. However Killing vectors need not be so simple. The Schwarzschild solution has two more Killing vectors

$$K_1 = \sin\phi \partial_\theta + \cot\theta \cos\phi \partial_\phi \,, \quad K_2 = \cos\phi \partial_\theta - \cot\theta \sin\phi \partial_\phi \,. \tag{2.31}$$

These two combine with K_3 to generate the SO(3) isometry of the spacetime: one can check that

$$[K_i, K_j] = \epsilon_{ijk} K_k \,, \tag{2.32}$$

which is indeed the Lie algebra $\mathfrak{so}(3)$.

The conserved charges for these are not nearly as simple as the previous two:

$$Q_1 = r^2\big(\sin\phi\dot{\theta} + \cos\theta\cos\phi\sin\theta\dot{\phi}\big)\,, \quad Q_2 = r^2\big(\cos\phi\dot{\theta} - \cos\theta\sin\theta\sin\phi\dot{\phi}\big)\,. \tag{2.33}$$

One can check upon application of the geodesic equation that these are indeed conserved. Recall that when we consider geodesics we use the rotational symmetry to set $\theta(0) = \frac{\pi}{2}$ and $\dot{\theta}(0) = 0$ which leads to motion in a plane. With these values the two charges Q_1, Q_2 vanish.

Aside: To find these Killing vectors one needs to solve the Killing equation (2.14) which gives a set of PDEs to solve. In this case, rather than trying to solve these PDEs, one can be slightly smarter and use the embedding of the S^2 into $\mathbb{R}^3$. The isometries are then the rotations about the three axes. We know that these are then:

$$X = z\partial_y - y\partial_z\,, \quad Y = z\partial_x - x\partial_z\,, \quad Z = x\partial_y - y\partial_x\,, \tag{2.34}$$

with the names of the vector fields given by the axis of rotation. Introducing the embedding coordinates

$$x = \sin\theta\cos\phi\,, \quad y = \sin\theta\sin\phi\,, \quad z = \cos\theta\,, \tag{2.35}$$

we find that

$$K_1 = X\,, \quad K_2 = Y\,, \quad K_3 = Z\,. \tag{2.36}$$

This drastically simplifies the problem in this case where we know the embedding into a simple space. This is not always possible and one must just bite the bullet and solve the PDEs.

The familiar conserved quantities along geodesics of the Schwarzschild solution are the ones where the Killing vector is of the form $K = \partial_\psi$ for some coordinate ψ. For any Killing vector we can find coordinates such that the Killing vector is of this form. However if there are other Killing vectors this transformation may ruin the nice form of these. For example the three Killing vectors of S^2 studied above, only one of the Killing vectors is in this nice adapted form. One could change coordinates to make either of the other two of this nice form however the sacrifice is that K_3 is no longer of this nice form. This is most easily seen from the embedding of the S^2 into $\mathbb{R}^3$. If we permuted the embedding coordinates in (2.35), whichever coordinate was just $\cos\theta$ would have the simple form. To see that in the case of SO(3) that only one of the Killing vectors can be of this nice form notice that $[\partial_\mu, \partial_\nu] = 0$ and therefore if two vectors were of this simple form the $\mathfrak{so}(3)$ algebra would not be satisfied.

2.5 Spherically symmetric, static and stationary spacetimes

In general when trying to solve Einstein's equations we need to make some simplifying assumptions. A set of simplifying assumptions you should already have seen when studying the Schwarzschild solution are spherically symmetric, static and stationary spacetimes. We will quickly review what this means in order to use these assumptions in the following section to describe cold stars.

2.5.1 Spherically symmetric spacetimes

You should be familiar with the isometries of a round two-sphere. One can rotate the two-sphere through any axis and it looks the same. This is of course the group SO(3). It can be further enhanced to O(3) if we also include reflections however we will not do this in the following. Any one-dimensional subgroup of SO(3) gives a one-parameter group of isometries and thus a Killing vector field. The rank of SO(3) is 3 and thus there are three independent Killing vectors which can be used to generate the full symmetry group.

If one puts the following metric on the round two-sphere

$$\mathrm{d}s^2(S^2) = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\,, \tag{2.37}$$

the three Killing vectors are

$$K_1 = \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi\,, \quad K_2 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\,, \quad K_3 = \partial_\phi\,. \tag{2.38}$$

One can check that these satisfy

$$[K_i, K_j] = \epsilon_{ijk}K_k\,, \tag{2.39}$$

which is the Lie algebra of $\mathfrak{so}(3)$.

Definition: *Spherically symmetric*
We say a spacetime is *spherically symmetric* if its isometry group contains an SO(3) subgroup whose orbits are 2-spheres. The orbit of a point p under a group of diffeomorphisms is the set of points that one obtains by acting on p with all the diffeomorphisms.

The statement about the orbits being two-dimensional is important. One can find $SO(3)$ orbits which are three-dimensional. To see this consider deforming an S^3 breaking the $SO(4)$ isometry to an $SO(3)$ subgroup. The orbits will be three-dimensional if chosen correctly.

Definition: *Area radius function*
In a spherically symmetric spacetime one can define the *area-radius function* $r : M \to \mathbb{R}$ defined to be

$$r(p) = \sqrt{\frac{A(p)}{4\pi}}\,, \tag{2.40}$$

where $A(p)$ is the area of the S^2 orbit through the point p. In other words the induced metric on the S^2 passing through the point p has induced metric $r(p)^2 \mathrm{d}s^2(S^2)$.

2.5.2 Static and stationary spacetimes

Definition: *Stationary*

A spacetime is *stationary* if it admits an everywhere timelike Killing vector K.

When the space-time is stationary it allows us to introduce a distinguished coordinate adapted to the Killing vector and write the metric in a simpler form. Given the Killing vector we may define the flow by finding the integral curve of the vector, let us parametrise the flow by t. We can pick a hypersurface Σ nowhere tangent to K^μ and introduce coordinates x^i on Σ. We may then assign coordinates (t, x^i) to the point a parameter distance t along the integral curve through the point on Σ with coordinates x^i. This gives a coordinate chart in which $K^\mu = \left(\partial_t\right)^\mu$. Since k^μ is a Killing vector field the metric is independent of t and therefore the metric takes the form

$$\mathrm{d}s^2 = g_{00}(x^k)\mathrm{d}t^2 + 2g_{0i}(x^k)\mathrm{d}t\mathrm{d}x^i + g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.41}$$

with $g_{00}(x^i) < 0$. Conversely given a metric of this form ∂_t is a timelike Killing vector field and thus the metric is stationary.

Definition: *Hypersurface-orthogonal*

Let Σ be a hypersurface in M specified by the equation $f(x) = 0$, with $f : M \to \mathbb{R}$ a smooth function. We require $\mathrm{d}f \neq 0$ on Σ, then $\mathrm{d}f$ is normal to Σ.[5] The dual vector to $\mathrm{d}f$, let us call it ξ, is said to be hypersurface orthogonal. If ξ is timelike the hypersurface is said to be spacelike; if ξ is spacelike the hypersurface is timelike and it ξ is null then the hypersurface is said to be null.

It follows that any other normal to the hypersurface can be written as $n = g\mathrm{d}f + fn'$ with g a smooth function which does not vanish anywhere on Σ, and n' a smooth one-form. Then we have

$$\mathrm{d}n = \mathrm{d}g \wedge \mathrm{d}f + \mathrm{d}f \wedge n' + f\mathrm{d}n' \quad \Rightarrow \quad \mathrm{d}n\big|_\Sigma = (\mathrm{d}g - n') \wedge \mathrm{d}f \quad \Rightarrow \quad n \wedge \mathrm{d}n\big|_\Sigma = 0\,. \tag{2.42}$$

Conversely Frobenius' theorem implies.

Theorem *Frobenius*:

If n is a non-zero one-form such that $n \wedge \mathrm{d}n = 0$ everywhere, then there exist functions f, g

[5] To see this consider a curve $\gamma(\lambda) \subset \Sigma$. By definition $f(\gamma(\lambda)) = 0$ for all λ, thus, $0 = \partial_\lambda f(\gamma(\lambda)) = \dot\gamma^\mu(\lambda)\partial_\mu f(\gamma(\lambda)) = \mathrm{d}f(\dot\gamma(\lambda))$. The latter is equivalent to $\mathrm{d}f$ being normal to the hypersurface.

such that $n = g\mathrm{d}f$ and therefore n is normal to the surfaces defined by $f(x) =$constant, and therefore hypersurface-orthogonal.

Definition: *Static*
A spacetime is *static* if it admits a hypersurface-orthogonal timelike Killing vector field.
Note: Static implies stationary, but the converse is not true.

For a static spacetime we know that K^μ is hypersurface orthogonal so when we define coordinates (t, x^i) as in the stationary case, we may further choose Σ to be orthogonal to K^μ. Σ is the hypersurface $t = 0$ and therefore it has normal $\mathrm{d}t$. Therefore at $t = 0$ we must have $K_\mu \propto (1, 0, ..., 0)$ since $K = \mathrm{d}f$. However we saw that in the stationary case that the Killing vector takes the form $K = \partial_t$. Since $k_i = g_{i\mu}k^\mu = g_{i0}$ it follows that we must have $g_{0i} = 0$. In these adapted coordinates a static metric takes the form

$$\mathrm{d}s^2 = g_{00}(x^k)\mathrm{d}t^2 + g_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j\,, \tag{2.43}$$

again with $g_{00}(x^k) < 0$.

Note that the metric now has a discrete time-reversal symmetry: $(t, x^i) \to (-t, x^i)$. Static then means time-independent and invariant under time reversal. Note that though static implies stationary, the converse is not necessarily true. A rotating body cannot be static because time-reversal changes the rotation (it rotates in the opposite direction if time is reversed).

We will impose that our spacetime is both stationary and spherically symmetric. The isometry group is then $\mathbb{R}\times\mathrm{SO}(3)$ with $\mathbb{R}$ the time-translations. It turns out that these conditions also imply static in this case, since a rotating star would have a preferred axis which would break the spherical symmetry.

3 Spherical cold stars and stellar collapse

Birkhoff's theorem proves that the Schwarzschild solution is the unique asymptotically flat, spherically symmetric solution of Einstein's equations in the absence of matter and cosmological constant. As such, away from any spherically symmetric static object such as a star, planet or black hole the metric is the Schwarzschild metric. There are a few questions we may want to ask at this point. What is the metric inside a star where the Schwarzschild solution is no longer valid (since there is now a non-trivial contribution from the energy momentum tensor)? Does GR tell us anything about the different types of stars: hot stars, white dwarfs, neutron stars? In this section we answer these questions by studying the extension of the Schwarzschild solution to describe a *cold star*.

As opposed to a hot star, where there is a thermal source of pressure generated by nuclear reactions in its core, a cold star must be supported from collapse by a non-thermal pressure source. When a star forms by condensation of a dust cloud due to gravitational attraction the pressure increases which leads to an increase in temperature. When the dust cloud has collapsed far enough and has reached a critical temperature, nuclear fusion in the core begins. The dominant process is the conversion of four protons to form a helium-4 nucleus. The emission of photons and neutrinos at this stage provides a thermal radiation which balances against the collapse of the star due to gravity. As the Hydrogen fuel is depleted a helium core builds up and the pressure from thermal radiation decreases and the star begins to collapse again.

If the star is massive enough as the core contracts it once again heats up and if a critical temperature is reached, helium can be fused, giving a thermal pressure which halts the collapse. If the star is not big enough the temperature which allows Helium to fuse is not reached and the star uses up its remaining fuel becoming a red dwarf. This process of a period of equilibrium followed by collapse can keep repeating with the formation of heavier nuclei in the core such as nickel and iron.

The crucial issue governing how far along this evolutionary sequence a star goes is whether electron degeneracy pressure becomes sufficient to support the star from further collapse. There is a critical mass M_C, (3.17), below which the collapse is halted by the electron degeneracy pressure. The Pauli exclusion principle states that two or more identical fermions[6] cannot occupy the same quantum state within a quantum system simultaneously.[7] Due to this a gas of cold fermions resists compression, producing a pressure known as *degeneracy pressure*. If the mass of the star is below the critical mass no further nuclear fusion will occur and the star will simply cool down forever in a stable *white dwarf* configuration. This is the fate of our sun. A white dwarf is much denser than a regular star: to get an idea about how much denser it is a matchbox sized piece of white dwarf material would weigh roughly

[6]A fermion is particle with half integer spin. Fermions obey Fermi–Dirac statistics. Quarks and leptons (electrons, muons and tau-ons and their neutrino versions) are examples of fermions.

[7]To get a feel of why this is true one needs to recall some facts about the wave-function in quantum mechanics. We construct a state by acting on the ground state with operators. Operators which give bosons (integer spin field) satisfy commutation relations, while operators which give rise to fermions satisfy anti-commutation relations. If we want to insert the same (all quantum numbers the same) fermion at the same point we must act with the same operator but due to the anti-commutator relations this vanishes and therefore the wave-function vanishes.

the same as t an elephant. Newtonian gravity is still applicable here and shows that a white dwarf cannot have a mass greater than the *Chandrasekhar limit*, 1.4 $\mathrm{M}_{\odot}$ with $M_{\odot}$ the mass of the Sun. A star more massive than this cannot end its life as a white dwarf unless it sheds some of its mass.

If M is greater than M_C then after a core of nickel and iron of mass M_C has formed it will be unable to support itself, electron degeneracy pressure is insufficient and no further nuclear fusion occurs. The core will undergo gravitational collapse . When the density of the core reaches nuclear density, the density of the nucleus of an atom, neutron degeneracy pressure and nuclear forces provide a significant cold matter pressure. At such high pressure one finds that beta decay is reversed, protons combine with electrons to produce neutrons. If the mass of the star is below the critical limit for cold matter M_{critical} $2M_{\odot}$ then the collapse will be halted leading to a *neutron star*. At this stage the Newtonian approximation is no longer applicable and one must use general relativity.

When the collapse of the core is halted or slowed at nuclear densities a shock wave is produced and this is expected to lead to the outer envelope of the star producing a supernova. The presence of pulsars (neutron stars with a hot spot rotating at high speed) at the sites of the Crab and Vela supernova remnants provides strong evidence that this supernovae are produced in conjunction with the collapse of the core of a star at the end-point of stellar evolution.

The final option is to have a star which has a mass larger than the critical mass M_{critical}. Equilibrium can never be achieved and complete gravitational collapse will occur. The end-point of such a collapse will be a Schwarzschild black hole. We find that for a massive enough star gravitational collapse into a black hole is inevitable.[8]

In this section we will show that general relativity predicts a maximum mass for a cold star. To reach this conclusion we will assume that the star is spherically symmetric and static, recall that this is one of the assumptions that goes into Birkhoff's theorem. The interior of the star can be modelled by a perfect fluid and we then need to solve Einstein's solutions with a perfect fluid source and match onto the Schwarzschild solution outside the star.

[8]One can formulate this more concretely following Penrose and Hawing that collapse becomes inevitable once a *trapped surface* forms. A trapped surface is a two-dimensional for which both the out-going and in-going future directed geodesics orthogonal to the surface converge. For example consider spheres with r, t constant in the Schwarzschild metric, these are trapped surfaces for $r < R_{\text{Schwarzschild}}$.

3.1 Tolman–Oppenheimer–Volkoff equations

Since we have a static spacetime we have a timelike Killing vector field K with which we can foliate our spacetime with the surfaces Σ_t which are orthogonal to K. The orbits of SO(3) through a point $p \in \Sigma_t$ lie within Σ_t. This allows us to define coordinates (r, θ, ϕ) such that the most general metric with our given assumptions takes the form

$$\mathrm{d}s^2 = -\mathrm{e}^{2\Phi(r)}\mathrm{d}t^2 + \mathrm{e}^{2\Psi(r)}\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2) \,. \tag{3.1}$$

We now need to specify the energy-momentum tensor. Outside the star this vanishes and it remains to come up with a suitable ansatz within the star. We can describe this as a perfect fluid. The energy momentum tensor for a perfect fluid takes the form

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} \,, \tag{3.2}$$

with u_μ the four-velocity of the fluid, normalised to $u_\mu u^\mu = -1$, ρ the energy density and p the pressure measured in the fluid's local rest frame. Since we are interested in time-independent and spherically symmetric stars the fluid should be at rest thus u points in the time-direction only and therefore

$$u = \mathrm{e}^{-\Phi(r)}\partial_t \,. \tag{3.3}$$

Moreover the time-independence and spherical symmetry imply that ρ and p only depend on r while the vanishing of the energy-momentum tensor outside of the star implies that ρ, p vanish when $r > R_c$ with R_c the radius of the star.

A fluid's equations of motion are determined by the conservation of the energy momentum tensor. This follows from the Einstein equations, ergo we need only consider the Einstein conditions in the following. Since the Einstein equations inherit the symmetries of the spacetime it follows that there are only three non-trivial independent conditions arising from the Einstein equations. We may take these to be the $tt, rr, \theta\theta$ components, see the mathematica file in moodle which does this computation.

The independent Einstein equations are

$$\begin{aligned}
E_{tt} &= \frac{\mathrm{e}^{2\Psi}}{r^2}\Big[\frac{\mathrm{d}}{\mathrm{d}r}\Big(r(1-\mathrm{e}^{-2\Psi})\Big) - 8\pi r^2\rho\Big] = 0 \,,\\
E_{rr} &= \frac{1}{r}\left[\mathrm{e}^{-2\Phi}\partial_r \mathrm{e}^{2\Phi} - \frac{\mathrm{e}^{2\Psi}-1}{r} - 8\pi r \mathrm{e}^{2\Psi}p\right] = 0 \,,\\
E_{\theta\theta} &= \mathrm{e}^{-2\Psi} r\left[\mathrm{e}^{\Psi-\Phi}\partial_r\Big(r\mathrm{e}^{-\Psi}\partial_r \mathrm{e}^{\Phi}\Big) - \partial_r\Psi - 8\pi e \mathrm{e}^{2\Psi}p\right] = 0 \,.
\end{aligned} \tag{3.4}$$

To proceed it is useful to introduce $m(r)$ via

$$e^{2\Psi(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1}, \tag{3.5}$$

with $2m(r) < r$. The tt component of the Einstein equation becomes

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r). \tag{3.6}$$

Moreover the rr component reduces to

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}. \tag{3.7}$$

In the Newtonian limit we have $r^3 p(r) \ll m(r)$ and $m(r) \ll r$ so (3.7) reduces to

$$\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} \approx \frac{m(r)}{r^2}, \tag{3.8}$$

this is just the spherically symmetric version of Poisson's equation for the Newtonian gravitational potential. We can see the other terms in (3.7) as relativistic corrections.

The final non-trivial component of the Einstein equations is the $\theta\theta$ component given above, however rather than using that equation, it is simpler to derive the final equation from the r-component of energy momentum conservation. This gives

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\big(p(r) + \rho(r)\big)\frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}. \tag{3.9}$$

One can check that this is implied by $E_{\theta\theta} = 0$ above, see the mathematica file. In the Newtonian limit ($P \ll \rho, m(r) \ll r$) it reduces to the Newtonian hydrostatic equilibrium equation

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} \approx -\frac{\rho(r)m(r)}{r^2}. \tag{3.10}$$

Note that general relativity has little effect on the equilibrium configurations of stars with $p \ll \rho$ and $m(r) \ll r$.

We have four unknowns $\big(m(r), \Phi(r), \rho(r), p(r)\big)$ and only three equations so the system is currently underdetermined. The one remaining condition comes from the fact that we are interested in a cold star, one which has a vanishing temperature. Thermodynamics implies that T, ρ, p are not independent, and therefore we may write $p = p(\rho)$. Moreover we should take $\rho > 0$ and $p > 0$ and that $p(\rho)$ is an increasing function of ρ.[9] The three equations (3.6), (3.7) and (3.9) are known as the *Tolman–Oppenheimer–Volkoff* equations.

[9] If this were not the case then the star would be unstable since a fluctuation in some region that led to an increased energy density would lead to a decrease in pressure. This would cause the fluid to more into this region which would lead to a further increase in ρ and the fluctuation would continue to grow.

Outside the star We know that in the absence of matter and with the imposed constraints, that the unique solution is the Schwarzschild solution:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-2}\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2)\,. \tag{3.11}$$

The constant M is the total mass of the star. Recall that $R_s = 2M$ is the Schwarzschild radius where an event horizon is located. We must therefore take the star to have a radius larger than the Schwarzschild radius: $R_c > R_s$. Regular stars have $R_c \gg R_s$, for the sun $R_s \approx 3\text{km}$ while $R_c \approx 7 \times 10^5\text{km}$.

Inside the star We now want to consider the interior of the star, and patch it with the exterior solution above such that the full metric is smooth at the patching surface at $r = R_c$. We can integrate (3.6) to give

$$m(r) = 4\pi \int_0^r \rho(r')r'^2\mathrm{d}r' + m_*\,, \tag{3.12}$$

with m_* and integration constant.

At $r = 0$ the solution should be smooth and look like flat Minkowski space, the net gravitational attraction at the centre is zero and is therefore equivalent to Minkowski space. This implies that at $r \to 0$ we have $\mathrm{e}^{2\Psi(0)} = 1$. Comparing with (3.5) we see that this is equivalent to $m(0) = 0$. From our integrated solution, (3.12) we see that this implies that the integration constant vanishes, $m_* = 0$.

At $r = R_c$, for our interior solution to match with the Schwarzschild solution, we need to impose the boundary condition

$$M = 4\pi \int_0^{R_c} \rho(r)r^2\mathrm{d}r\,. \tag{3.13}$$

There is a slight subtlety here in that the total energy of the matter should include the correct volume measure when integrating over a spacelike hypersurface, the energy for the spacelike hypersurface Σ_t defined to be

$$E = \int_{\Sigma_t} \rho(r)\mathrm{dvol}(\Sigma_t) = \int_{\Sigma_t} \rho(r)\mathrm{e}^{\Psi(r)}r^2\sin\theta\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi = 4\pi \int_0^{R_c} \rho(r)\mathrm{e}^{\Psi(r)}r^2\mathrm{d}r\,. \tag{3.14}$$

Note that this differs with the total mass of the star due to the $\mathrm{e}^{\Psi(r)}$ factor. Since $\mathrm{e}^{\Psi(r)} > 1$ it follows that $E > M$ and one can associate the positive difference $E - M$ to be the gravitational binding energy of the star. This would be the amount of energy needed to disperse the matter to infinity, for spherical stars this is a well-defined concept but does not always make sense in GR.

Note that due to the constraint that $2m(r) > r$ for all r, so that $\mathrm{e}^{\Psi(r)} > 0$ it follows that there is a upper bound on the possible mass of the star: $2M < R_c$. There is no Newtonian analogue of this condition. Reinstating the factors of c and G_N we have $2G_N M < c^2 R_c$ and in the $c \to \infty$ limit this is trivial, hence why this constraint is not seen in the Newtonian theory.

This upper bound can be improved. From equation (3.9) after some algebra and assuming $\rho \geq 0$ and $\rho'(r) \leq 0$, which you will do in sheet 1, one finds that

$$\frac{m(r)}{r} \leq \frac{2}{9}\left[1 - 6\pi r^2 p(r) + \sqrt{1 + 6\pi r^2 p(r)}\right]. \tag{3.15}$$

Evaluating on the radius of the star where $p = 0$, one finds

$$R_c \geq \frac{9M}{4}. \tag{3.16}$$

Note that this is actually independent of the equation of state and so it applies equally to hot stars and cold stars which satisfy these assumptions. Stars of uniform constant density can get arbitrarily close to saturating the bound but as they get closer to the bound the pressure at the centre diverges.

In order to solve the TOV equations we should use numerical integration. We view (3.6) and (3.9) as a coupled set of ODEs for $m(r)$ and $\rho(r)$ for some given equation of state. These can be solved, at least numerically on a computer once initial conditions for the mass and density are given. We have that $m(0) = 0$ and therefore we ned only specify a density $\rho_c = \rho(0)$ at the centre of the star.

Given these initial conditions we can numerically solve (3.6) and (3.9). Since the latter equation shows that p decreases with r there must be some point where the pressure vanishes, this is the surface of the star and the radius is determined by $p(R_c) = 0$. We can invert this to determine R_c as a function of ρ_c. From (3.13) we can determine M as a function of ρ_c. Finally we may determine $\Phi(r)$ inside the star by integrating (3.7) from the surface of the star with initial condition that $2\Phi(R_c) = \log(1 - 2M/R_c)$, i.e. it gives the Schwarzschild solution potential. Hence for a given equation of state, static, spherically symmetric cold stars are form a 1-parameter family of solutions labelled by the central density ρ_c.

3.2 Bounds on the mass

If one follows the above procedure one finds that as ρ_c increases then M increases to a maximum before decreasing again for larger ρ_c. One can see this from (3.15). Due to the minus sign in the first term as we crank up $p(r)$ the contribution from the positive square root term will no longer be dominant and the upper bound on the mass will start to get smaller. It follows that there is a maximum mass that a cold star can attain.

The maximum mass depends heavily on the details of the equation of state of cold matter. For the equation of state of a white dwarf where electron degeneracy pressure is the dominant outward force, one reproduces the *Chandrasekhar bound*:

$$M_C \approx 1.4\Big(\frac{2}{\mu_N}\Big)^2 M_\odot\,, \tag{3.17}$$

where μ_N is the number of nucleons per electron. The calculation for this bound does not require general relativity, Newtonian gravity is good enough, and the two bounds agree to a good precision. Experimentally we know the equation of state up to some density ρ_0 which is nuclear density, past this we no longer know the density.[10] One may guess that with some crazy configuration one could arrange for a star which is arbitrarily heavy, subject to the above bound. General relativity says that there is in fact a maximal bound independent of the equation of state. This is around $5M_\odot$.

To see why this is true observe that ρ is a decreasing function of r. We may define the *core* of the star as the region in which $\rho > \rho_0$ where we do not know the equation of state, and the *envelope* (since it envelopes the core) as the region $\rho < \rho_0$ where we do know the equation of state. Let r_0 be the radius of the core, so that the core is the region $r < r_0$ and the envelope is the region $r_0 < r < R_c$. The mass of the core is $m_0 = m(r_0)$. Since the density in the core is bigger than the density on the boundary with the envelope we must have that

$$m_0 \geq \frac{4\pi r_0^3 \rho_0}{3}\,. \tag{3.18}$$

Note that Newtonian gravity would also predict this inequality, however in GR we also have the additional constraint (3.15) which we should evaluate at $r = r_0$ where we know the equation of state and may therefore determine $p_0 = p\big(\rho(r_0)\big)$:

$$\frac{m_0}{r_0} < \frac{2}{9}\Big[1 - 7\pi r_0^2 p_0 + \sqrt{1 + 6\pi r_0^2 p_0}\Big]\,. \tag{3.19}$$

Since the RHS is a decreasing function of p_0 evaluating at $p_0 = 0$ we get the weaker bound

$$m_0 < \frac{4r_0}{9}\,. \tag{3.20}$$

[10] Past this density this becomes a strongly coupled phenomenon described by a strongly coupled QFT. Since we typically make progress with understanding QFTs using perturbation theory, when they are strongly coupled this technique fails and we need new ones. Recently there has been some work using AdS/CFT to work out a realistic equation of state. (This is by no means the only technique, but it has a nice connection to string theory.)

These two inequalities define a finite region in the $m_0 - r_0$ plane. Hence, even though we are ignorant of the equation of state within the core, GR predicts that its mass cannot be arbitrarily large.

Using (3.18) to eliminate r_0 and plugging this into (3.20) we have

$$m_0 < \frac{4}{9\sqrt{3\pi\rho_0}}\,. \tag{3.21}$$

Hence, even though we do not know the equation of state inside the core GR predicts that its mass cannot be indefinitely large. Experimentally we know the equation of state of cold matter at densities much higher than the density of atomic nucei so we take $\rho_0 = 5 \times 10^{14}\mathrm{g/cm}^3$. Plugging this into the above gives the bound $m_0 < 5M_\odot$.

If we are given a core with mass m_0 and radius r_0 we can solve (numerically) for the envelope region using the known equation of state and the equations for $m(r)$ and $p(r)$ with the initial conditions given by the core. If one plugs this into a computer programme one finds that the maximal mass M as a function of ρ_0, m_0. One can then vary this over the allowed region for (m_0, r_0) one finds that the largest mass is attained for the maximum of m_0. At this maximum the envelope contributes less than 1% of the total mass so the maximum mass of M is at almost the same as the maximum of m_0 and we have $M \leq 5M_\odot$.

This is an upper bound for *any* physically reasonable equation of state for $\rho > \rho_0$. Any equation of state will have a smaller upper bound than the one given here. One may put further constraints on what we call a physically reasonable equation of state. A natural demand is that the speed of sound through the mass should not exceed the speed of light, so that $\frac{\mathrm{d}p}{\mathrm{d}\rho} \leq 1$, then the upper bound is further reduced to about $3M_\odot$.

3.3 Summary

What have we learnt from this exercise? Firstly we see once again that GR predicts something that Newtonian gravity cannot, we find an upper bound on the maximal size of any cold star, independent of its composition. Secondly this has an extremely important consequence for the ultimate fate of a star. Ordinary hot starts are supported against collapse under their own weight by ideal gas pressure resulting from the high temperature. This pressure is much higher than the pressure that can be produced by cold matter at comparable densities and so the above upper limits do not apply. However since a hot star radiates energy, just look out the nearest window during the day, if this energy is not replenished hydrostatic equilibrium cannot be maintained. As the fuel source is used up the hydrostatic equilibrium is lost and it begins to contract until the cold matter pressure dominates the remaining thermal pressure. If the star was small enough a stable equilibrium may be reached using cold matter pressure and will remain like this forever. However if the mass is greater than the cold matter upper limit equilibrium can never be achieved and the star would have to undergo complete gravitational collapse unless they shed some of their mass to bring their total mass below the upper bound.

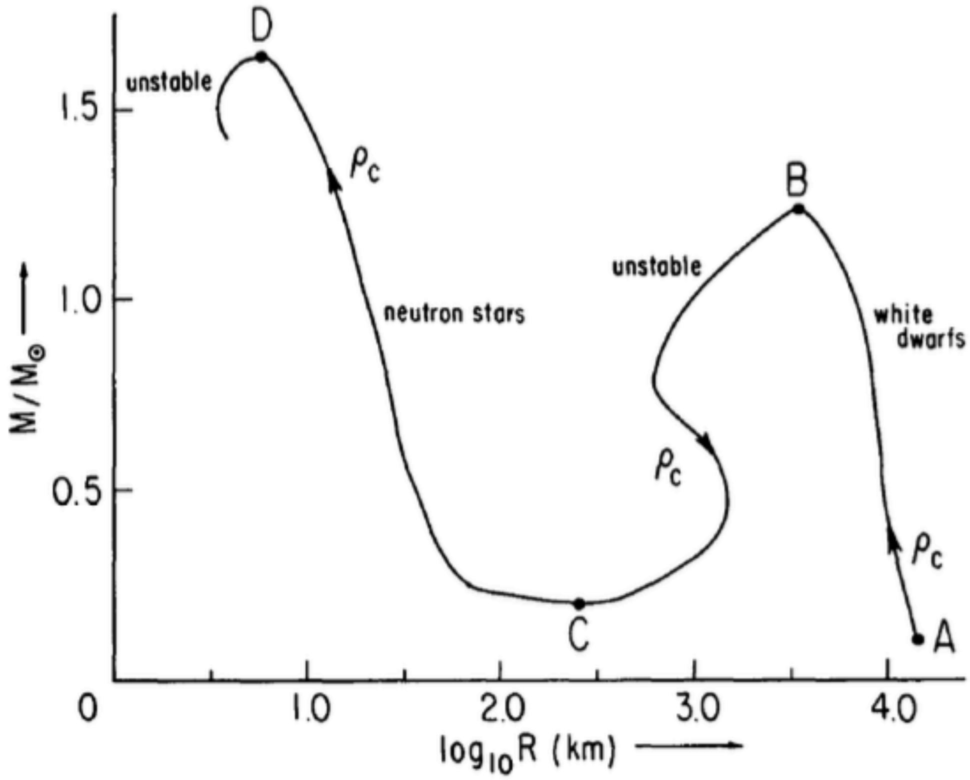

Figure 1: The equilibrium configurations of cold matter. Given an equation of state the equilibrium configuration is uniquely determined by the central density ρ_c. The radii and masses of these configurations are shown for values of ρ_c ranging from $\approx 10^5$g cm^{-3} at point A to $\approx 10^17$g cm^{-3} beyond point D. In the white dwarf regime the values of M and R_c depend somewhat on the assumed composition of the star. The neutron star regime is far more dependant on the assumptions that go into the equation of state, and interactions between the fundamental constituents of the matter. In the latter regime this is just a rough sketch of the qualitative features. The point B is the Chandrasekhar limit and beyond this the white dwarf must undergo further gravitational collapse to become a neutron star. It is at this point that the electron degeneracy pressure is insufficient to prevent gravitational collapse and therefore the equation of state changes past this point.
Figure taken from Wald based on a figure by Harrison, Thorne, Wakano and Wheeler.

4 Causality and Penrose Diagrams

Let us consider a spacetime M. One of the postulates that we demand General Relativity satisfies is that it is causal. A signal can be sent between two distinct points if and only if the points can be joined by a non-spacelike curve. Our goal in this section is to investigate the properties of causality on spacetime. Given that our spacetimes are generically infinite in extent this can be difficult to understand on a piece of paper. There is a useful way of resolving this issue called *conformal compactification.*

Definition: *Conformal transformation*
A *conformal transformation* is a map from a spacetime (M, g) to a spacetime $(M, \tilde{g})$ such that

$$\tilde{g}_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x)\,, \tag{4.1}$$

where $\Omega(x)$ is a smooth function of the spacetime coordinates and $\Omega(x) \neq 0$ for all $x \in M$.

Two spacetimes whose metrics are related by a conformal transformation have the same null geodesics. However, timelike and spacelike geodesics in one metric will not necessarily be geodesics in the other. You will prove this in problem sheet 1. One reason why conformal transformations are useful is because they preserve the causal structure of spacetime. Consider a vector V^μ on M, not necessarily a geodesic. Then since $\Omega(x)^2 > 0$ it follows that

$$\begin{aligned} g_{\mu\nu}V^\mu V^\nu > 0 \quad &\Leftrightarrow \quad \tilde{g}_{\mu\nu}V^\mu V^\nu > 0\,, \\ g_{\mu\nu}V^\mu V^\nu = 0 \quad &\Leftrightarrow \quad \tilde{g}_{\mu\nu}V^\mu V^\nu = 0\,, \\ g_{\mu\nu}V^\mu V^\nu < 0 \quad &\Leftrightarrow \quad \tilde{g}_{\mu\nu}V^\mu V^\nu < 0\,. \end{aligned} \tag{4.2}$$

Hence curves which are timelike, null or spacelike with respect to one metric remain timelike, null or spacelike respectively in the conformally rescaled metric.

We may use this to our advantage when studying the causal structure of spacetime. By using a suitably chosen conformal factor we may bring "infinity" to a finite coordinate distance which allows us to draw the causal structure on a finite piece of paper. This is known as a Penrose diagram and encodes the causal structure of the spacetime.

The general procedure for drawing a Penrose diagram is to perform the following steps. First change coordinates on (M, g) such that "infinity" is brought to finite coordinate distance. This then allows us to draw the spacetime on a finite piece of paper. The points at "infinity" will become the edges of the finite diagram. Typically the metric will diverge at these points. To remedy this we perform a conformal transformation on g to obtain $\tilde{g}$ which is regular on the edges. The new pair $(M, \tilde{g})$ is a good representation of the original spacetime (M, g)

for understanding the causal structure: they have the exact same causal structure. It is customary to add the points at infinity to the spacetime to form a new manifold $\tilde{M}$ (with boundary now). The resulting spacetime $(\tilde{M}, \tilde{g})$ is oft called the *conformal compactification* of (M, g).

Note that this has some limitations. Conformal transformations generically change the curvature tensors so that $\tilde{R}_{\mu\nu\rho\sigma} \neq R_{\mu\nu\rho\sigma}$, $\tilde{R}_{\mu\nu} \neq R_{\mu\nu}$, $\tilde{R} \neq R$ ... and so forth, therefore the conformally compactified spacetime is unphysical, it does not satisfy the Einstein field equations anymore. Moreover, timelike and spacelike geodesics of (M, g) are not geodesics in $(M, \tilde{g})$. The utility of the conformal compactification is for understanding the causal structure.

To understand this better let us consider some examples.

4.1 Minkowski Space in two-dimensions

First consider Minkowski space in two-dimensions. The metric in rectangular coordinates is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 \,, \tag{4.3}$$

where $-\infty < t, x < \infty$. The null geodesics are given by $t \pm x =$constant. We may introduce light-cone coordinates $u = t - x$ and $v = t + x$ which makes the null geodesics pretty simple. In these coordinates the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v \,. \tag{4.4}$$

The coordinates are still infinite and so we have not really done much yet. To proceed we want to shrink infinity down to a finite distance away. Define

$$u = \tan \tilde{u} \,, \qquad v = \tan \tilde{v} \,, \tag{4.5}$$

where $-\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2}$. Note that the range is open because strictly $u, v \to \pm\infty$ are not in the spacetime. The line-element with these coordinates is now

$$\mathrm{d}s^2 = -\frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}} \mathrm{d}\tilde{u}\mathrm{d}\tilde{v} \,. \tag{4.6}$$

It diverges as $\tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$. We can now define a new metric conformally related to the one above. The obvious conformal factor to use is chosen to remove the prefactor. We take

$$\tilde{g} = \cos^2 \tilde{u} \cos^2 \tilde{v} g = -\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} \,. \tag{4.7}$$

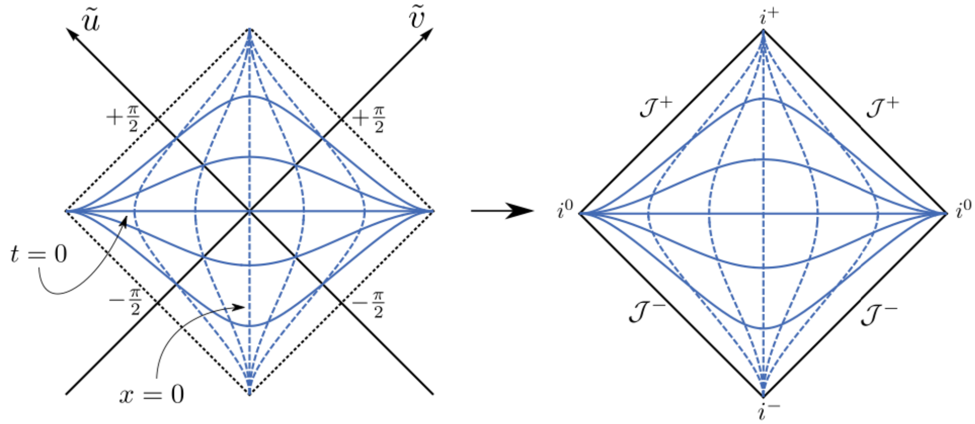

Figure 2: **Left:** On the left we have Minkowski space, (M, g) in $(\tilde{u}, \tilde{v})$ coordinates. The boundaries $\tilde{u}, \tilde{v} = \pm\frac{\pi}{2}$ are not part of M and g diverges there. Lines with $r = \text{const}$ are given by dashed lines, while the solid lines are those with $t = \text{const}$. **Right:** On the right is the Penrose diagram of the conformally compactified spacetime. Future past timelike infinity $i^{\pm}$, future/past null infinity is denoted $\mathcal{J}^{\pm}$ while spacelike infinity is denoted i^0.

This metric is now regular at the points at infinity where either $\tilde{u}$ $\tilde{v}$ are equal to $\pm\frac{\pi}{2}$. Since it is regular there we may now add these points to the spacetime. The resulting spacetime $(\tilde{M}, \tilde{g})$ is the *conformal compactification* of (M, g). We may now draw this, see figure 2

The two points $(\tilde{u}, \tilde{v}) = (-\frac{\pi}{2}, -\frac{\pi}{2})$ and $(\tilde{u}, \tilde{v}) = (\frac{\pi}{2}, \frac{\pi}{2})$ are denoted by $i^{\mp}$ respectively. All past and future directed timelike curves end up at $i^{\mp}$ so we refer to i^-/i^+ as past/future timelike infinity. Future directed null geodesics either end up at $\tilde{v} = \frac{\pi}{2}$ with constant $|\tilde{u}| < \frac{\pi}{2}$ or at $\tilde{u} = \frac{\pi}{2}$ with constant $|\tilde{v}| < \frac{\pi}{2}$. This set of points is denoted by $\mathscr{I}^+$ (called scri-plus) and referred to as *future null infinity*. An analogous definition holds for past null infinity $\mathscr{I}^-$ (scri-minus). Spacelike infinity, i^0 denotes the set of end-points of spacelike geodesics, which correspond to $(\tilde{u}, \tilde{v}) = (\frac{\pi}{2}, -\frac{\pi}{2})$ and $(\tilde{u}, \tilde{v}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

4.2 Minkowski Space in $d > 2$

We have just seen the Penrose diagram for $d = 2$, it turns out that this is some-what special in dimension, Minkowski space in $d > 2$ is somewhat different. Consider Minkowski space in $d > 2$ dimensions. We may use the "rectangular" metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{d-1} (\mathrm{d}x^i)^2 \,, \tag{4.8}$$

where the coordinates have ranges $t \in (-\infty, \infty)$, $x^i \in (-\infty, \infty)$. To proceed we may a change of coordinates going to spherical polar coordinates so that the spacelike part of the metric is equivalent to

$$\sum_{i=1}^{d-1} (\mathrm{d}x^i)^2 = \mathrm{d}r^2 + r^2 \mathrm{d}s^2(S^{d-2})\,, \tag{4.9}$$

with S^{d-2} the unit $(d-2)$-dimensional sphere and $\mathrm{d}s^2(S^{d-2})$ the round metric on it. This exhibits the spacetime as a cone centred at $x^i = 0$. We take $r \geq 0$. In these coordinates the Minkowski metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}s^2(S^{d-2})\,. \tag{4.10}$$

We can define light-cone coordinates

$$u = t - r\,, \quad v = t + r\,, \tag{4.11}$$

which puts the metric into the form

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \frac{(v-u)^2}{4}\mathrm{d}s^2(S^{d-2})\,. \tag{4.12}$$

Note that since $r \geq 0$ we have $u \leq v$. We now want to bring infinity to finite coordinate length, to do this we change coordinates to

$$u = \tan \tilde{u}\,, \quad v = \tan \tilde{v}\,, \tag{4.13}$$

where

$$\tilde{u} \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,, \qquad \tilde{v} \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,. \tag{4.14}$$

Note that the range is open since the points at $\pm\infty$ in the original coordinates are not part of the spacetime. We still need to impose that $\tilde{u} \leq \tilde{v}$. In these coordinates the metric reads

$$\mathrm{d}s^2 = -\frac{1}{4\cos^2\tilde{u}\cos^2\tilde{v}}\Big[-4\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\mathrm{d}s^2\big(S^{d-2}\big)\Big]\,. \tag{4.15}$$

We may now use a conformal transformation to remove the overall pre-factor and we are left with

$$\tilde{g} = 4\cos^2\tilde{u}\cos^2\tilde{v} g = -4\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \sin^2(\tilde{v}-\tilde{u})\mathrm{d}s^2\big(S^{d-2}\big)\,. \tag{4.16}$$

As before, after the conformal transformation $\tilde{u}, \tilde{v} = \pm\frac{\pi}{2}$ is no longer a problem and we may compactify the space by including these points. We therefore have the coordinate ranges $-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq \frac{\pi}{2}$. At fixed point on the sphere the metric is the same as that of 2d Minkowski space, the difference is in the ranges of $\tilde{u}, \tilde{v}$. We only include the half which is

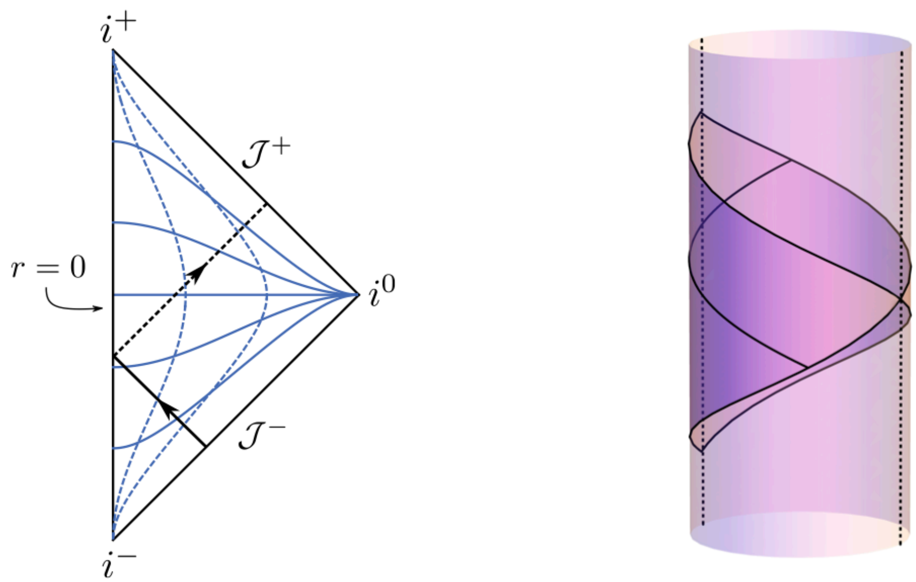

Figure 3: **Left:** On the left we have Minkowski space in general dimension > 2. Each point represents a $d-2$-dimensional sphere. As the null geodesic passes through $r = 0$ it emerges on another copy of the Penrose diagram whose points represent the anti-podes (diametrically opposite point) on the spheres. **Right:** The right digram shows the conformal compactification for $d = 4$ as a portion of the Einstein static universe. The curved line represents that same null geodesic as on the left-hand-side.

right of the vertical line. Every point on the sphere represents a $d-2$ dimensional sphere of radius $\sin(\tilde{v}-\tilde{u})$. The Penrose diagram is drawn in figure 3

In 4d, we can picture this differently. Define the coordinates $T = \tilde{v}+\tilde{u}$ and $\chi = \tilde{v}-\tilde{u}$. The coordinate ranges are then $-\pi < T < \pi$ and $0 < \chi < \pi$, with the added constraint $|T| + \chi \leq \pi$. The metric reads

$$\hat{g} = -\mathrm{d}T^2 + \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}s^2(S^2)\,. \tag{4.17}$$

The spatial part is just the round metric of a three-sphere. This therefore represents a static universe with spherical spatial slices corresponding to a finite portion of the Einstein static universe. See the right-hand side of figure 3 there this is plotted. Note that the vertical direction of the cylinder is T while the angular direction is χ. At each point there is a two-sphere with radius $\sin^2\chi$. We have

$$\begin{aligned} i^+ &= \text{ future timelike infinity } (T=\pi, \chi=0)\,, \\ i^0 &= \text{ spatial infinity } (T=0, \chi=\pi)\,, \\ i^- &= \text{ past timelike infinity } (T=-\pi, \chi=0)\,, \\ \mathscr{I}^+ &= \text{future null infinity} (T=\pi-\chi, 0<\chi<\pi)\,, \\ \mathscr{I}^- &= \text{past null infinity} (T=-\pi+\chi, 0<\chi<\pi)\,. \end{aligned} \tag{4.18}$$

Note that $i^\pm, i^0$ are actually points since $\chi = 0$ and $\chi = \pi$ are the north and south poles of S^3. Meanwhile $\mathscr{I}^\pm$ are null surfaces with the topology of $\mathbb{R}\times S^2$.

There are a number of features to observe. Radial null geodesics are at $\pm 45^\circ$ in the diagram. All timelike geodesics begin at i^- and end at i^+. All null geodesics begin at $\mathscr{I}^-$ and end at $\mathscr{I}^+$.

4.3 Rindler spacetime in 1+1 dimensions

Rindler space is a subregion of Minkowski space associated with observers who are eternally accelerated at a constant rate. It appears often when looking at the near-horizon region of black holes. Consider the two-dimensional Minkowski metric and an observer moving at a uniform acceleration of magnitude ξ^{-1} in the x-direction. Their trajectory is

$$t(\tau) = \xi\sinh(\tau)\,, \qquad x(\tau) = \xi\cosh(\tau)\,, \tag{4.19}$$

which has constant acceleration α. Note that the trajectory of the observer satisfies

$$x^2(\tau) - t^2(\tau) = \xi^2\,, \tag{4.20}$$

which describes a hyperboloid asymptoting to null paths $x=-t$ in the past and $x=t$ in the future. The accelerated observer travels from past null infinity to future null infinity, rather than timelike infinity as would be reached by geodesic observers. The region $x \le t$ is forever hidden to them which makes the line $x=t$ a horizon to these observers. This horizon is of a different flavour to the Schwarzschild horizon since that is an observer independent object while this horizon is associated with a special family of observers, see figure 4.

Rindler space corresponds to the right wedge $x > |t|$ foliated by the worldlines of the accelerated observers.

We can choose new coordinates (η, ξ) on 2d Minkowski space that is adapted to uniformly accelerated motion. Let

$$t = \xi\sinh(\eta)\,, \quad x = \xi\cosh a\eta)\,, \tag{4.21}$$

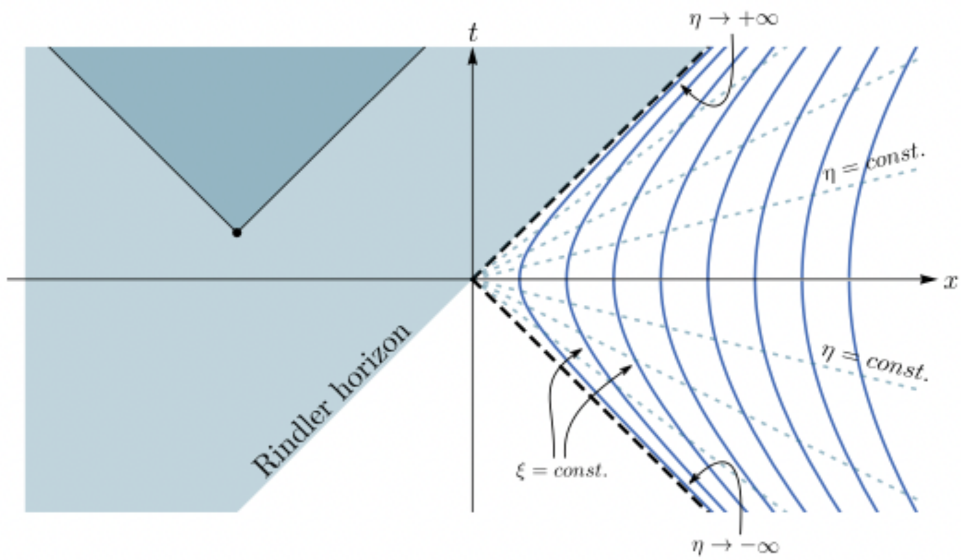

Figure 4: Eternally accelerating observers in Minkowski space. Their worldlines are in blue and labelled by ξ. Events in the shaded region such as the black dot are hidden to them. The Rindler horizon is the boundary between the shaded and unshaded regions. Rindler space is the right wedge bounded by the dashed black lines which are null. The straight lines are lines of constant Rindler time.

with coordinate range $0 < \xi < \infty$ and $-\infty < \eta < \infty$. In these coordinates the Minkowski metric in (η, ξ) coordinates is

$$\mathrm{d}s^2 = -\xi^2 \mathrm{d}\eta^2 + \mathrm{d}\xi^2 \,. \tag{4.22}$$

The proper time measured by an accelerated observer, i.e. a stationary (ξ =constant) observer in Rindler coordinates is $\tau = \xi\eta$. Since Rindler space is just a subregion of Minkowski space the Penrose diagram is just a piece of figure 2.

4.4 Kruskal Space

Recall that we could extend the Schwarzschild solution beyond the horizon by using Kruskal coordinates. The metric in these coordinates reads

$$\mathrm{d}s^2 = -\frac{32M^3}{r} \exp\Big(-\frac{r}{2M}\Big) \mathrm{d}U \mathrm{d}V + r^2 \mathrm{d}s^2(S^2) \,. \tag{4.23}$$

Recall that the range of the coordinates is $-\infty < U, V < \infty$. We need to define a new set of null coordinates to bring infinity to a finite coordinate distance. We transform as

$$U = \tan \tilde{U} \,, \quad V = \tan \tilde{V} \,, \tag{4.24}$$

such that $-\frac{\pi}{2} < \tilde{U}, \tilde{V} < \frac{\pi}{2}$. The line element becomes

$$\mathrm{d}s^2 = \frac{1}{4\cos^2\tilde{U}\cos^2\tilde{V}}\left[-\frac{128M^3}{r}\exp\left(-\frac{r}{2M}\right)\mathrm{d}\tilde{U}\mathrm{d}\tilde{V} + r^2\cos^2\tilde{U}\cos^2\tilde{V}\mathrm{d}s^2(S^2)\right]. \tag{4.25}$$

We perform the usual conformal transformation

$$\tilde{g} = 4\cos^2\tilde{U}\cos^2\tilde{V}g = -\frac{128M^3}{r}\exp\left(-\frac{r}{2M}\right)\mathrm{d}\tilde{U}\mathrm{d}\tilde{V} + r^2\cos^2\tilde{U}\cos^2\tilde{V}\mathrm{d}s^2(S^2). \tag{4.26}$$

The curvature singularity at $r = 0$ is at $UV = 1$ in U, V coordinates and now corresponds to

$$1 = UV = \tan\tilde{U}\tan\tilde{V} \quad \Leftrightarrow \quad \sin\tilde{U}\sin\tilde{V} - \cos\tilde{U}\cos\tilde{V} = 0 \quad \Leftrightarrow \quad \cos(\tilde{U} + \tilde{V}) = 0. \tag{4.27}$$

This implies that it is located at $\tilde{U} + \tilde{V} = \pm\frac{\pi}{2}$. To make this simpler it is useful to define $\tilde{U} = T - X$ and $\tilde{V} = T + X$. The Penrose diagram includes the points at infinity and the singularity, we draw it in figure 5.

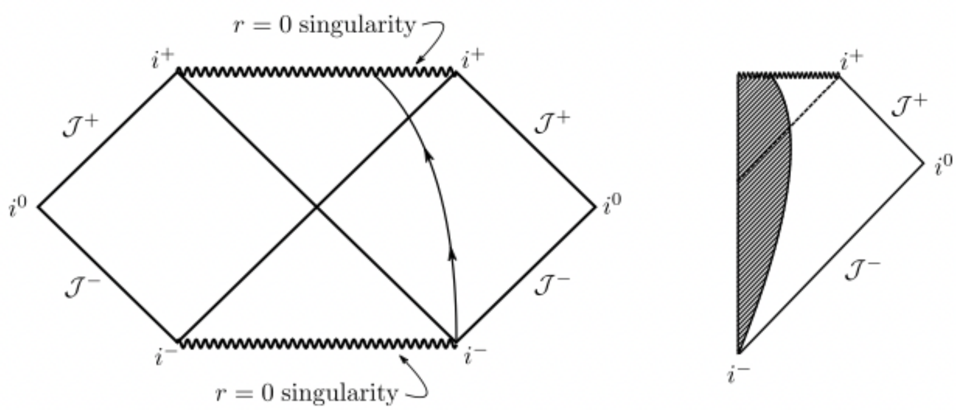

Figure 5: **Left:** The Penrose diagram for Kruskal spacetime. The possible trajectory of the surface of a collapsing star is plotted, the parts to the left correspond to the interior of the star and is described by a metric (at fixed time slice) to the metric we constructed in section 3. **Right:** The Penrose diagram for a collapsing star. The curved surface represents the surface of the star with the shaded area corresponding to the interior of the star. The horizon corresponds to the dashed line and appears in spacetime once the star has collapsed sufficiently.

In contrast to the conformal compactification of Minkowski space the conformally related metric is singular at $i^{\pm}$. Lines of constant r meet at $i^{\pm}$ and this includes the curvature singularity at $r = 0$. Less obviously, it turns out that one cannot choose Ω to make the conformally rescaled metric smooth at i^0.

We can also plot the Penrose diagram of a spherically symmetric collapsing star. The interior of the star is excluded since the stress energy tensor does not vanish there. We end up with only the two regions 1 and 3 of Kruskal spacetime, there is no white hole region.

5 Charged Black holes

At this point we have almost beaten to death the Schwarzschild solution, we need some new solutions to play with. There is a generalisation to the Schwarzschild solution that we can study: we can give it some charge. This will retain the static and spherically symmetric properties of the Schwarzschild solution but couple Einstein gravity to electromagnetism. The charged black hole is known as the *Reissner–Nordström* (RN) black hole.

In nature large imbalances of charge do not occur, it is favourable for the charged object to attract particles of opposite charge and gradually lose its charge. We would therefore expect matter undergoing gravitational collapse to be neutral and so the presence of charged black holes in nature does not seem particularly relevant. Nevertheless the solution exhibits some interesting features. Moreover, for those doing string theory, RN black holes occasionally appear, though probably not in your course.

5.1 Einstein gravity coupled to electromagnetism

We want to couple Einstein gravity to Electromagnetism. Recall that the general prescription for coupling matter to gravity is through minimal coupling.[11] Minimal coupling says we replace the Minkowski metric with the curved metric of spacetime, we replace regular derivatives with covariant derivatives and add in the correct volume measure.

Electromagnetism in terms of forms

Recall that Electromagnetism is governed by Maxwell's equations:

$$\begin{aligned} \nabla \times \vec{B} - \partial_t \vec{E} &= \vec{J}\,, \\ \nabla \cdot \vec{E} &= \rho\,, \\ \nabla \times \vec{E} + \partial_t \vec{B} &= 0\,, \\ \nabla \cdot \vec{B} &= 0\,. \end{aligned} \tag{5.1}$$

Here $\vec{B}$ and $\vec{E}$ are the electric and magnetic field 3-vectors, $\vec{J}$ is a current, ρ is the charge density. These equations are invariant under Lorentz transformations, even though they do not look invariant. We can write these in a manifestly invariant way by introducing the two-form field strength F and its one-form potential A.

[11] One can also add non-minimal terms but we will not consider these here.

Writing the Maxwell's equations in component notation we have

$$\begin{aligned} \epsilon^{ijk}\partial_j B_k - \partial_0 E^i &= J^i \,, \\ \partial_i E^i &= J^0 \,, \\ \epsilon^{ijk}\partial_j E_k + \partial_0 B^i &= 0 \,, \\ \partial_i B^i &= 0 \,. \end{aligned} \tag{5.2}$$

We have introduced the current 4-vector $J = (\rho, \vec{J})$ to rewrite the first two conditions. Let us define the field strength tensor $F_{\mu\nu}$ to be

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}_{\mu\nu} \tag{5.3}$$

We have

$$F^{0i} = E^i \,, \qquad F^{ij} = \epsilon^{ijk} B_k \,. \tag{5.4}$$

Therefore the first two equations in (5.3) can be rewritten as

$$\begin{aligned} \partial_j F^{ij} - \partial_0 F^{0i} &= J^i \,, \\ \partial_i F^{0i} &= J^0 \,, \end{aligned} \tag{5.5}$$

which may be rewritten as

$$\partial_\mu F^{\mu\nu} = -J^\nu \,. \tag{5.6}$$

Similarly the bottom two equations in (5.3) may be rewritten as

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \,. \tag{5.7}$$

Writing F as a two-form we have the two equations

$$\mathrm{d} \star F = -J \,, \qquad \mathrm{d}F = 0 \,. \tag{5.8}$$

The first equation is known as the Maxwell equation, while the second is the Bianchi identity. Since $\mathrm{d}F = 0$ this means that locally F can be written as a closed form,

$$F = \mathrm{d}A \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{5.9}$$

The one-form A is known as the gauge field. Note that it is not unique, $A + \mathrm{d}\Lambda$ gives the same field strength F when Λ is a smooth function. Adding the term $\mathrm{d}\Lambda$ to the potential is known as a gauge transformation, it is a redundancy/symmetry in our description of the the theory. Physical quantities will generally be expressed in terms of the field strength F. On the other hand we view the gauge field as the dynamical field of the theory, i.e. the field we vary an action with respect to.

We can write an action for electromagnetism by using the gauge field A and defining the field strength F to be $F = \mathrm{d}A$. Then the action giving rise to Maxwell's equations with sources is

$$S_{\text{Maxwell}} = \int \mathrm{d}^4x \; \mathcal{L}_{\text{EM}} = \int \mathrm{d}^4x \Big[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \Big] . \tag{5.10}$$

We have

$$\frac{\partial \mathcal{L}_{\text{EM}}}{\partial A_\nu} = J^\nu \,, \tag{5.11}$$

and

$$\frac{\partial \mathcal{L}_{\text{EM}}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \,. \tag{5.12}$$

Putting everything together, the Euler Lagrange equations give

$$\partial_\mu F^{\mu\nu} = -J^\nu \,, \tag{5.13}$$

as we found above from Maxwell's equations. The Bianchi identity arises because we define $F = \mathrm{d}A$ and by using that d$^=$0.

The Lagrangian for electromagnetism in the absence of sources in flat space is

$$\mathcal{L}_{\text{EM}} = -F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma} \,. \tag{5.14}$$

To couple this to gravity we will replace the Minkowski metric with the curved metric, add the volume measure and replace derivatives with covariant derivatives. Derivatives appear in the field strength as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{5.15}$$

where the latter follows when using the Levi–Civita connection. The action for Einstein–Maxwell theory is then

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g} \Big(R - F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \Big) \equiv \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g} \Big(R - F_{\mu\nu}F^{\mu\nu} \Big) \,. \tag{5.16}$$

The equations of motion derived from the variation of the Einstein–Maxwell action are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 2 \bigg(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma} \bigg) \,, \\ \nabla_\mu F^{\mu\nu} &= 0 \,, \end{aligned} \tag{5.17}$$

and should be accompanied by the Bianchi identity $\mathrm{d}F = 0$. *Exercise: Check the equations of motion are indeed those derived from the action.*

5.2 Reissner–Nordström black hole

There is a generalisation of Birkhoff's theorem for four-dimensional Einstein–Maxwell theory.

Theorem:

The Unique spherically symmetric solution of the Einstein–Maxwell equations with non-constant area radius function r is the *Reissner–Nordström* solution:

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2)\,, \\ A &= -\frac{Q}{r}\mathrm{d}t - P\cos\theta\mathrm{d}\phi\,, \\ f(r) &= 1 - \frac{2M}{r} + \frac{e^2}{r^2}\,, \qquad e^2 = Q^2 + P^2\,. \end{aligned} \tag{5.18}$$

The solution has three parameters: M, P, Q. We will show later that these are the mass, magnetic charge and electric charge of the solution. Note that there is no evidence for the existence of magnetic monopoles (which the P describes) in nature, however it is a valid solution of the equations of motion. The non-constant area radius function is important, if this is not enforced then one can find an additional solution, see problem sheet 2.

There are several properties which are similar to the Schwarzschild solution. The solution is static with the timelike Killing vector ∂_t. It is also asymptotically flat, like the Schwarzschild solution and has a curvature singularity at $r = 0$. Note that we may smoothly recover the Schwarzschild solution by sending $Q, P \to 0$.

To discuss the properties of the solution it is convenient to define

$$\Delta(r) = r^2 f(r) = r^2 - 2Mr + e^2 = (r - r_+)(r - r_-)\,, \qquad r_\pm = M \pm \sqrt{M^2 - e^2}\,. \tag{5.19}$$

The metric then takes the form

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{r^2}\mathrm{d}t^2 + \frac{r^2}{\Delta(r)}\mathrm{d}r^2 + r^2\mathrm{d}s^2(S^2)\,. \tag{5.20}$$

Since $r = 0$ is a genuine curvature singularity we would like to hide it behind a horizon, much in the same way that the curvature singularity is hidden in the Schwarzschild solution behind a horizon. This is determined by having a root of $\Delta(r)$, where the metric has a coordinate singularity.[12] There are then three distinct behaviours for the metric depending on the possible roots $r_\pm$, which in turn are determined by the sign of $M^2 - e^2$.

[12]We will make the existence of a horizon more concrete later but let us use the rule of thumb that there is some degeneration of the metric at the horizon.

5.2.1 Super extremal RN: $e^2 > M^2$

If $M^2 - e^2 < 0$ the roots $r_\pm$ are complex and therefore $\Delta(r)$ does not have any real zeros. Thus the curvature singularity is not hidden behind a horizon and we have a naked singularity.[13] There is no obstruction to an observer travelling to the singularity, studying it and then returning to us to tell us all about it. If one studies the geodesics one finds that the naked singularity is repulsive, timelike geodesics never intersect $r = 0$, rather they approach $r = 0$ but reverse course and move away. Null geodesics can reach the singularity as can non-geodesic timelike curves.

As $r \to \infty$ the solution approaches flat spacetime and the causal structure looks normal everywhere. The conformal diagram will therefore be just like that of Minkowski space, except now $r = 0$ is a singularity. The nakedness of the singularity should be offensive to you. We should never expect to find a black hole with $M^2 < e^2$ as a result of gravitational collapse. Roughly, the condition states that the total energy of the hole is less than the contribution to the energy of the electromagnetic fields alone, and therefore we must have something with negative mass. Therefore we consider this unphysical. The Penrose diagram is given in figure 6.

5.2.2 Sub-extremal RN: $M^2 > e^2$

In this case Δ has two real simple roots and there are consequently two coordinate singularities. The surfaces defined by $r = r_\pm$ are both null hypersurfaces and are both event horizons (for the moment we define an event horizon as a hypersurface separating spacetime points to those which are connected to infinity by a timelike path from those that are not). The singularity at $r = 0$ is a timelike line (contrast this with Schwarzschild where it was spacelike).

To see that they are coordinate singularities we can proceed in a similar manner as we did for the Schwarzschild solution and define tortoise like coordinates. Let us begin with $r > r_+$ and define

$$\mathrm{d}r_* = \frac{r^2}{\Delta(r)}\mathrm{d}r\,. \tag{5.21}$$

Integrating gives

$$r_* = r + \frac{1}{2\kappa_+}\log\frac{r - r_+}{r_+} + \frac{1}{2\kappa_-}\log\frac{r - r_-}{r_-} + \text{const}\,, \tag{5.22}$$

where

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}\,. \tag{5.23}$$

[13] We will discuss this in more detail later.

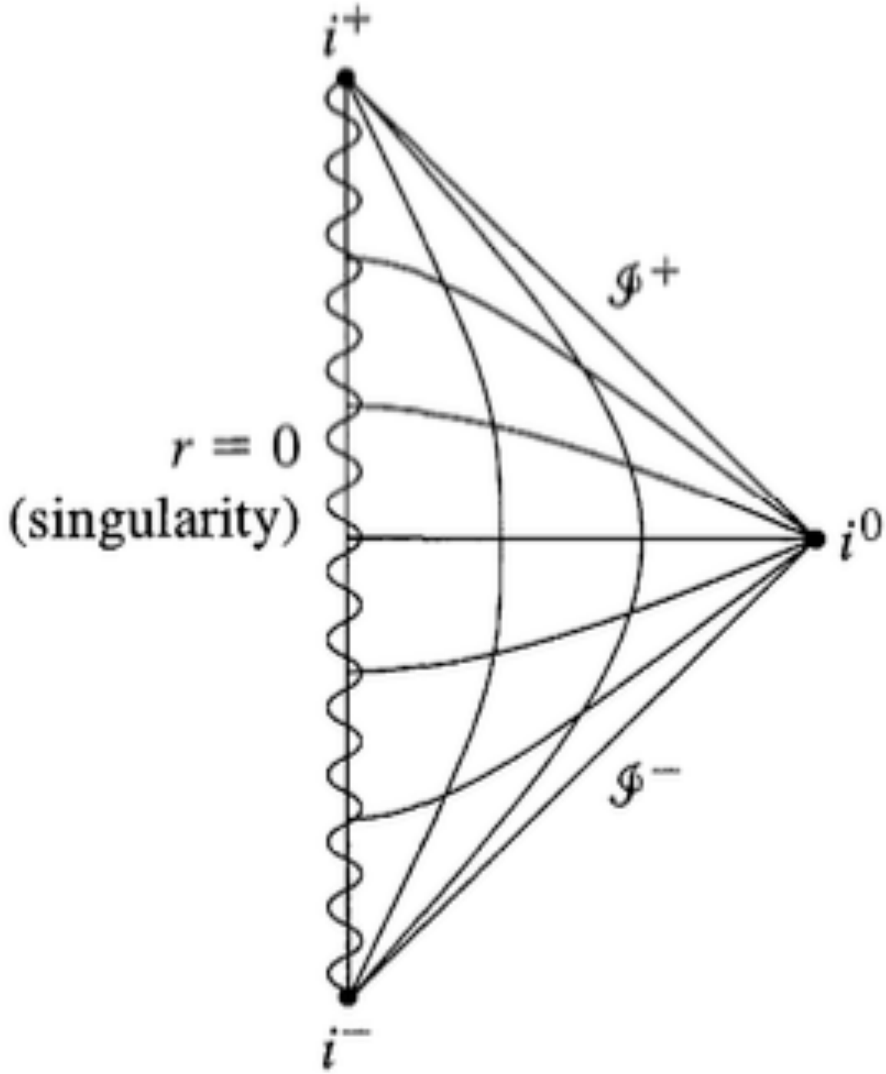

Figure 6: The Penrose diagram for the super-extremal Reissner–Nordström solution.

Now let

$$u = t - r_* \,, \qquad v = t + r_* \,. \tag{5.24}$$

In ingoing Eddington–Finkelstein coordinates the RN metric is

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{r^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}s^2(S^2) \,. \tag{5.25}$$

This is now smooth for any $r > 0$ hence we can analytically continue the metric into the new region $0 < r < r_+$. There is a curvature singularity at $r = 0$ and there is a null hypersurface at $r = r_\pm$.

It follows that no point in the region $r < r_+$ can send a signal to $\mathscr{I}^+$, hence it describes a black hole. The black hole region is $r \leq r_+$ and the future event horizon is the null hypersurface $r = r_+$.

To understand the global structure we need to define Kruskal-like coordinates:

$$U^\pm = -\mathrm{e}^{-\kappa_\pm u}\,, \qquad V^\pm = \pm \mathrm{e}^{\kappa_\pm v}\,. \tag{5.26}$$

Starting in the region $r > r_+$ we use coordinates (U^+, V^+, θ, ϕ) to obtain the metric

$$\mathrm{d}s^2 = -\frac{r_+ r_-}{\kappa_+^2 r^2}\mathrm{e}^{-2\kappa_+ r}\left(\frac{r - r_-}{r_-}\right)^{1+\kappa_+/|\kappa_-|}\mathrm{d}U^+\mathrm{d}V^+ + r^2\mathrm{d}s^2(S^2)\,. \tag{5.27}$$

where $r(U^+, V^+)$ is defined implicitly by

$$-U^+V^+ = \mathrm{e}^{2\kappa_+ r}\left(\frac{r - r_+}{r_+}\right)\left(\frac{r_-}{r - r_-}\right)^{\kappa_+/|\kappa_-|}\,. \tag{5.28}$$

The RHS is a monotonically increasing function of r from $r > r_-$. initially we have $U^+ < 0$ and $V^+ > 0$ which gives $r > r_+$, but now we can analytically continue to $U^+ \geq 0$ or $V^+ \leq 0$. In particular the metric is smooth and non-degenerate when $U^+ = 0$ or $V^+ = 0$. We obtain a diagram very similar to the Kruskal diagram we had for Schwarzschild, see figure 7.

Just as for Kruskal we have a pair of null hypersurfaces which intersect in the bifurcation 2-sphere located at $U^+ = V^+ = 0$. Surfaces of constant t are Einstein–Rosen bridges which connect regions I and IV. The major difference to the Schwarzschild solution is that we no longer have a curvature singularity in regions II and III because $r(U^+, V^+) > r_-$. However we know that it is possible to extend our metric into the $r < r_-$ region, hence the above spacetime must be extendable. In other words we know from the EF coordinates that radial null geodesics reach $r = r_-$ in finite affine parameter so we have to investigate what happens there.

To do this we should start in region II and use ingoing EF coordinates (v, r, θ, ϕ), since we know that these cover regions I and II. We can now define a retarded time coordinate u in region II. First define a time coordinate $t = v - r_*$ in region II with r_* as defined in (5.22). The metric in coordinates (t, r, θ, ϕ) takes the static RN form given above with $r_- < r < r_+$. Now define $u = t - r_* = v - 2r_*$. Having define u in region II we can now define Kruskal coordinates $U^- < 0$ and $V^- < -$ in region II using the formula above. In these coordinates

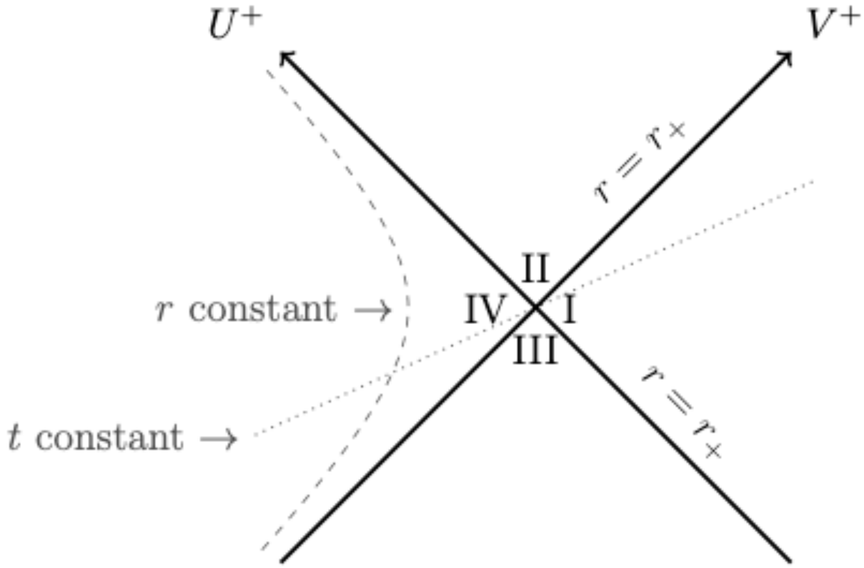

Figure 7: The Reissner–Nordström solution in (U^+, V^+) coordinates.

the metric is

$$\mathrm{d}s^2 = -\frac{r_+ r_-}{\kappa_-^2 r^2} \mathrm{e}^{2|\kappa_-|r} \left(\frac{r_+ - r}{r_+}\right)^{1+|\kappa_-|/\kappa_+} \mathrm{d}U^- \mathrm{d}V^- + r^2 \mathrm{d}s^2(S^2)\,, \tag{5.29}$$

where $r(U^-, V^-) < r_+$ is given implicitly by

$$U^- V^- = \mathrm{e}^{-2|\kappa_-|r} \left(\frac{r - r_-}{r_-}\right) \left(\frac{r_+}{r_+ - r}\right)^{|\kappa_-|/\kappa_+}\,. \tag{5.30}$$

We may as before analytically continue to $U^- \geq 0$ and $V^- \geq 0$ which gives the diagram 8.

We now have the regions V and VI in which $0 < r < r_-$. These regions contain the curvature singularity at $r = 0$ ($U^- V^- = -1$) which is timelike. Region III' is isometric to region III and so by ontroducing new coordinates $(U^{+\prime}, V^{+\prime})$ this can be analytically continued to the future to give further new regions I', II', and IV' as shown in figure 9. In this diagram the regions I' and IV' are new asymptotically flat regions isometric to I and IV. We may repeat this procedure indefinitely to the future and past, so that the maximal analytic extension of the RN solution contains infinitely many regions. The resulting Penrose diagram is given in figure 10. It extends to infinity in both the past and future.

This seems a bit crazy, infinite universes, what is happening here? Notice that if you are an observer falling into the black hole from far away r_+ is just like the Schwarzschild horizon.

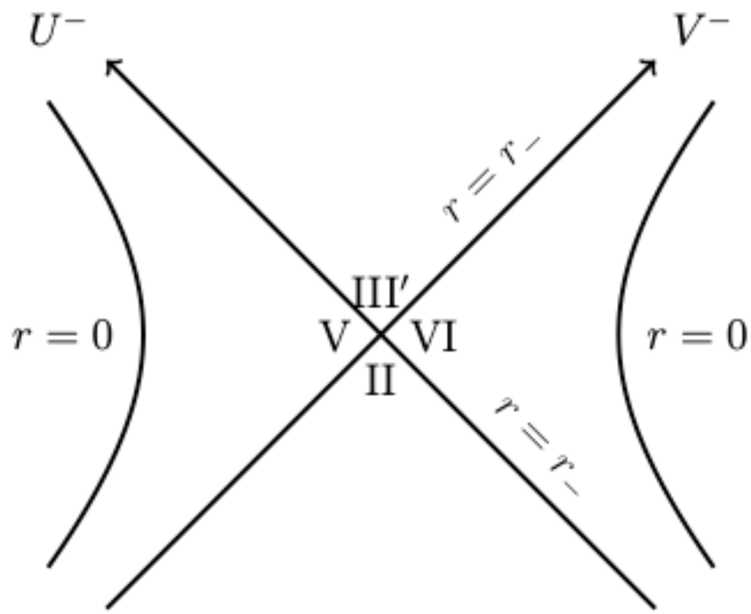

Figure 8: The Reissner–Nordström solution in (U^-, V^-) coordinates.

At this radius r switches from being a spacelike coordinate to a timelike one and therefore you necessarily move in the direction of decreasing r. Witnesses outside the black hole see the same phenomena that they would for the Schwarzschild solution, the infalling observer is seen to move more and more slowly and is increasingly redshifted.

The inevitable fall from r_+ to ever-decreasing radii only lasts until you reach the null surface at $r = r_-$ where r switches from being a timelike coordinate back to being spacelike. You need not continue travelling on a trajectory of decreasing r and therefore your inevitable doom of hitting the singularity can be stopped. Indeed $r = 0$ is a timelike line and you are therefore and therefore not necessarily in your future.

At this point you can continue on to $r = 0$ or begin to move in the direction of increasing r back through the null surface at $r = r_-$. Then r will once again be a timelike coordinate, however now the orientation is reversed and you must travel in the direction of increasing r until you are spat out of the event horizon at $r = r_+$. This is like emerging from a white hole into the rest of the universe. From here you can choose to go back into the black hole, this time a different one to the one you initially entered. You may then repeat this to your hearts content.

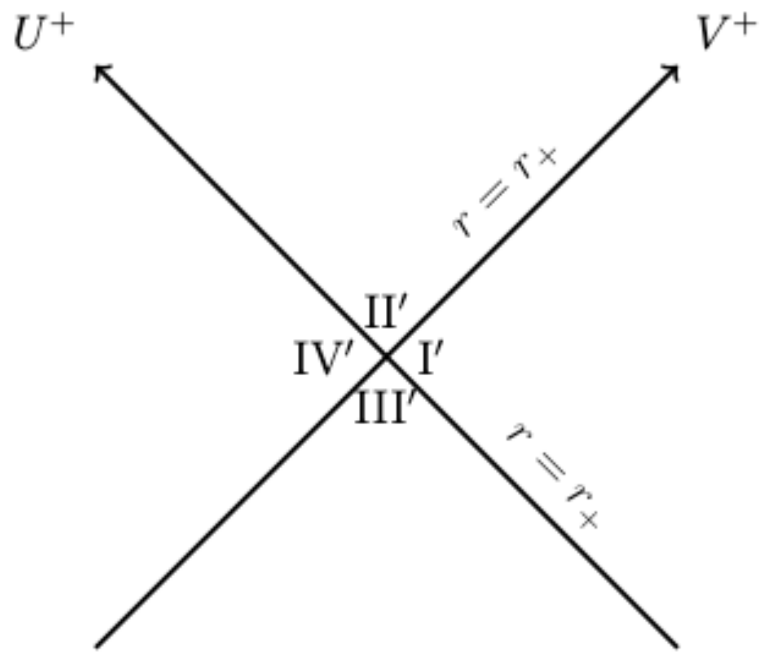

Figure 9: The regions I', II', IV' of the Reissner–Norström solution.

How much of this story is actually science over science fiction? Well, not much. Viewing the universe from the point of an observer inside the black hole tho is about to cross the event-horizon at $r = r_-$ you notice that the observer can look back in time to see the entire history of the external universe, at least as seen from the black hole. They see this infinitely long history in a finite proper time thus any signal that gets to them as they approach $r = r_-$ is infinitely blue-shifted. Therefore it is likely that any non-spherically symmetric perturbation that comes into an RN black hole will violently disturb the geometry. For this reason it is difficult to say exactly what the actual geometry inside the horizon looks like, but there is no good reason why it must contain and infinite number of asymptotically flat regions connecting to each other via various wormholes.

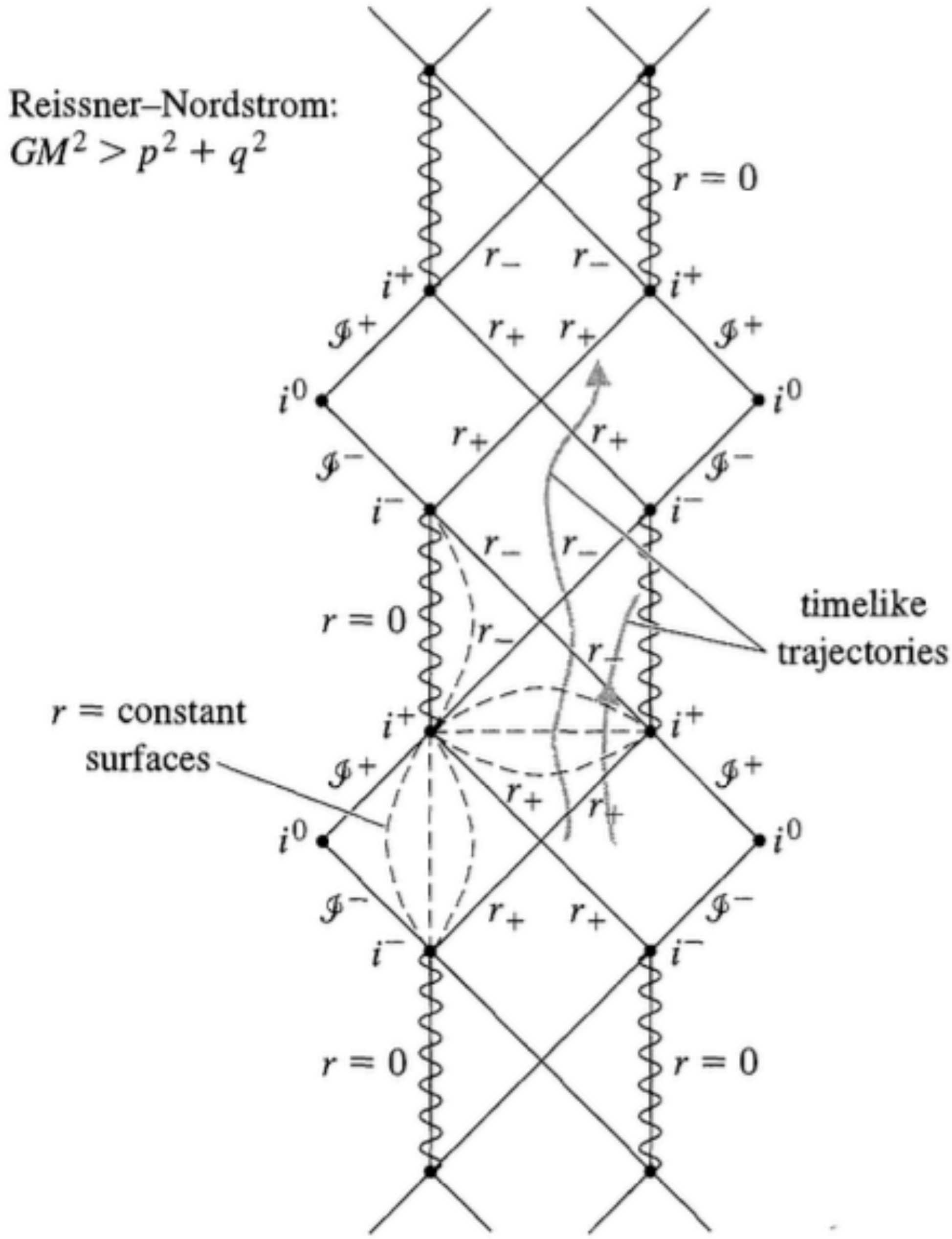

Figure 10: The Penrose diagram of the RN black hole.

5.2.3 Extremal RN: $M^2 = e^2$

Finally let us consider the extremal RH when the two roots become equal and we obtain a double root. The metric of the RN extremal solution is

$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2 \mathrm{d}t^2 + \left(1-\frac{M}{r}\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}s^2(S^2)\,, \tag{5.31}$$

which has a coordinate singularity at $r = r_+ = r_- = M$.

The coordinate r is never time-like, it becomes null on the horizon at $r = r_+ = r_-$ but is spacelike either side of the horizon. The singularity at $r = 0$ is once again a timelike line and as in the other cases may be avoided. You may avoid the singularity and continue to move to the future to extra copies of the asymptotically flat region, the singularity is always to the "left". The Penrose diagram is given in figure 11

Extremal black holes appear frequently when considering supersymmetric theories, they are generally the black holes which preserve supersymmetry. As we will see extremal implies that the temperature of the black hole vanishes. The solution seems unstable since adding a little matter will take us to the sub extremal solution. In the extremal case the mass is balanced by the charge, this can be reformulated when considering a supersymmetric theory as saturating the Bogomol'nyi–Prasad–Sommerfield bound. Two extremal black holes with the same sign charges will attract each other gravitationally but repel each other electromagnetically and the two forces precisely cancel. We can find exact solutions to the coupled Einstein–Maxwell equations representing any number of such black holes in a stationary configuration.

To see this it is useful to first rewrite the RN solution and to focus on just electric charges for simplicity. Define the radial coordinate

$$\rho = r - M \tag{5.32}$$

then the metric takes the isotropic form

$$\mathrm{d}s^2 = -H(\rho)^{-2}\mathrm{d}t^2 + H(\rho)^2\Big[\mathrm{d}\rho^2 + \rho^2\mathrm{d}s^2(S^2)\Big]\,, \tag{5.33}$$

where

$$H(\rho) = 1 + \frac{M}{\rho}\,. \tag{5.34}$$

Since the bracketed part of the metric is just the metric on $\mathbb{R}^3$ we may rewrite the metric as

$$\mathrm{d}s^2 = -H(\vec{x})^{-2}\mathrm{d}t^2 + H(\vec{x})^2\Big[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\Big]\,, \tag{5.35}$$

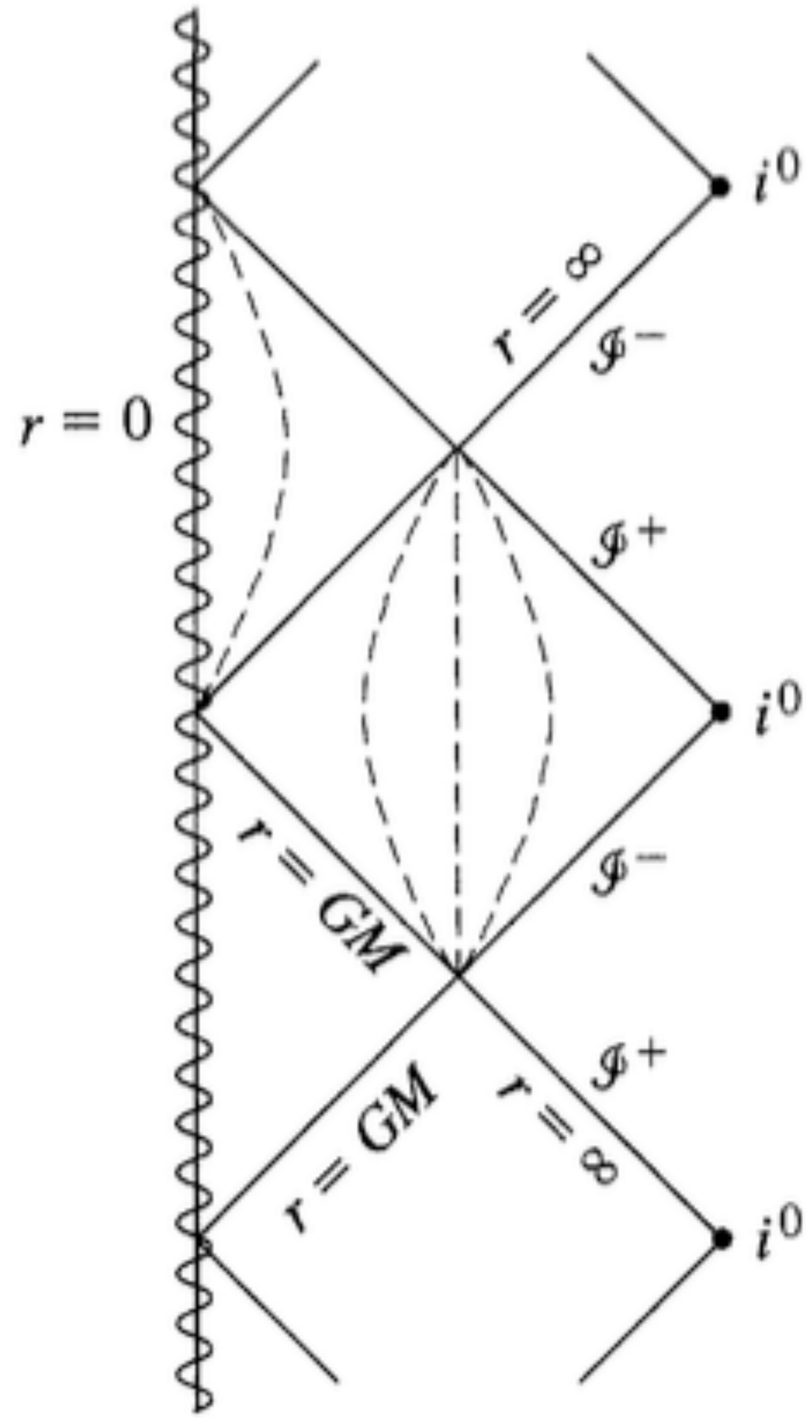

Figure 11: The Penrose diagram of the extremal RN black hole.

with

$$H(\vec{x}) = 1 + \frac{M}{|\vec{x}|}\,. \tag{5.36}$$

In the original components the electric field of the extremal solution can be expressed in terms of a vector potential A as

$$F_{rt} = \frac{Q}{r^2} = \partial_r A_0\,, \qquad A_0 = -\frac{Q}{r}\,. \tag{5.37}$$

We may rewrite this as

$$A_0 = H^{-1} - 1\,. \tag{5.38}$$

We can now forget that H takes the form above and just plug the metric into the field equations and we find that we have a solution provided

$$\nabla^2 H = 0\,, \tag{5.39}$$

with ∇^2 the Laplacian on $\mathbb{R}^3$. It is straightforward to write down all solutions that are well behaved at infinity, they take the form

$$H = 1 + \sum_{a=1}^{N} \frac{M_a}{|\vec{x} - \vec{x}_a|}\,, \tag{5.40}$$

for some set of N spatial points $\vec{x}_a$. These are the locations of the N extremal RN black holes with masses M_a and electric charges $Q_a = M_a$.

5.3 Charges in curved spacetime

We now want to see how to compute the electric and magnetic charges of the solution and check that they do indeed agree with the parameters Q and P in the RN solution. Consider Maxwell's equation in the presence of a current density J:

$$\mathrm{d} \star F = -4\pi \star J\,, \quad \mathrm{d}F = 0\,. \tag{5.41}$$

The first implies that $\mathrm{d} \star J = 0$, which in components is equivalent to $\nabla_\mu J^\mu = 0$ which is the definition of a conserved current.

Consider a spacelike hypersurface Σ. We define the total electric charge on Σ to be

$$Q = -\int_\Sigma \star J\,. \tag{5.42}$$

Using Maxwell's equations we can write

$$Q = \frac{1}{4\pi}\int_\Sigma \mathrm{d} \star F\,, \tag{5.43}$$

and assuming Σ has boundary $\partial\Sigma$ Stoke's theorem gives

$$Q = \frac{1}{4\pi}\int_{\partial\Sigma} \star F\,. \tag{5.44}$$

This is the analogue of Gauss' law $Q \sim \int \vec{E}\cdot \mathrm{d}\vec{S}$.

Consider Minkowski spacetime in spherical polar coordinates and choose the orientation so that the volume form is

$$\mathrm{dvol} = r^2 \sin\theta \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{5.45}$$

Take Σ to be the surface at fixed $t = 0$.[14] We may view Σ as the boundary of the region $t \leq 0$ then Stoke's theorem fixes the orientation of Σ as $r^2 \sin\theta \mathrm{d}r\wedge \mathrm{d}\theta\wedge \mathrm{d}\phi$. Let Σ_R be the region of Σ with $r \leq R$ of Σ, the boundary is then the two-sphere with radius R: S^2_R. Stokes' theorem fixes the orientation of the two-sphere to be $\mathrm{d}\theta\wedge\mathrm{d}\phi$. Consider the Coulomb potential

$$A = -\frac{q}{r}\mathrm{d}t\,, \qquad F = -\frac{q}{r^2}\mathrm{d}t\wedge \mathrm{d}r\,. \tag{5.46}$$

Taking the Hodge dual gives

$$\star F = q\sin\theta \mathrm{d}\theta\wedge\mathrm{d}\phi\,, \tag{5.47}$$

and hence the charge on Σ_R is

$$S[\Sigma_R] = \frac{1}{4\pi}\int_{S^2_R} \star F = \frac{1}{4\pi} q\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi = q\,. \tag{5.48}$$

Our definition of Q gives the expected result.

For an asymptotically flat hypersurface in Minkowski spacetime we can take the limit $R \to \infty$ to express the total charge on Σ as an integral at infinity. Motivated by this we define the charges at asymptotic infinity to be

$$Q = \frac{1}{4\pi}\lim_{r\to\infty}\int_{S^2_r} \star F\,, \qquad P = \frac{1}{4\pi}\lim_{r\to\infty}\int_{S^2_r} F\,, \tag{5.49}$$

where S^2_r is the sphere with radius r.

Note that even when there is no charged matter, $J = 0$ we can still obtain a non-trivial charge, for example the RN solution above. The total charge on a spacelike hypersurface vanishes, since $J = 0$, however when we convert the integral to a surface integral at infinity we obtain two terms because the surface has two asymptotically flat ends. The charges on each of these boundary pieces can be non-zero, so long as they cancel each other when summed.

[14] We could of course choose any value of t, 0 is just notationally simpler.

It remains to be seen why we call this a conserved charge. Consider two spacelike surfaces Σ_1 and Σ_2. Consider the cylindrical surface, V which is bounded by Σ_1 and Σ_2, and large enough to contain all of the sources, see figure 12. From this latter condition it follows that $J = 0$ on the boundaries and outside V. We then have

$$\begin{aligned} 0 &= \int_V \mathrm{d} \star J \\ &= \int_{\partial V} \star J \\ &= \int_{\Sigma_1} \star J - \int_{\Sigma_2} \star J \\ &= \frac{1}{4\pi} \int_{\partial\Sigma_1} \star F - \frac{1}{4\pi} \int_{\partial\Sigma_2} \star F \\ &= Q[\Sigma_1] - Q[\Sigma_2]\,. \end{aligned} \tag{5.50}$$

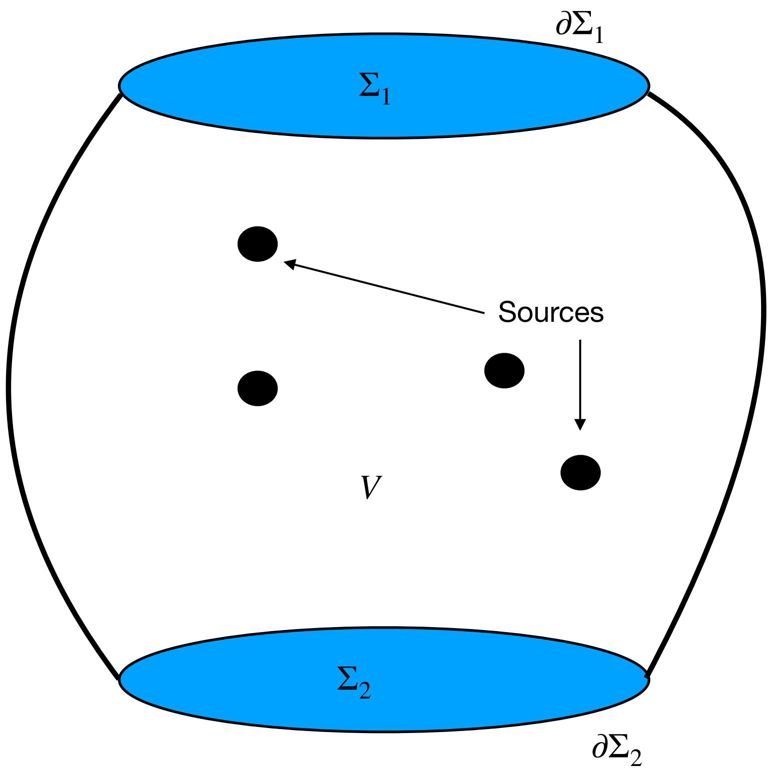

Figure 12: The region V bounded by the two spacelike hypersurfaces Σ_i and containing all the sources.

We can also define magnetic charges similarly. Since they are already defined on the spacelike hypersurface we just need to integrate F, as opposed to $\star F$. A similar argument for showing that it is conserved holds as well.

Let us do this for the RN black hole. We have

$$F = \frac{Q}{r^2}\mathrm{d}r \wedge \mathrm{d}t + P \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi \,. \tag{5.51}$$

The magnetic charge is defined to be

$$P[S^2] = \frac{1}{4\pi}\int_{S^2} F = \frac{P}{4\pi}\int_{S^2} \mathrm{dvol}(S^2) = P \,. \tag{5.52}$$

For the electric charge we need the Hodge dual. We have

$$\star F = \frac{1}{r^2}\mathrm{d}t \wedge \mathrm{d}r + Q \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi \,. \tag{5.53}$$

Therefore the electric charge is

$$Q[S^2] = \frac{1}{4\pi}\int_{S^2} Q \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi = Q \,. \tag{5.54}$$

We find that indeed the parameters Q and P are the electric and magnetic charges.

6 Rotating black holes

All the solutions we have seen so far have been static and spherically symmetric, though these are nice testing grounds for us to learn things from they are not likely to be objects that we will see in our universe. Observational evidence seems to suggest that black holes should rotate. Our goal in this section is to study rotating black holes.

Since the black holes are rotating we must give up our spherical symmetry, they can however be axisymmetric: symmetric under rotations about an axis. Moreover we must give up our metric being static, and reduce to the weaker stationary class of metric. This follows since if we were to dun time in the opposite direction we must see rotation in the opposite direction, clearly this cannot be static, we should then impose the weaker stationary condition. These generalisations make the metric a lot more complicated. Although the Schwarzschild solution and Reissner–Nordström solutions were discovered shortly after general relativity was invented, the metric we will study, known as the Kerr(–Newman) metric was first found in 1963. Kerr originally found the rotating metric without any charges but was later extended by Newman to include charges.

6.1 The Kerr–Newman solution

The Kerr–Newman solution in *Boyer-Lindquist coordinates* is

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{\Delta(r) - a^2\sin^2\theta}{\rho(r,\theta)^2}\mathrm{d}t^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta(r))}{\rho(r,\theta)^2}\mathrm{d}t\mathrm{d}\phi \\
&\quad + \frac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta(r)}{\rho(r,\theta)^2}\sin^2\theta\mathrm{d}\phi^2 + \frac{\rho(r,\theta)^2}{\Delta(r)}\mathrm{d}r^2 + \rho^2(r,\theta)\mathrm{d}\theta^2\,, \\
A &= -\frac{1}{\rho(r,\theta)^2}\Big(Qr(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi) + P\cos\theta(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)\Big)\,.
\end{aligned} \tag{6.1}$$

The functions are

$$\rho(r,\theta)^2 = r^2 + a^2\cos^2\theta\,, \quad \Delta(r) = r^2 - 2Mr + a^2 + e^2\,, \quad e^2 = Q^2 + P^2\,. \tag{6.2}$$

At larger r the above coordinates reduce to the spherical polar coordinates in Minkowski spacetime, θ, ϕ have the usual interpretation as angles on S^2, so we have $0 < \theta < \pi$ and $\phi \in [0, 2\pi]$. This depends on 4 parameters, a, M, Q and P. You may guess that M is the mass, Q the electric charge, P the magnetic charge and a related to the angular momentum. We will show how to compute the angular momentum and mass soon, but for the moment let us just give the result. The parameter a is the angular momentum per unit mass,

$$a = \frac{J}{M}\,, \tag{6.3}$$

with J the Komar angular momentum.

Note that the metric can be rearranged into the form

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{\rho^2(r,\theta)}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 + \frac{\rho^2(r,\theta)}{\Delta(r)}\mathrm{d}r^2 + \frac{\sin^2\theta}{\rho^2(r,\theta)}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2 + \rho^2(r,\theta)\mathrm{d}\theta^2\,, \tag{6.4}$$

which makes clear that the $a = 0$ limit recovers the Reissner–Nordstrom solution of the previous section.

This is the unique stationary black hole solution of the Einstein–Maxwell theory. An equilibrium black hole in the presence of the elctromagnetic field is therefore fully characterised by the three numbers M, J and Q.

6.2 The Kerr solution

Since all of the essential phenomena persist in the absence of charge we will set $Q = P = 0$ in the remainder of this section. If we set $a \to 0$ the metric reduces to the Schwarzschild solution. If instead we keep a fixed but set $M \to 0$ then we recover flat space, but in funky coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\mathrm{d}r^2 + (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta \mathrm{d}\phi^2 \,. \tag{6.5}$$

The spatial part of the metric is flat three-dimensional space written in ellipsoidal coordinates, see figure 13. They are related to Cartesian coordinates in three-dimensional space by

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \,, \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \,, \\ z &= r\cos\theta \,. \end{aligned} \tag{6.6}$$

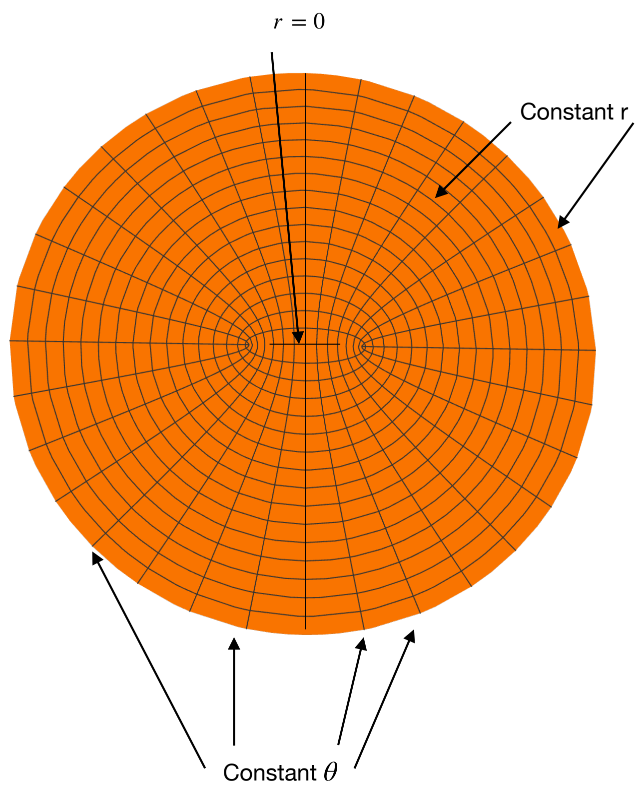

Figure 13: The structure of the ellipsoidal coordinates of the Kerr metric. The region $r = 0$ is a two-dimensional disc with length $2a$.

There are two Killing vectors of the metric both of which are manifest since the metric is independent of both t and ϕ. Both $K = \partial_t$ and $R = \partial_\phi$ are Killing vectors. K is not orthogonal to t =constant hypersurfaces and hence the metric is stationary and not static. This makes sense since the black hole is rotating, so not static, but it is spinning in exactly the same way at all times so it is stationary. R expresses the axial symmetry of the solution, we have a symmetry rotating the solution around the axis of rotation.

Besides the Killing vectors the Kerr metric also has a Killing tensor. A Killing tensor is any symmetric $(0,n)$ tensor $\sigma_{\mu_1\ldots\mu_n}$ satisfying

$$\nabla_{(\nu}\sigma_{\mu_1\ldots\mu_n)} = 0\,. \tag{6.7}$$

For the Kerr geometry we can define the $(0,2)$ tensor

$$\sigma_{\mu\nu} = 2\rho^2 l_{(\mu}n_{\nu)} + r^2 g_{\mu\nu}\,, \tag{6.8}$$

where

$$l^\mu = \frac{1}{\Delta}(r^2+a^2,\Delta,0,a)\,,\quad n^\mu = \frac{1}{2\rho^2}(r^2+a^2,-\Delta,0,a)\,, \tag{6.9}$$

both vectors are null and satisfy

$$l^\mu l_\mu = 0\,,\quad n^\mu n_\mu = 0\,,\quad l^\mu n_\mu = -1\,. \tag{6.10}$$

The coordinates have been chosen so that the event horizons occur at those fixed values of r for which $g^{rr}=0$. Since $g^{rr}=\Delta/\rho^2$ we have zeroes when

$$\Delta(r) = r^2 - 2Mr + a^2 = 0\,. \tag{6.11}$$

We may then write

$$\Delta = (r-r_+)(r-r_-)\,,\quad r_\pm = M \pm \sqrt{M^2-a^2}\,. \tag{6.12}$$

The solutions with $M^2 - a^2 < 0$ describe a naked singularity, and the $M^2 = a^2$ solution is unstable, so lets assume that $M^2 > a^2$ from now on. The metric is also singular at $\theta = 0, \pi$ but these are just coordinate singularities of spherical polars so we will ignore these. There is also a singularity at $\rho^2 = 0$ when $r = 0$ and $\theta = \frac{\pi}{2}$.

Let us show that $r = r_+$ is just a coordinate singularity. To do this we define *Kerr coordinates* (v, r, θ, χ) for $r > r_+$ by

$$\mathrm{d}v = \mathrm{d}t + \frac{r^2+a^2}{\Delta(r)}\mathrm{d}r\,,\quad \mathrm{d}\chi = \mathrm{d}\phi + \frac{a}{\Delta(r)}\mathrm{d}r\,. \tag{6.13}$$

In the new coordinates we have $\chi \sim \chi + 2\pi$ and the Killing vectors are

$$K = \frac{\partial}{\partial v}\,,\qquad R = \frac{\partial}{\partial \chi}\,. \tag{6.14}$$

The new metric in these coordinates is

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta(r) - a^2\sin^2\theta}{\rho^2}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r - \frac{2a\sin^2\theta(r^2+a^2-\Delta(r))}{\rho^2}\mathrm{d}v\mathrm{d}\chi \\ &- 2a\sin^2\theta\mathrm{d}\chi\mathrm{d}r + \frac{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\chi^2 + \rho^2\mathrm{d}\theta^2\,. \end{aligned} \tag{6.15}$$

This change of coordinates shows that the metric is non-degenerate at $r = r_+$. We can analytically continue through the surface $r = r_+$ into a new region with $0 < r < r_+$.

The surface $r = r_+$ is a null hypersurface with normal

$$\xi^\mu = K^\mu + \Omega_H R^\mu\,, \tag{6.16}$$

with

$$\Omega_H = \frac{a}{r_+^2 + a^2}\,. \tag{6.17}$$

Note that one-form

$$\xi = \frac{\rho(r,\theta)}{r^2+a^2}\mathrm{d}r\,, \tag{6.18}$$

vanishes on the $r = r_+$ surface and is therefore normal to this hypersurface. The dual one-form is

$$\xi = \partial_v + \frac{\Delta(r)}{r^2+a^2}\partial_r + \frac{a}{r^2+a^2}\partial_\chi\,, \tag{6.19}$$

which agrees with ξ above on the horizon where $\Delta(r_+) = 0$. The norm of ξ is

$$\xi^\mu\xi_\mu = \frac{\rho^2(r,\theta)\Delta(r)}{(r^2+a^2)^2}\,, \tag{6.20}$$

which clearly vanishes at $r = r_+$ and therefore the vector ξ is a null Killing vector on $r = r_+$. The region $r \le r_+$ part of the black hole region of this spacetime with $r = r_+$ is the future event horizon $\mathcal{H}^+$. In Boyer–Lindquist coordinates the Killing vector is

$$\xi = \frac{\partial}{\partial t} + \Omega_H\frac{\partial}{\partial\phi}\,. \tag{6.21}$$

Observe that $\xi^\mu\partial_\mu(\phi - \Omega_H t) = 0$ and therefore $\phi = \Omega_H t + \text{const}$ on integral curves of ξ^μ. Conversely integral curves of K have $\phi = \text{const}$. We see that particles moving on orbits of ξ rotate with angular velocity Ω_H with respect to a stationary observer (someone on an orbit of K). In particular they rotate with this angular velocity with respect to a stationary observer at infinity. Since ξ is tangent to the generators of $\mathcal{H}^+$, then these generators rotate with angular velocity Ω_H with respect to a stationary observer at infinity so we can interpret Ω_H as the angular velocity of the black hole.

6.3 Komar Integrals

In the above we have claimed that the Kerr black hole is rotating and has angular momentum $J = aM$, we would like to back up this claim. This relies on us defining a Komar integral, which is essentially a charge associated to a Killing vector.

We have seen that we can define conserved electric and magnetic charges given a gauge field, one can understand the need for a charge associated to a Killing vector by playing a little game with Kaluza–Klein reduction.

Consider Einstein gravity in five-dimensions without a cosmological constant. Let us take an ansatz for the metric of the form

$$\mathrm{d}s^2 = \phi^2(x)\big(\mathrm{d}\psi + A_\mu \mathrm{d}x^\mu\big)^2 + g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{6.22}$$

where ∂_ψ is a Killing vector and the one-form A is defined only on the base with coordinates x. Note that gauge transformations are just coordinate transformations in this formalism.

We can now plug this into the five-dimensional vacuum Einstein equations. One finds that there are three conditions one must impose in order for the metric to satisfy the five-dimensional Einstein vacuum equations:

$$\begin{aligned}
\Box\phi &= \frac{1}{4}\phi^3 F^{\mu\nu}F_{\mu\nu}\,,\\
\nabla_\mu\Big(\phi^3 F^{\mu\nu}\Big) &= 0\,,\\
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}\phi^2\Big(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi)\,,
\end{aligned} \tag{6.23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and everything is a four-dimensional object defined by the metric $g_{\mu\nu}$. For a constant ϕ we can see the Maxwell equation and Einstein equation of the Einstein–Maxwell theory, of course setting ϕ =constant imposes a non-trivial relation on the F but let us forget about this for the moment.

We see that if 5-dimensional spacetime has a circle which is small, then we see a four-dimensional spacetime which is Einstein gravity plus electromagnetism. Now we know that in the four-dimensional theory we can define electric (and magnetic) charges, but there should be some remnant of these electric charges in the five-dimensional theory. In the five-dimensional theory it must enter through the gauge field A and therefore it is connected to the Killing vector ∂_ψ: there must be a way of defining a conserved charge to a Killing vector which is the analogue of the electric charge in the dimensionally reduced theory.

Let k be a Killing vector, recall that this implies that $\nabla_{(\mu}k_{\nu)} = 0$, and therefore $\nabla_\mu k_\nu$ is anti-symmetric. We can define the two-form

$$K_{\mu\nu} = \nabla_\mu k_\nu \,, \qquad K = \mathrm{d}k \,, \tag{6.24}$$

were we have abused notation to write k for the form and also the vector. For any vector X recall that we have

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) X^\sigma = R^\sigma{}_{\rho\mu\nu} X^\rho \,. \tag{6.25}$$

Let us use this with the Killing vector and contract the σ and μ indices, we have

$$\begin{aligned} \nabla_\mu \nabla_\nu k^\mu - \nabla_\nu \nabla_\mu k^\mu &= R_{\rho\nu} k^\rho \\ &= \nabla_\mu \nabla_\nu k^\mu \\ &= \nabla^\mu K_{\nu\mu} \,, \end{aligned} \tag{6.26}$$

and therefore we have *(Exercise)*

$$\nabla^\mu K_{\mu\nu} = -2 R_{\nu\mu} k^\mu \,. \tag{6.27}$$

In form notation we have

$$\mathrm{d} \star \mathrm{d}k = 8\pi G_N \star J = 2 \star R_{\mu\nu} k^\mu \mathrm{d}x^\nu \,. \tag{6.28}$$

This should look reminiscent of how we defined electric charges in the previous section. We may rewrite the above using Einstein's equations:

$$R_{\mu\nu} = 8\pi G_N \Big(T_{\mu\nu} - \frac{1}{2} T^\rho{}_\rho g_{\mu\nu} \Big) \,, \tag{6.29}$$

to find the current

$$J_\mu = 2 \Big(T_{\mu\nu} - \frac{1}{2} T^\rho{}_\rho g_{\mu\nu} \Big) k^\nu \,. \tag{6.30}$$

Thus $\mathrm{d} \star J = 0$. In analogy to how we defined a charge in electromagnetism, on a spatial hypersurface Σ, we may define the conserved charge

$$Q_k(B) = -\int_\Sigma \star J = \frac{1}{8\pi} \int_\Sigma \mathrm{d} \star \mathrm{d}k = \frac{1}{8\pi} \int_{\partial\Sigma} \star \mathrm{d}k \tag{6.31}$$

We define the charge to be taken at asymptotic infinity.

Definition: *Komar mass*

Let Σ be a spacelike hypersurface with boundary S_r^2 in an asymptotically flat stationary spacetime, with time-like Killing vector k. The *Komar mass* (or Komar energy) is

$$M_{\text{Komar}} = -\frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r^2} \star \mathrm{d}k \,. \tag{6.32}$$

This is a measure of the total energy of the spacetime. This energy comes from both matter and the gravitational field. You have seen in GR1 exercises that even when computing the Komar mass for the Schwarzschild solution, which is a vacuum solution with no matter, we find a non-zero Komar mass which is equal to M.

Since the only property of k we used was that it is a Killing vector we can also define the angular momentum.

Definition: *Komar angular momentum*

Let Σ be a spacelike hypersurface with boundary S_r^2 in an asymptotically flat stationary spacetime with killing axisymmetric vector k. Then the *Komar angular momentum* is

$$J_{\text{Komar}} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r^2} \star \mathrm{d}k \,. \tag{6.33}$$

6.4 Maximal extension

The Kerr coordinates are analogous to the ingoing Eddington–Finkelstein coordinates that we used for the Reisner–Nordström solution. One can similarly define retarded EF coordinates and study the white hole region, before constructing Kruskal like coordinates which cover the various regions of the metric.

Just as for the RN solution, the spacetime can be extended across the null hypersurfaces at $r = r_-$ in regions II and III. The resulting maximal extension is similar to that of RN except for the behaviour near the singularity. There is no longer a singularity at $r = 0$ but rather at $\rho = 0$, this is where the Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges. Since $\rho^2 = r^2 + a^2\cos^2\theta$ is the sum of two manifestly non-negative quantities it can only vanish when both vanish, this is then at

$$r = 0\,, \qquad \theta = \frac{\pi}{2}\,. \tag{6.34}$$

For fixed v, r, θ the metric is

$$\mathrm{d}s^2\Big|_{v,r,\theta\text{ Fixed}} = \frac{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}{\rho^2}\sin^2\theta \mathrm{d}\chi^2 \tag{6.35}$$

We see that as we take $r \to 0$ we have

$$\mathrm{d}s^2\Big|_{v,r,\theta\text{ Fixed}} = a^2\sin^2\theta\mathrm{d}\chi^2. \tag{6.36}$$

This then defines a disc parametrised by θ and χ. When we also take $\theta = \frac{\pi}{2}$ we end up with the metric $\mathrm{d}s^2 = a^2\mathrm{d}\chi^2$ which is the metric on a ring of radius a. Therefore in the Kerr metric, the curvature singularity has the structure of a ring. The rotation has softened the Schwarzschild singularity, spreading it out over a ring. If you travel toward $r = 0$ from any other angle other than $\theta = \frac{\pi}{2}$ you will not encounter the singularity and will instead pass through and enter a new asymptotically flat region. This is not an identical copy of the spacetime you came from though, instead it is described by the Kerr metric with $r < 0$. As a result Δ never vanishes and there are no horizons in this space.

This spacetime with $r < 0$ has an unusual feature. One finds that $R = \partial_\phi$ becomes time-like near the singularity, the metric at fixed $t, r < 0$ and $\theta = \frac{\pi}{2}$ is

$$\mathrm{d}s^2 = \Big(r^2 + a^2 + \frac{2Ma}{r}\Big)\mathrm{d}\chi^2\,, \tag{6.37}$$

which close enough to the singularity is negative. Since χ is 2π-periodic we end up with *closed timelike curves*. You may sometimes hear these referred to as time-machines. It is a curve that is everywhere timelike and that eventually returns to where it started in spacetime. You can then travel on this CTC and meet yourself in the past!

This region is unphysical. Much like in the case of sub-extremal RN the inner horizon at $r = r_-$ becomes a curvature singularity in the presence of the smallest perturbations to the Kerr metric: at the inner horizon perturbations are infinitely blueshifted, which leads to divergences in the curvature scalars.

When we considered Schwarzschild we saw that it describes the metric outside a spherical star. This was a consequence of Birkhoff's theorem. In contrast the Kerr solution does *not* describe the spacetime outside a rotating star. This solution is expected to describe only the final state of gravitational collapse. One can't obtain a solution describing gravitational collapse to form a Kerr black hole by simply gluing in a ball of collapsing matter as was possible for Schwarzschild. Additionally, the spacetime during collapse would not even be stationary as gravitational waves must be emitted.

Theorem *Carter 1971, Robinson 1975*

If (M, g) is a stationary, axisymmetric, asymptotically flat vacuum spacetime suitably regular on, and outside a connected event horizon then (M, g) is a member of the 2-parameter Kerr family of solutions. The parameters are the angular momentum and mass.

This result says that the final state of gravitational collapse is generically a Kerr black hole and is fully characterised by just 2 numbers. In contrast the initial state can be arbitrarily complicated. Nearly all information about the initial state is lost during gravitational collapse: either by radiation to infinity, or by falling into the black hole, and just 2 numbers are required to describe the final state on and outside the event horizon. There is an extension of this theorem for the 4-parameter Kerr–Newman solution.

To draw the Penrose diagram it is now more difficult because the metric is no longer spherically symmetric. Since the curvature singularity will appear only for $\theta = \frac{\pi}{2}$ the Penrose diagram will look different for $\theta \neq \frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$. To represent both cases it is customary to draw a Penrose diagram that is an amalgamation of the Penrose diagram for an observer falling in from the north pole and along the equatorial plane at fixed χ. Notice that $\chi = \text{const}$ means that ϕ is not constant so the observer falling in at $\theta = \frac{\pi}{2}$ rotates about the polar axis. See figure 14 for the Penrose diagram.

6.5 Ergosphere and Penrose process (or how to steal energy from a black hole)

By definition a black hole is a region of space where no matter nor light can escape from. It may come as a surprise that we can extract energy from a black hole if it has an ergosphere.

The norm of the Killing vector K is

$$K^\mu K_\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta)\,, \tag{6.38}$$

which we see does not vanish on the horizon, instead on the horizon it is spacelike. This Killing vector is already spacelike at the outer horizon, except at the north and south poles at $\theta = 0, \pi$ where it is null. The locus of points where $K^\mu K_\mu = 0$ is called the stationary limit surface and is given by

$$(r - M)^2 = M^2 - a^2\cos^2\theta\,, \tag{6.39}$$

while the outer event horizon is given by

$$(r_+ - M)^2 = M^2 - a^2\,. \tag{6.40}$$

Thus there is a region between these two surfaces, which is called the *ergosphere*, where K is spacelike, see figure 15. Therefore since in the ergosphere ∂_t is not time-like one cannot travel along its integral curves and remain stationary with respect to observers at infinity. A stationary observer is someone whose 4-velocity is parallel to K, since this is spacelike in the ergosphere they cannot be stationary. Recall that in order to be timelike we need to satisfy $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ inside the ergosphere. However each of the terms of the metric are positive definite inside the ergosphere except the term $g_{t\phi}$, and therefore $\dot{\phi} \neq 0$ and so must rotate. Since $\dot{t} > 0$ for a future directed worldline, we must have $\dot{\phi} > 0$ and therefore the timelike worldline is dragged around in the direction of the rotation of the black hole. This effect is an example of *frame dragging*.

We may exploit this to obtain energy from the black hole. Consider a particle with 4-momentum $P^\mu = m\dot{x}^\mu$ with m the rest of the particle. Recall that the existence of Killing

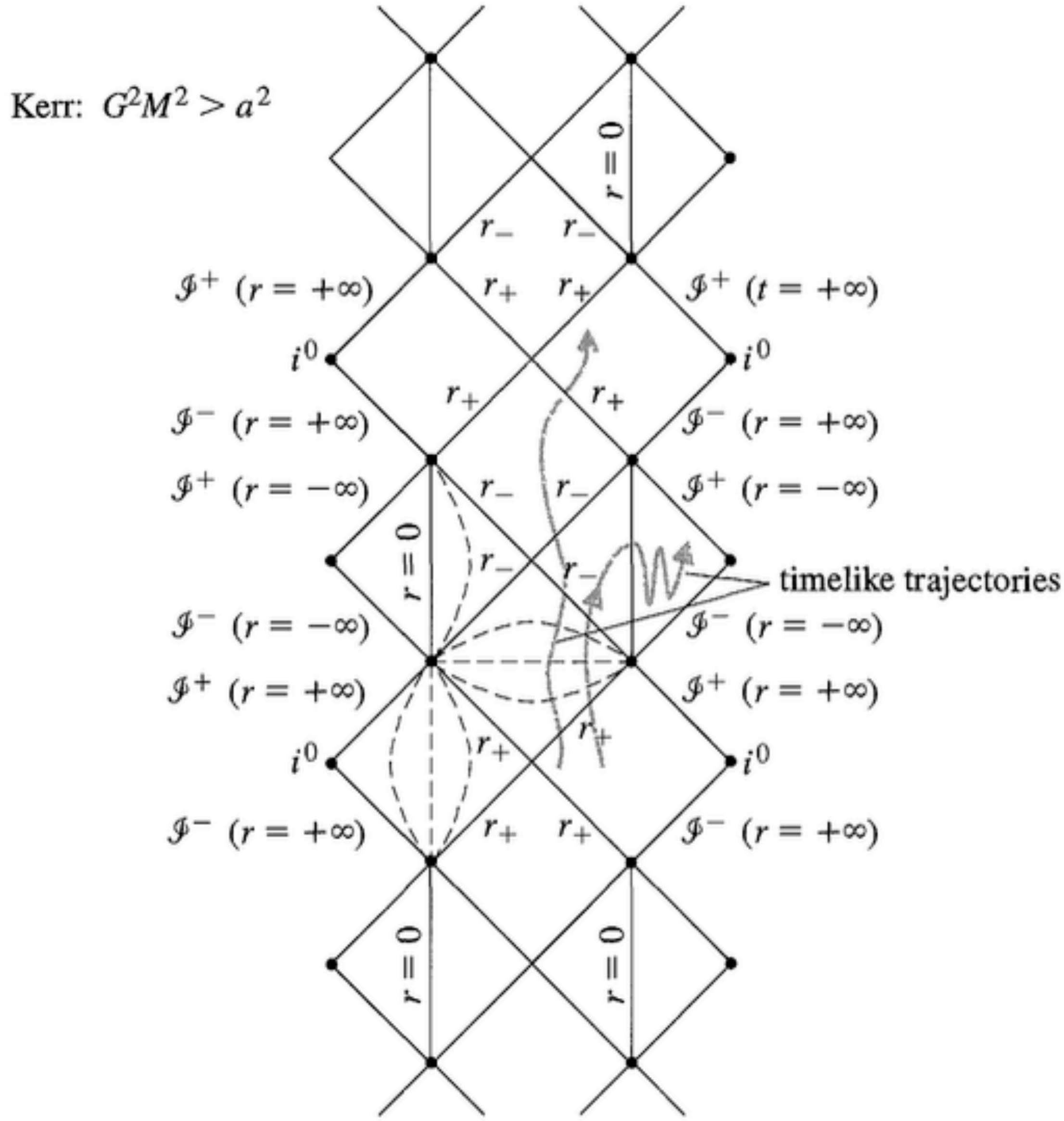

Figure 14: The Penrose diagram for sub-extremal Kerr. There are and infinite number of copies of the region outside the black hole. The singularity at $r = 0$ only appears for $\theta = \frac{\pi}{2}$ and is absent for other values of θ. The regions beyond the singularity are where we have CTCs.

vectors implies the existence of conserved quantities along geodesics. We have the two con-

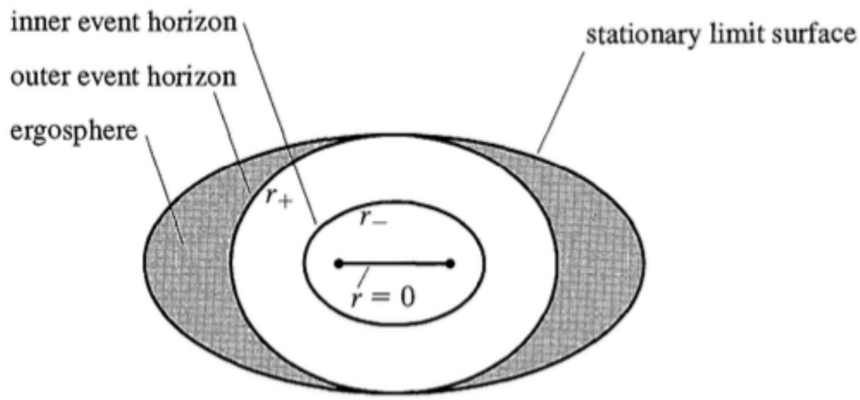

Figure 15: The horizon structure around the Kerr solution. The event horizons are null surfaces that separate points past which it is impossible to return to a certain region of space. The stationary limit surface, is timelike everywhere except where it is tangent to the event horizon at the poles. It represents the place past which it is impossible to be a stationary observer. The ergosphere between the stationary limit surface and the outer event horizon is a region in which it is possible to enter and leave again but not to remain stationary.

served quantities:

$$\begin{aligned} E = -K_\mu p^\mu &= m\left[\left(1 - \frac{2Mr}{\rho^2}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{2Mar}{\rho^2}\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right], \\ l = R_\mu p^\mu &= m\left[-\frac{2Mar}{\rho^2}\sin^2\theta\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{(r^2+a^2)^2 - \Delta(r)a^2\sin^2\theta}{\rho^2}\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]. \end{aligned} \tag{6.41}$$

These differ slightly with the definitions before where we had energy and angular momentum per unit mass, here we have multiplied by the mass of the particle. They are of course still conserved. The minus sign in E is because at infinity both K and p are timelike and so their inner product is negative and we want energy to be positive.

Let the particle approach a Kerr black hole along a geodesic. The energy of the particle according to a stationary observer at infinity is conserved along the geodesic. Inside the ergosphere, since K becomes spacelike we can imagine particles for which

$$E = -K_\mu p^\mu < 0\,. \tag{6.42}$$

This may bother you slightly that there is a particle with negative energy however, one can find that all particles have positive energy outside the ergosphere, those with negative energy must remain in the ergosphere or be accelerated until its energy is positive if it is to escape.

This allows for a way of extracting energy from a rotating black hole. Let us start away far from the black hole and throw something into the black hole along a geodesic. Let us denote the 4-momentum to be p_0, then its total energy that we measure is

$$E_0 = -p_0^\mu K_\mu \,, \tag{6.43}$$

which is conserved. Let the object enter the ergosphere. We arrange for the object to eject a mass, in a smart way, whilst in the ergosphere. Conservation of momentum gives

$$p_0 = p_1 + p_2 \,, \tag{6.44}$$

with p_1 the momentum of the object and p_2 the momentum of the ejected mass. Contracting with the Killing vector K we have the expected relation

$$E_0 = E_1 + E_2 \,. \tag{6.45}$$

If we arrange for $E_2 < 0$ by a clever choice of way of ejecting the mass, then we must have $E_1 > E_0$. Penrose showed that the ejected mass with negative energy must fall into the black hole, while the object can now escape with more energy than it initially began with. This is the *Penrose process* and is a method for extracting energy from a rotating black hole.

So can a rotating black hole be used as an infinite source of energy? There is no such thing as a free lunch (though cafe pi occasionally has free lunch samples), so the energy must comes from somewhere, and the only candidate is that it comes from the black hole. The Penrose process extracts energy from the black hole by decreasing the black holes angular momentum. When the mass is ejected we need to it to be ejected against the black hole's rotation. Recall that we saw that the event horizon was a Killing horizon for the Killing vector

$$\xi^\mu = K^\mu + \Omega_H R^\mu \,. \tag{6.46}$$

On the outer event horizon this indeed becomes null. The statement that the object with momentum p_2 crosses the event horizon by moving forward in time, is simply that

$$p_2^\mu \xi_\mu < 0 \,. \tag{6.47}$$

Plugging in the definitions of E and l, we see that this is equivalent to

$$l_2 < \frac{E_2}{\Omega_H} \,. \tag{6.48}$$

Since E_2 is negative and Ω_H positive it follows that $l_2 < 0$ and therefore the particle has negative angular momentum, therefore it is moving against the rotation of the black hole.

Once our object has escaped the ergosphere and the mass has fallen inside the event horizon the mass and the angular momentum of the black hole are changed. They are now the initial values plus the negative contributions from the in-falling mass:

$$\delta M = E_2 \,, \qquad \delta J = l_2 \,, \tag{6.49}$$

with $J = Ma$ the angular momentum of the black hole. The inequality 6.48 then translates into a limit on the amount the angular momentum can decrease

$$\delta J < \delta M \Omega_H^{-1} \,. \tag{6.50}$$

The ideal process would be when we have equality, in this case the mass thrown into the black hole becomes more and more null (since in this limit we have $p_2^\mu \xi_\mu \to 0$).

There is now a slight curiosity that appears, we can use the Penrose process to reduce the mass of the black hole, however there is a non-decreasing quantity: the area of the horizon. Let us compute the area of the event horizon at $r = r_+$. To do this we look at the induced metric on the horizon by setting $t =$const $r = r_+$. The induced metric is

$$\begin{aligned} \mathrm{d}s^2(\text{horizon}) = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j &= \mathrm{d}s^2(\mathrm{d}t = 0 \,, \mathrm{d}r = 0 \,, r = r_+) \\ &\frac{(r_+^2 + a^2)^2}{r_+^2 + a^2\cos^2\theta}\sin^2\theta\mathrm{d}\phi^2 + (r_+^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 \,, \end{aligned} \tag{6.51}$$

The area of the horizon is then simply

$$A = \int \mathrm{dvol}(\text{horizon}) \,. \tag{6.52}$$

For the metric at hand the determinant is

$$\begin{aligned} \det(\gamma) &= \frac{(r_+^2 + a^2)^2}{r_+^2 + a^2\cos^2\theta}\sin^2\theta \times (r_+^2 + a^2\cos^2\theta) = (r_+^2 + a^2)^2\sin^2\theta \,, \\ \mathrm{dvol}(\text{horizon}) &= (r_+^2 + a^2)\sin\theta\mathrm{d}\theta \wedge \mathrm{d}\phi \,. \end{aligned} \tag{6.53}$$

The area is then

$$A_{\text{horizon}} = (r_+^2 + a^2)\int \sin\theta\mathrm{d}\phi\mathrm{d}\theta = 4\pi(r_+^2 + a^2) \,. \tag{6.54}$$

To show that this does not decrease we work with the so called *irreducible mass* defined by

$$M^2_{\text{irreducible}} = \frac{A}{16\pi} \,. \tag{6.55}$$

Then we have

$$\begin{aligned} M^2_{\text{irreducible}} &= \frac{r_+^2 + a^2}{4} \\ &= \frac{1}{2}\Big(M^2 + \sqrt{M^4 - M^2a^2}\Big) \\ &= \frac{1}{2}\Big(M^2 + \sqrt{M^4 - J^2}\Big) \,. \end{aligned} \tag{6.56}$$

We can differentiate to obtain ho $M_{\text{irreducible}}$ is affected by changes in the mass or angular momentum:

$$\delta M_{\text{irreducible}} = \frac{a}{4M_{\text{irreducible}}\sqrt{M^2 - a^2}}(\Omega_H^{-1}\delta M - \delta J)\,. \tag{6.57}$$

We see that the inequality (6.50) becomes

$$\delta M_{\text{irreducible}} > 0\,. \tag{6.58}$$

The irreducible mass can never be reduced, hence the name. It follows that the maximum amount of energy that can be extracted from the black hole is

$$\max(E) = M - M_{\text{irreducible}} = M - \frac{1}{\sqrt{2}}\sqrt{M^2 + \sqrt{M^4 - J^2}}\,. \tag{6.59}$$

The result after a complete extraction of this amount of energy is a Schwarzschild solution with mass $M_{\text{irreducible}}$. The most efficient process is to start with an extremal Kerr black hole and then one can extract out approximately 29% of its total energy.

The irreducibility of $M_{\text{irreducible}}$ immediately shows that the surface area is non-decreasing. We have

$$\delta A = \frac{8\pi a}{\Omega_H\sqrt{M^2 - a^2}}\big(\delta M - \Omega_H \delta J\big)\,. \tag{6.60}$$

This may be recast as

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J\,, \tag{6.61}$$

where κ is

$$\kappa = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})}\,. \tag{6.62}$$

The quantity κ is the surface gravity of the Kerr solution. This is the force that an observer at infinity would have to exert in order to keep a unit mass at the horizon.

For every Killing horizon we can associate a quantity called the surface gravity. Given the Killing horizon we have an associated Killing vector, ξ which is null on the horizon. Since ξ is a normal vector to the Killing horizon it obeys the geodesic equation

$$\xi^\mu \nabla_\mu \xi^\nu = -\kappa \xi^\nu\,. \tag{6.63}$$

It turns out that κ is constant over the horizon (we will prove this later).

The above equation first started people thinking about a correspondence between the laws of thermodynamics and black holes. The first law of thermodynamics is

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V\,, \tag{6.64}$$

where T is the temperature, S the entropy, p the pressure and V the volume, thus $p\mathrm{d}V$ is the work done on the system. It is then natural to think of the term $\Omega_H \delta J$ as the work we do on the black hole by throwing our mass into the black hole. It is then natural to construct the dictionary

$$E \leftrightarrow M\,, \quad S \leftrightarrow \frac{A}{4G_N}\,, \quad T \leftrightarrow \frac{\kappa}{2\pi}\,. \tag{6.65}$$

This observation leads nicely on towards studying black hole thermodynamics. Before we get there we need to introduce some more formal definitions of what a black hole is and a little more technology.

7 Causality and singularities

Many physical questions can be rephrased as an initial value problem. Given the state of a system at some moment in time what will be the state of the system at some later time. The fact that this has a definitive answer is due to causality: future events can be understood as consequences of initial conditions plus the laws of physics. Initial value problems are as common in GR as in Newtonian physics or special relativity, however the dynamical nature of the spacetime background introduces new ways in which an initial value formulation could break down.

For the moment we will look at the problem of evolving matter fields on a fixed background spacetime rather than the evolution of the metric. The guiding principle is that no signals can travel faster than the speed of light; therefore information can only flow along timelike or null paths, not necessarily geodesics. We will define a *causal curve* to be any path which is timelike or null everywhere. Given any subset S of a manifold M, we can define the *causal future* of S denoted $J^+(S)$ to be the set of points that can be reached from S following a future directed causal curve. The *chronological future* $I^+(S)$ is the set of points that can be reached by following a future directed timelike curve. A point p will always be in its causal future $J^+(S)$ but not necessarily its own chronological future $I^+(p)$, though it could be. The causal past J^- and chronological past I^- are defined analogously.

A subset $S \subset M$ is called *achronal* if no two points in S are connected by a time-like curve. For example any edgeless spacelike hypersurface in Minkowski space is achronal. Given a closed (the complement is open) achronal set we define the *future domain of dependence* of S, $D^+(S)$ to be the set of all points p such that *every* past moving inextendible causal curve through p must intersect S. By inextendible we mean that the curve goes on forever and does not end at some finite point. Elements of S are elements of $D^+(S)$. A similar definition of the past domain of dependence, $D^{-}(S)$ holds by replacing future with past. We define the boundary of $D^{+}(S)$ to be the *future Cauchy horizon* $H^{+}(S)$ and likewise the boundary of $D^{-}(S)$ to be the *past Cauchy horizon* $H^{-}(S)$. They are both null surfaces. We have sketched this in figure 16.

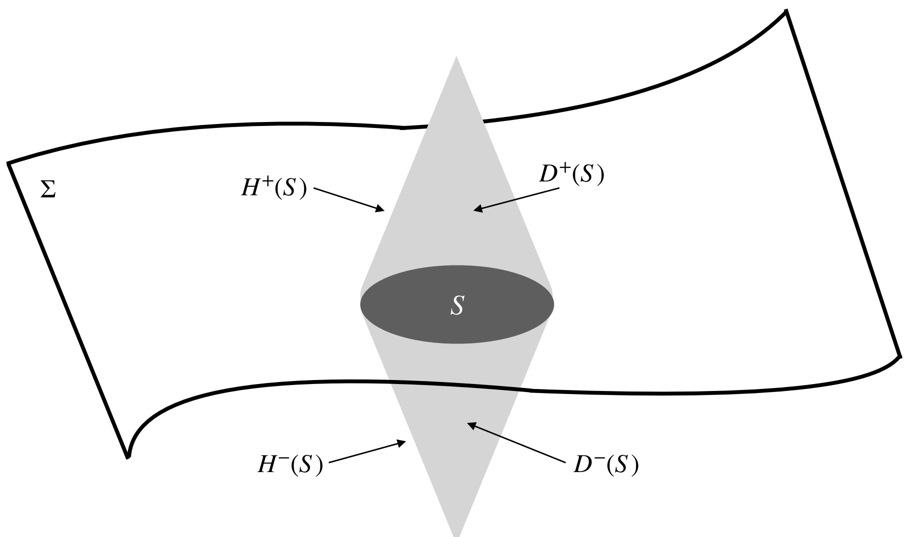

Figure 16: A depiction of the domains of dependence of the set S on the achronal surface Σ.

If nothing moves faster than light, signals cannot propagate outside the lightcone of any point p. Therefore if every curve that remains inside the lightcone must intersect S then information specified on S should be sufficient to predict what the situation is at p. That is, initial data for matter fields on S can be used to solve for the matter fields at p. The set of all points for which we can predict what happens by knowing what happens on S is the union $D(S) = D^{+}(S) \cup D^{-}(S)$ is called the *domain of dependence*. A closed achronal surface Σ is said to be a *Cauchy surface* if the domain of dependence $D(\Sigma)$ is the entire manifold. Information given on the Cauchy surface can be used to predict what happens throughout all of spacetime. If a spacetime has a Cauchy surface (it need not) it is said to be *globally hyperbolic*.

Therefore a globally hyperbolic spacetime is one in which one can predict what happens everywhere from data on Σ. Minkowski spacetime is an example of a globally hyperbolic spacetime as is the Kruskal spacetime. Examples of non globally hyperbolic spacetimes is 2d Minkowski space with the origin removed. In this case for any partial Cauchy surface Σ, there will be some inextendible causal curves which don't intersect Σ because they end at the origin.

8 Singularity theorem

We have seen that a spherically symmetric gravitational collapse results in the formation of a singularity. One can ask whether this is an artefact of the spherical symmetry or if it is something more generic?In Newtonian gravity the collapse of a spherically symmetric ball of matter produces a singularity with infinite density at the origin, however a tiny perturbation of the spherical symmetry does not lead to a singularity, rather a bouncing solution. One could ask whether this is the same for GR. Work by Roger Penrose answered this question and showed that singularities are a generic prediction of general relativity.[15]

8.1 Singularities

We have seen numerous different types of singularity so far. We have defined a metric singularity to arise in some basis if its components are not smooth or the determinant vanishes. A *coordinate singularity* can be eliminated by a change of coordinates, for example $r = 2M$ in the Schwarzschild spacetime in Schwarzschild coordinates. These singularities are unphysical and can be removed by a better choice of coordinates. If it is not possible to remove the singularity by a change of coordinates then we have a physical singularity. A *scalar curvature singularity* is a singularity where some scalar constructed from the Riemann tensor blows up.

Not all physical singularities are curvature singularities however, we have seen one in problem sheet 2. Consider the manifold $M = \mathbb{R}^2$ and introduce plane polar coordinates (r, ϕ) with $\phi \sim \phi + 2\pi$ and define the Riemannian metric

$$g = \mathrm{d}r^2 + \lambda^2 r^2 \mathrm{d}\phi^2 \,, \tag{8.1}$$

with $\lambda > 0$. The metric determinant vanishes at $r = 0$, however for $\lambda = 1$ this is just Euclidean space in polar coordinates so we can convert to Cartesian coordinates to see that $r = 0$ is just a coordinate singularity. However for $\lambda \neq 1$ and define $\phi' = \lambda\phi$. Then the metric is

$$g = \mathrm{d}r^2 + r^2 \mathrm{d}\phi'^2 \,, \tag{8.2}$$

[15]It is sometimes said that GR predicts its own downfall. This is because GR predicts singularities but is ill equipped to deal with them. To fully understand them we need a theory of quantum gravity, which GR is not.

which is locally isometric to Euclidean space and therefore curvature singularity free. However, it is not globally isometric to Euclidean space because the period of ϕ' is $2\pi\lambda$. Consider a circle $r = \epsilon$, this has

$$\frac{\text{circumference}}{\text{radius}} = \frac{2\phi\lambda\epsilon}{\epsilon} = 2\pi\lambda\,, \tag{8.3}$$

which does not tend to 2π as $\epsilon \to 0$. Recall that any smooth Riemannian manifold is locally flat, i.e. one recovers results of Euclidean geometry on sufficiently small length scales. The above shows that this is not true for the above metric for small circles around $r = 0$ and therefore the metric cannot be extended smoothly to $r = 0$. This is an example of a conical singularity.

A problem in defining singularities is that they are not places, they do not belong to the spacetime manifold because we define spacetime as the pair (M, g) where g is a smooth Lorentzian metric. This is the reason we remove $r = 0$ from the Kruskal spacetime, the metric is no longer smooth there. Similarly in the above example if we want a smooth manifold we should take $M = \mathbb{R}^2/(0,0)$ so that $r = 0$ is not part of the spacetime M.

In both examples the existence of the singularity implies that some geodesics cannot be extended to arbitrarily large affine parameter because they end at the singularity. We will use this property to define what we mean by a *singular* spacetime.

First eliminate the trivial case where a geodesic ends because we haven't taken the range of its parameter to be large enough. A curve is a smooth map $\gamma : (a, b) \to M$. Sometimes a curve can be *extended*, that is it is part of a bigger curve. If this happens then the first curve will have an endpoint, which is defined as follows.

Definition *future endpoint*

The point $p \in M$ is a *future end-point* of a future-directed causal curve $\gamma : (a, b) \to M$ if, for any neighbourhood O of p there exists t_0 such that $\gamma(t) \in O$ for all $t > t_0$. We say that γ is *future inextendible* if it has no future endpoint. Similarly for past endpoints and past inextendibility. The curve γ is *inextendible* if it is both future and past inextendible.

Example Let (M, g) be Minkowski spacetime and let $\gamma : (-\infty, 0) \to M$ be $\gamma(t) = (t, 0, 0, 0)$. Then the origin is a future end-point of γ. However if we instead let (M, g) be Minkowski spacetime with the origin removed then γ is future inextendible.

Definition *Complete*

A geodesic is *complete* if an affine parameter for the geodesic extends to $\pm\infty$. A *spacetime is geodesically complete* if all inextendible causal geodesics are complete.

Example Minkowski spacetime is goedesically complete as is the spacetime describing a spherically symmetric star. Kruskal spacetime on the other hand is goedesically *incomplete* because some geodesics have $r \to 0$ in finite affine parameter and hence cannot be extended to infinite affine parameter.

A spacetime which is extendible will also be geodesically incomplete. The incompleteness in this case is because we are not considering the full spacetime. We will therefore regard a spacetime as singular if it is geodesically incomplete and inextendible. This is true for the Kruskal spacetime, Kruskal extension of the RN and Kerr–Newman black holes.

8.2 Null hypersurfaces

To begin let us define what it means for a hypersurface to be null.

Definition: *Null hypersurface*
A *null hypersurface* is a hypersurface whose normal is everywhere null.

Example Consider surfaces of constant r in Schwarzschild spacetime. The one-form $n = \mathrm{d}r$ is normal to such surfaces. The norm is

$$n^2 = 1 - \frac{2M}{r}\,. \tag{8.4}$$

We see that the hypersurface $r = 2M$ is a null hypersurface.

Let n_μ be normal to a null hypersurface $\mathcal{N}$. Then any (non-zero) vector X^μ tangent to the hypersurface obeys $n_\mu X^\mu = 0$. Therefore, either X^μ is spacelike or X^μ is parallel to n^μ. In particular note that n^μ is tangent to the hypersurface, since it is null, hence on $\mathcal{N}$ the integral curves of n^μ lie within $\mathcal{N}$.

Proposition: *The integral curves of n are null geodesics. They are called the generators of $\mathcal{N}$.*

Proof: Let $\mathcal{N}$ be given by the equation $f =$constant for some function f with $\mathrm{d}f \neq 0$ on $\mathcal{N}$. Then we have $n = h\mathrm{d}f$ for h some function which does not vanish on $f =$constant. Let $N = \mathrm{d}f$, the integral curves of n and N are the same up to a choice of reparametrisation.[16] Since $\mathcal{N}$ is null we have that $N^\mu N_\mu = 0$ on $\mathcal{N}$ which implies that the gradient of this function

[16]To see this consider the integral curves defined by n^μ:

$$n^\mu = \frac{\mathrm{d}\tilde{x}^\mu(\tilde{\lambda})}{\mathrm{d}\tilde{\lambda}}\,.$$

We have that the integral curves for N are then

$$N^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = h^{-1}n^\mu = h^{-1}\frac{\mathrm{d}\tilde{x}^\mu}{\mathrm{d}\tilde{\lambda}} = \left[h^{-1}\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}\right]\frac{\mathrm{d}\tilde{x}^\mu}{\mathrm{d}\lambda}\,.$$

By choosing the parameter $\lambda(\tilde{\lambda})$ so that $\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}} = h$ we may make the bracket in the last term become unity and therefore we have shown that the integral curves are the same up to a choice of reparametrisation.

is normal to $\mathcal{N}$:

$$\nabla_\mu(N^\nu N_\nu)\Big|_{\mathcal{N}} = 2\alpha N_\mu\,, \tag{8.5}$$

with α some function on $\mathcal{N}$. Now since $\nabla_\mu N_\nu = \nabla_\mu\nabla_\nu f = \nabla_\nu\nabla_\mu f = \nabla_\nu N_\mu$ we have

$$N^\nu\nabla_\mu N_\nu = N^\nu\nabla_\nu N_\mu \quad\Rightarrow\quad N^\nu\nabla_\nu N_\mu\Big|_{\mathcal{N}} = \alpha N_\mu\,. \tag{8.6}$$

This is nothing but the geodesic equation for a non-affinely parametrised geodesic. Hence on $\mathcal{N}$ the integral curves of N, and therefore also n are null geodesics.

Consider Kruskal spacetime, with metric (1.69). Let $N = \mathrm{d}U$, this is null everywhere (since $g^{UU} = 0$) and is normal to a family of null hypersurfaces defined by U =constant. Since $N^2 = 0$ everywhere it follows that N is tangent to affinely parametrised null geodesics. Raising an index gives

$$N^\mu = -\frac{r}{16M^3}\mathrm{e}^{\frac{r}{2M}}\left(\frac{\partial}{\partial V}\right)^\mu. \tag{8.7}$$

Let $\mathcal{N}$ be the surface $U = 0$. Since $U = 0$ corresponds to $r = 2M$ on $\mathcal{N}$ we have that N is simply a constant multiple of $\frac{\partial}{\partial V}$. Thus V is an affine parameter for the generators of $\mathcal{N}$. Similarly U is an affine parameter of for the generators of the null hypersurface $V = 0$.

Black holes are characterised by the fact that you can enter them but never exit. The most important feature is therefore not the singularity but rather than event horizon. An event horizon is a hypersurface separating those spacetime points that are connected to infinity by a timelike path from those that are not.

8.3 Geodesic Deviation

Definition *one-parameter family of geodesics*

A one-parameter family of geodesics is a map $\gamma : I \times I' \to M$ where both I and I' are open intervals of $\mathbb{R}$ such that: for fixed σ, $\gamma(\sigma,\lambda)$ is a geodesic with affine parameter λ; the map $(\sigma,\lambda) \mapsto \gamma(\sigma,\lambda)$ is smooth, one to one and with a smooth inverse. The latter condition implies that the family of geodesics sweeps out a 2d surface $\Sigma \subset M$.

Let X be the tangent vector to the geodesics and S the vector tangent to the curves of constant λ, which are parametrised by σ. In a coordinate chart x^μ the geodesics are specified by $x^\mu(\sigma,\lambda)$ with

$$S^\mu = \frac{\partial x^\mu}{\partial\sigma}\,,\quad X^\mu = \frac{\partial x^\mu}{\partial\lambda}\,. \tag{8.8}$$

Hence

$$x^\mu(\sigma+\delta\sigma,\lambda) = x^\mu(\sigma,\lambda) + S^\mu(s,\lambda)\delta s + \mathcal{O}(\delta s^2)\,. \tag{8.9}$$

We see that S^μ points from one geodesic to an infinitesimally nearby one in the family. The vector S is called the *deviation vector* or *separation vector*.

On the surface Σ we can use σ, λ coordinates and therefore we have

$$S = \frac{\partial}{\partial \sigma}\,, \qquad X = \frac{\partial}{\partial \lambda}\,. \tag{8.10}$$

Hence S and X commute:

$$[X, S] = 0\,, \quad \Leftrightarrow \quad X^\mu \nabla_\mu S^\nu = S^\mu \nabla_\mu X^\nu\,. \tag{8.11}$$

From here we find the geodesic deviation equation:

$$X^\rho \nabla_\rho \Big(X^\mu \nabla_\mu S^\nu \Big) = R^\nu{}_{\mu\rho\tau} X^\mu X^\rho S^\tau\,. \tag{8.12}$$

8.4 Geodesic congruences

A more comprehensive picture of the behaviour of neighbouring geodesics comes from considering not just a one-parameter family but an entire *congruence* of geodesics. Let U be an open region of M. A congruence of U is a set of geodesics such that every point in U lies on precisely one curve. A geodesic congruence can be thought of as a tracing out the paths of a set of non-interacting particles moving through spacetime with non-intersecting paths. If the geodesics cross then the congruence comes to an end at that point. Consider a congruence for which all the geodesics are of the same type, (timelike, spacelike, null). We can then arrange such that the tangent vector, X^μ is normalised to $X^\mu X_\mu = \pm 1, 0$ depending on the type.

Let $X^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ and consider a 1-parameter family of geodesics belonging to a congruence. Then $[S, X] = 0$ can be written as

$$X^\mu \nabla_\mu S^\nu = B^\nu{}_\mu S^\mu\,, \qquad B^\nu{}_\mu = \nabla_\mu X^\nu\,, \tag{8.13}$$

and measures the failure for S to be parallely transported along the geodesic with tangent X. It therefore measures the extent to which neighbouring geodesics deviate from remaining parallel. Note that due to X being a geodesic and normalised to have fixed constant norm, we have

$$X_\nu B^\nu{}_\mu = B^\nu{}_\mu X^\mu = 0\,. \tag{8.14}$$

This implies that

$$X^\mu \nabla_\mu (X^\nu S_\nu) = X^\mu B_{\mu\nu} S^\nu = 0\,. \tag{8.15}$$

Thus $S_\mu X^\mu$ is constant along the congruence.

Even after normalising the norm $X^\mu X_\mu$ by an appropriate choice of the affine parameter we still have the freedom to shift by a constant. We can define the constant to be different on different geodesics, allowing it to depend on σ: $\lambda' = \lambda - a(\sigma)$ for some function $a(\sigma)$. This changes the deviation vector to

$$S'^\mu = S^\mu + \frac{\mathrm{d}a(\sigma)}{\mathrm{d}\sigma} X^\mu \,. \tag{8.16}$$

This is still a deviation vector pointing to the same geodesic as S^μ. Now we have

$$X^\mu S'_\mu = X^\mu S_\mu + \frac{\mathrm{d}a(\sigma)}{\mathrm{d}\sigma} X^\mu X_\mu \,, \tag{8.17}$$

and therefore in the time-like and space-like case we may fix $X^\mu S_\mu = 0$, which is a sort of "gauge" freedom. Since $X^\mu S_\mu$ is constant we have $X^\mu S_\mu = 0$ everywhere. We can define the projector

$$P^\mu{}_\nu = \delta^\mu_\nu - |X|^{-2} X^\mu X_\nu \,, \tag{8.18}$$

which projects onto the vector space of the tangent space of a point p vectors normal to X.

Null case This of course does not work for null geodesics since $X^\mu X_\mu = 0$ and therefore $X^\mu S'_\mu = X^\mu S_\mu$. We can instead fix the gauge freedom by picking a spacelike hypersurface Σ which intersects each geodesic once. Let N^μ be a vector field defined on Σ obeying $N^2 = 0$ and $N^\mu X_\mu = -1$ on Σ. Now we can extend N off of Σ by parallel transport along the geodesics: $X^\mu \nabla_\mu N^\nu = 0$. This implies that $N^2 = 0$ and $N^\mu X_\mu = -1$ everywhere. We have therefore constructed a vector field such that

$$N^2 = 0 \,, \quad X^\mu N_\mu = -1 \,, \quad X^\mu \nabla_\mu N^\nu = 0 \,. \tag{8.19}$$

We can decompose any deviation vector uniquely as

$$S^\mu = \alpha X^\mu + \beta N^\mu + \hat{S}^\mu \,, \tag{8.20}$$

where

$$U^\mu \hat{S}_\mu = N^\mu \hat{S}_\mu = 0 \,, \tag{8.21}$$

which implies that $\hat{S}$ is either spacelike or zero. Now $U^\mu S_\mu = -\beta$ and therefore β is constant along each geodesic. Therefore we can write the the deviation vector S as the sum of a part $\alpha X^\mu + \hat{S}^\mu$ orthogonal to X and a part βN that is parallely transported along each geodesic.

An important case is when the congruence contains the generators of a null hypersurface $\mathcal{N}$ and we are interested only in the behaviour of these generators. In this case we pick a one-parameter family of geodesics contained within $\mathcal{N}$ then the deviation vector S will be tangent to $\mathcal{N}$ and hence obey $X^\mu S_\mu = 0$ since X is normal to $\mathcal{N}$, in the decomposition above this is equivalent to $\beta = 0$.

We can write

$$\hat{S}^\mu = \tilde{P}^\mu{}_\nu S^\nu \,, \tag{8.22}$$

where $\tilde{P}$ is the projector

$$\tilde{P}^\mu_{\ \nu} = \delta^\mu_\nu + N^\mu X_\nu + X^\mu N_\nu \,, \tag{8.23}$$

acting on the tangent space at p onto $T_\perp$ the 2d space of vectors at p orthogonal to X and N. Since both X and N are parallely transported so is P,

$$X^\mu \nabla_\mu \tilde{P}^\nu{}_\sigma = 0 \,. \tag{8.24}$$

8.5 Expansion, rotation and shear

$B_{\mu\nu}$ is a $(0,2)$ tensor and so we may decompose it into its: anti-symmetric part, symmetric traceless part, and trace part.

Let us restrict to the null case. Then we may act on B with the projector $\tilde{P}$ as

$$\hat{B}^\mu{}_\nu = \tilde{P}^\mu{}_\rho B^\rho{}_\sigma \tilde{P}^\sigma{}_\nu \,, \tag{8.25}$$

which is restricted to the 2d space $T_\perp$. We define

Definition: *Expansion, shear and rotation*

The *expansion, shear* and *rotation* of the null geodesic congruence are

$$\theta = \hat{B}^\mu{}_\mu \,, \quad \sigma_{\mu\nu} = \hat{B}_{(\mu\nu)} - \frac{1}{2}\theta \tilde{P}_{\mu\nu} \,, \quad \omega_{\mu\nu} = \hat{B}_{[\mu\nu]} \,. \tag{8.26}$$

We have

$$\hat{B}^\mu{}_\nu = \frac{1}{2}\theta \tilde{P}^\mu{}_\nu + \sigma^\mu{}_\nu + \omega^\mu{}_\nu \,, \tag{8.27}$$

and

$$\theta = g^{\mu\nu} B_{\mu\nu} = \nabla_\mu X^\mu \,, \tag{8.28}$$

the latter shows that it is independent of the vector N and therefore is an intrinsic property of the congruence. Moreover the scalar invariants of the rotation and shear, for example $\omega_{\mu\nu}\omega^{\mu\nu}$ or the eigenvalues of σ are also independent of the choice of N.

Proposition If the congruence contains the generators of a (null) hypersurface $\mathcal{N}$ then $\omega_{\mu\nu} = 0$ on $\mathcal{N}$. Conversely if $\omega_{\mu\nu} = 0$ everywhere then X is everywhere hypersurface orthogonal, that is orthogonal to a family of null hypersurfaces.

proof The definition of $\hat{B}$ and $X^\mu \hat{B}_{\mu\nu} = 0 = \hat{B}_{\mu\nu} X^\nu$ implies

$$\hat{B}^\mu{}_\nu = B^\mu{}_\nu + X^\mu N_\rho B^\rho{}_\nu + X_\nu B^\mu{}_\rho N^\rho + X^\mu X_\nu N_\rho B^\rho{}_\tau N^\tau \, . \tag{8.29}$$

From here we have

$$X_{[\mu}\omega_{\nu\rho]} = X_{[\mu}\hat{B}_{\nu\rho]} = X_{[\mu}B_{\nu\rho]} \, , \tag{8.30}$$

since the other terms drop in the anti-symmetrisation. Using the definition of B we have

$$X_{[\mu}\omega_{\nu\rho]} = X_{[\mu}\nabla_\rho X_{\nu]} = -\frac{1}{6}(X \wedge \mathrm{d}X)_{\mu\nu\rho} \, . \tag{8.31}$$

If X is normal to $\mathcal{N}$ then $X \wedge \mathrm{d}X = 0$ on $\mathcal{N}$. Hence on $\mathcal{N}$

$$0 = X_{[\mu}\omega_{\nu\rho]} = \frac{1}{3}\big(X_\mu \omega_{\nu\rho} + X_\nu \omega_{\rho\mu} + X_\rho \omega_{\mu\nu}\big) \, . \tag{8.32}$$

Using that $X^\mu N_\mu = -1$ and $\omega_{\mu\nu} N^\nu = 0$ we have $\omega_{\mu\nu} = 0$ on $\mathcal{N}$. Conversely if $\omega = 0$ everywhere then the above shows that X is hypersurface orthogonal using Frobenius' theorem.

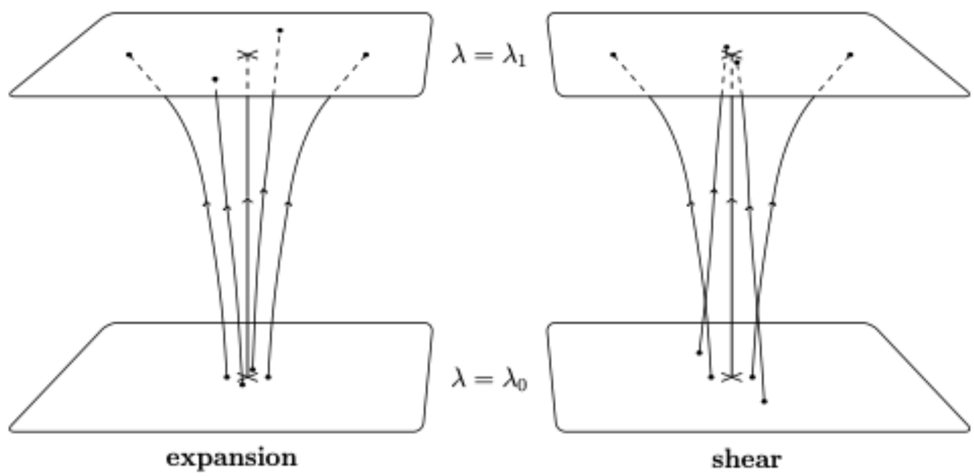

Figure 17: A depiction of the shear and expansion for a null hypersurface.

8.6 Expansion and shear of a null hypersurface

Assume that we have a congruence which includes the generators of a null hypersurface $\mathcal{N}$. The generators of a null hypersurface have $\omega = 0$. To understand how these generators behave let us restrict to deviation vectors tangent to $\mathcal{N}$, i.e. a one-parameter family of generators of $\mathcal{N}$. Consider the evolution of the generators of $\mathcal{N}$ as a function of affine parameter λ as shown in figure 17.

Qualitatively θ corresponds to neighbouring generators moving apart for $\theta > 0$, together for $\theta < 0$. Shear on the other hand corresponds to geodesics moving apart in one direction and together in the orthogonal direction whilst preserving the cross-sectional area.

To make this more precise let us introduce *Gaussian null coordinates* near $\mathcal{N}$ as follows. Pick a spacelike 2-surface S within $\mathcal{N}$ and let y^i $(i = 1, 2)$ be coordinates on this surface. Assign coordinates (λ, y^i) to the point affine parameter distance λ from S along the generator of $\mathcal{N}$ with tangent X^μ which intersects the surface S at the point with coordinates y^i. Now we have coordinates (λ, y^i) on $\mathcal{N}$ such that the generators are lines of constant y^i and $X = \partial_\lambda$.

Let V be a null vector field on $\mathcal{N}$ satisfying $V \cdot \partial_{y^i} = 0$ and $V \cdot X = 1$. Assign coordinates (r, λ, y^i) to the point affine parameter distance r along the null geodesic which starts at the point on $\mathcal{N}$ with coordinates (λ, y^i) and has tangent vector V there.

This defines a coordinate chart in a neighbourhood of $\mathcal{N}$ such that $\mathcal{N}$ is at $r = 0$ with $X = \partial_\lambda$ on $\mathcal{N}$ and ∂_r is tangent to affinely parametrised null geodesics. The latter implies that $g_{rr} = 0$ everywhere.

At $r = 0$ we have $g_{r\lambda} = X \cdot V = 1$ since $V = \partial_r$ on $\mathcal{N}$, and $g_{ri} = V \cdot \partial_{y^i} = 0$. Since $g_{r\mu}$ is independent of r these results are valid for all r. We also know that $g_{\lambda\lambda} = 0$ at $r = 0$, since X is null, and $g_{\lambda i} = 0$ at $r = 0$ (since ∂_{y^i} is tangent to $\mathcal{N}$ and hence orthogonal to X). Therefore we can write $g_{\lambda\lambda} = rF$ and $g_{\lambda i} = rh_i$ for some smooth functions F and h_i. Therefore the metric takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}r\mathrm{d}\lambda + rF\mathrm{d}\lambda^2 + 2rh_i\mathrm{d}y^i\mathrm{d}\lambda + h_{ij}\mathrm{d}y^i\mathrm{d}y^j \,. \tag{8.33}$$

On $\mathcal{N}$ the metric is

$$g\Big|_{\mathcal{N}} = 2\mathrm{d}r\mathrm{d}\lambda + h_{ij}\mathrm{d}y^i\mathrm{d}y^j \,, \tag{8.34}$$

so $X^\mu = (0, 1, 0, 0)$ on $\mathcal{N}$ implies that $U_\mu = (1, 0, 0, 0)$ on $\mathcal{N}$. Now $X \cdot B = B \cdot X = 0$ implies that $B^r{}_\mu = B^\mu{}_\lambda = 0$. We saw above that $\theta = B^\mu{}_\mu$. Hence on $\mathcal{N}$ we have

$$\theta = B^i{}_i = \nabla_i X^i = \partial_i X^i + \Gamma^i{}_{i\mu} X^\mu = \frac{1}{2} g^{i\mu} \big(\partial_\lambda g_{\mu i} + \partial_i g_{\mu\lambda} - \partial_\mu g_{i\lambda} \big) \,. \tag{8.35}$$

In the final expression note that $g^{i\mu}$ is non-zero only when $\mu = j$ and that $g^{ij} = h^{ij}$ where h^{ij} is the inverse of h_{ij}. Therefore on $\mathcal{N}$

$$\theta = \frac{1}{2} h^{ij} (\partial_\lambda g_{ij} + \partial_i g_{j\lambda} - \partial_j g_{i\lambda}) = \frac{1}{2} h^{ij} \partial_\lambda h_{ij} = \frac{1}{\sqrt{h}} \partial_\lambda \sqrt{h} \,, \tag{8.36}$$

where we used $g_{i\lambda} = 0$ on $\mathcal{N}$ and defined $h = \det h_{ij}$. Therefore we have

$$\frac{\partial}{\partial \lambda} \sqrt{h} = \theta \sqrt{h} \,. \tag{8.37}$$

From the form of the metric we see that $\sqrt{h}$ is nothing but the area element on a surface of constant λ within $\mathcal{N}$ so θ measures the rate of increase of this area element with respect to affine parameter along the geodesics.

8.7 Trapped surfaces

Consider a 2d spacelike surface S i.e. a 2d submanifold for which all tangent vectors are spacelike. For any $p \in S$ there will be precisely two future directed null vectors X_1 and X_2 orthogonal to S, up to rescalings. If we assume S is orientable then X_1^μ and X_2^μ can be defined continuously over S. This defines two families of null geodesics which start on S and are orthogonal to S. These null geodesics form two null hypersurfaces $\mathcal{N}_1$ and $\mathcal{N}_2$. In simple situations these correspond to the set of out-going and in-going light rays that start on S. Consider a null congruence that contains the generators of $\mathcal{N}_i$. By the proposition above we will have $\omega_{\mu\nu} = 0$ on $\mathcal{N}_1$ and $\mathcal{N}_2$.

Example Let S be a two-sphere $U = U_0$ and $V = V_0$ in the Kruskal spacetime. By symmetry the generators of $\mathcal{N}_i$ will be radial null geodesics, see figure 19

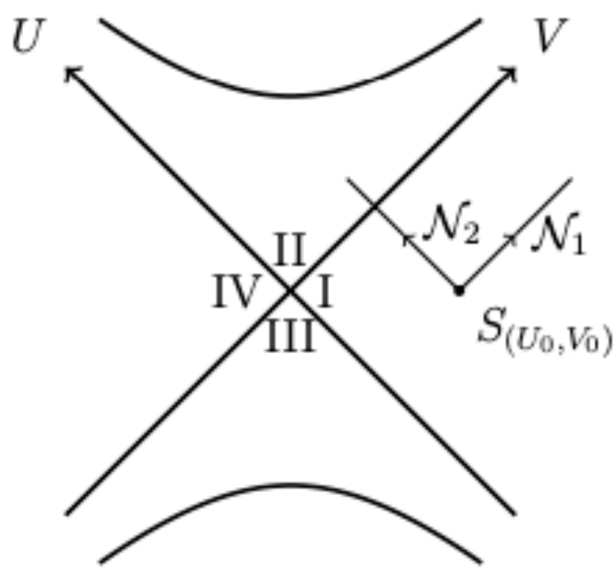

Figure 18: Null hypersurfaces orthogonal to a sphere S ($U = U_0$ $V = V_0$) in the Kruskal spacetime.

Hence $\mathcal{N}_i$ must be surfaces of constant U or constant V with generators tangent to dU

and $\mathrm{d}V$ respectively. We saw above that $\mathrm{d}U$ and $\mathrm{d}V$ correspond to affine parametrisation. Raising an index we find

$$X_1 = r\mathrm{e}^{r/2M}\frac{\partial}{\partial V}\,, \quad X_2 = r\mathrm{e}^{r/2M}\frac{\partial}{\partial U}\,, \tag{8.38}$$

where we have discarded an overall constant and fixed the signs so that X_1 and X_2 are future-directed. The vectors $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial V}$ are future-directed because they are globally null and hence define time-orientations. In region I they both give the same time orientation as the one defined by K.

We can compute the expansion of these congruences:

$$\theta_1 = \nabla_\mu X_1^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}X_1^\mu\right) = r^{-1}\mathrm{e}^{r/2M}\partial_V\left(r\mathrm{e}^{-r/2M}r\mathrm{e}^{r/2M}\right) = 2\mathrm{e}^{r/2M}\partial_V r\,. \tag{8.39}$$

The right-hand side can be calculated by using the implicit definition of U, V, this gives

$$\theta_1 = -\frac{8M^2}{r}U\,, \tag{8.40}$$

and a similar calculation gives

$$\theta_2 = -\frac{8M^2}{r}V\,. \tag{8.41}$$

We can now set $U = U_0$ and $V = V_0$ to study the expansion on S of the null geodesics normal to S. For S in region I we have $\theta_1 > 0$ and $\theta_2 < 0$ i.e. the outgoing null geodesics normal to S are expanding and the ingoing geodesics are converging. Similarly in region IV we have $\theta_2 > 0$ and $\theta_1 < 0$ so again we have an expanding family and a converging family. However in region II we have $\theta_1 < 0$ and $\theta_2 < 0$: both families of geodesics normal to S are converging. In region III we have $\theta_1 > 0$ and $\theta_2 > 0$ so both families are expanding.

Definition: *Trapped*

A compact orientable spacelike 2-surface is *trapped* if both families of null geodesics orthogonal to S have negative expansion everywhere on S. It is *marginally trapped* if both families have non-positive expansion everywhere on S.

In Kruskal spacetime all 2-spheres ($U = U_0$, $V = V_0$) in region II are trapped and 2-spheres on the event horizon ($U_0 = 0$, $V_0 > 0$) are marginally trapped.

8.8 Raychaudhuri's equation

We now want to understand how the expansion evolves along the geodesic of a null geodesic congruence.

Proposition: *Raychaudhuri's equation*

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}X^\mu X^\nu\,. \tag{8.42}$$

Proof: From the definition of θ we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = X^\mu \nabla_\mu \theta = X^\mu \nabla_\mu B^\rho{}_\sigma \tilde{P}^\sigma{}_\rho = \tilde{P}^\sigma{}_\rho X^\mu \nabla_\mu B^\rho{}_\sigma = \tilde{P}^\sigma{}_\rho X^\mu \nabla_\mu \nabla_\sigma X^\rho \,. \tag{8.43}$$

Now commute derivatives using the definition of the Riemann tensor:

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} &= \tilde{P}^\sigma{}_\rho X^\mu \Big(\nabla_\sigma \nabla_\mu X^\rho + R^\rho{}_{\tau\mu\sigma} X^\tau \Big) \\ &= \tilde{P}^\sigma{}_\rho \Big(\nabla_\sigma \big(X^\mu \nabla_\mu X^\rho \big) - \big(\nabla_\sigma X^\mu \big) \big(\nabla_\mu X^\rho \big) \Big) + \tilde{P}^\sigma{}_\rho R^\rho{}_{\mu\nu\sigma} X^\mu X^\nu \\ &= -B^\rho{}_\nu P^\nu{}_\mu B^\mu{}_\rho - R_{\mu\nu} X^\mu X^\nu \,,\end{aligned} \tag{8.44}$$

where we used the geodesic equation and in the final term the anti-symmetry of the Riemann tensor allows us to replace $\tilde{P}^\sigma{}_\rho$ with δ^σ_ρ. Finally we can rewrite the first term so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\hat{B}^\mu{}_\nu \hat{B}^\nu{}_\mu - R_{\mu\nu} X^\mu X^\nu \,. \tag{8.45}$$

The result then follows by using the expansion of $\hat{B}^\mu{}_\nu$ in equation (8.27).

Similar calculations give equations governing the evolution of shear and rotation

8.9 Energy conditions

Raychaudhuri's equation involves the Ricci tensor and so is purely geometric. Through the Einstein equation this is related to the energy-momentum tensor of matter. We want to consider only physical matter which implies that the energy-momentum tensor should satisfy some conditions. For example, an observer with 4-velocity u^μ would measure and energy momentum current $j^\mu = -T^\mu{}_\nu u^\nu$. We would expect that physically reasonable matter should not move faster than light, this current should be non-spacelike. This motivates:

Dominant energy condition For all future-directed timelike vectors V^μ the vector $-T^\mu{}_\nu V^\nu$ is a future-directed causal vector or zero.

For matter satisfying the dominant energy condition, if $T_{\mu\nu}$ is zero in some closed region S of a spacelike hypersurface Σ then $T_{\mu\nu}$ will be zero within $D^+(S)$.

Example

Consider a massless scalar field

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} \partial_\rho\phi\partial^\rho\phi \,. \tag{8.46}$$

Let

$$j^\mu = -T^\mu{}_\nu V^\nu = -V^\nu \partial_\nu\phi\partial^\mu\phi + \frac{1}{2} V^\mu \partial_\rho\phi\partial^\rho\phi \,, \tag{8.47}$$

then for timelike V^μ

$$j^2 = \frac{1}{4} V^2 \Big(\partial_\rho\phi\partial^\rho\phi \Big)^2 \leq 0 \tag{8.48}$$

so j is indeed causal or zero. Now consider

$$\begin{aligned} V^\mu j_\mu &= -(V^\mu \partial_\mu \phi)^2 + \frac{1}{2} V^2 (\partial_\rho \phi \partial^\rho \phi) \\ &= -\frac{1}{2}(V^\mu \partial_\mu \phi)^2 + \frac{1}{2} V^2 + \frac{1}{2} V^2 \left(\partial_\rho \phi - \frac{V \cdot \partial \phi}{V^2} V_\rho \right) \left(\partial^\rho \phi - \frac{V \cdot \partial \phi}{V^2} V^\rho \right) \end{aligned} \tag{8.49}$$

the final expression in brackets is orthogonal to V and hence must be spacelike or zero so its norm is non-negative. We then have $V \cdot j \leq 0$ using $V^2 < 0$. Hence j^μ is future directed or zero.

A less restrictive condition requires only that the energy density measured by all observers is positive:

Weak energy condition For any causal vector V we have

$$T_{\mu\nu} V^\mu V^\nu \geq 0 \,. \tag{8.50}$$

Null energy condition For any null vector V we have

$$T_{\mu\nu} V^\mu V^\nu \geq 0 \,. \tag{8.51}$$

The dominant energy condition implies the weak energy condition, which implies the null energy condition. Another energy condition is:

Strong energy condition For all causal vector V we have

$$\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho{}_\rho \right) V^\mu V^\nu \geq 0 \,. \tag{8.52}$$

Using the Einstein equation this is equivalent to

$$R_{\mu\nu} V^\mu V^\nu \geq 0 \,, \tag{8.53}$$

or *gravity is attractive*. Despite its name the strong energy condition does not imply any of the other conditions. The strong energy condition is needed to prove some of the singularity theorems, but the dominant energy condition is the most important physically. For example our universe appears to contain a positive cosmological constant. This violates the strong energy condition but respects the dominant energy condition.

8.10 Conjugate points

In a spacetime satisfying Einstein's equation with matter obeying the null energy condition, the generators of a null hypersurface satisfy

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \leq -\frac{1}{2} \theta^2 \,. \tag{8.54}$$

Consider the RHS of Raychaudhuri's equation. The generators of a null hypersurface have $\omega = 0$. Since vectors in $T_\perp$ are all spacelike, so the metric restricted to $T_\perp$ is positive definite. Hence $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$. Einstein's equations gives $R_{\mu\nu}X^\mu X^\nu = 8\pi T_{\mu\nu}X^\mu X^\nu$ because X is null. Hence the null energy condition implies $R_{\mu\nu}X^\mu X^\nu \geq 0$ and the result follows from Raychaudhuri's equation.

Corollary If $\theta = \theta_0 < 0$ at a point p on a generator γ of a null hypersurfaces then $\theta \to -\infty$ along γ within an affine parameter distance $2|\theta_0|^{-1}$ provided γ extends this far.

Proof: Let $\lambda = 0$ at p. Then equation (8.54) implies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\theta^{-1} \geq \frac{1}{2}\,. \tag{8.55}$$

Integrating gives $\theta^{-1} - \theta_0^{-1} \geq \frac{\lambda}{2}$ which can be rearranged to give

$$\theta \leq \frac{2\theta_0}{2 + \lambda\theta_0}\,. \tag{8.56}$$

If $\theta_0 < 0$ then the RHS goes to ∞ as $\lambda \to \frac{2}{|\theta_0|}$.

Definition: *Conjugate*

Points p, q on a geodesic γ are *conjugate* if there exists a solution of the geodesic deviation equation along γ that vanishes at p and q but it not identically zero.

Since gravity is an attractive force it focuses geodesics and if the generated curvature is strong enough conjugate points always develop provided we can extend the geodesics arbitrarily far in the past and in the future.

8.11 Definition of a black hole and the event horizon

We can now define a black hole and its event horizon. Consider a manifold with metric (M, g) and its conformal compactification $(\bar{M}, \bar{g})$. Recall that the causal past J^- of a region is the set of all points we can reach from that region by moving along a past-directed timelike paths. We can define the causal past of scri-plus $J^-(\mathscr{I}^+) \subset \bar{M}$. The set of points of M that can send a signal to $\mathscr{I}^+$ is $M \cap J^-(\mathscr{I}^+)$. We define the black hole region to be the complement of this region, and the future event horizon to be the boundary of the black hole region:

Definition: *Black hole region, future event horizon*

Let (M, g) be a spacetime that is asymptotically flat at null infinity. The *Black hole region* is

$$\mathcal{B} = M \backslash [M \cap J^-(\mathscr{I}^+)]\,, \tag{8.57}$$

where $J^-(\mathscr{I}^+)$ is defined using the unphysical spacetime $(\bar{M}, \bar{g})$. The *future event horizon* is $\mathcal{H}^+ = \partial\mathcal{B}$.

Similarly the white hole region is

$$\mathcal{W} = M \backslash [M \cap J^+(\mathscr{I}^-)] \,, \tag{8.58}$$

and the *past event horizon* is $\mathcal{H}^- = \partial \mathcal{W}$.

Definition *Killing horizon*

A null hypersurface $\mathcal{N}$ is a *Killing horizon* if there exists a Killing vector field ξ defined in a neighbourhood of $\mathcal{N}$ such that ξ is normal to $\mathcal{N}$.

Theorem: *Hawking 1972*

In a stationary, analytic, asymptotically flat vacuum black hole spacetime $\mathcal{H}^+$ is a Killing horizon.

A *naked singularity* is a singularity from which signals can reach $\mathscr{I}^+$, i.e. one that is not hidden behind an event horizon.

Definition: *Strong cosmic conjecture*

Naked singularities cannot form in gravitational collapse from generic initially non-singular states in an asymptotically flat spacetime obeying the dominant energy conditions.

8.12 Penrose Singularity Theorem

Theorem *Penrose 1965*

Let (M, g) be globally hyperbolic with a non-compact Cauchy surface Σ. Assume that the Einstein equation and the null energy condition are satisfied and that M contains a trapped surface T. Let $\theta_0 < 0$ be the maximum value of θ on T for both sets of null geodesics orthogonal to T. Then at least one of these geodesics is future inextendible and has an affine lengths no greater than $2/|\theta_0|$.

This implies that if there is a trapped surface then the maximal development is not geodesically complete. Such incompletenesss might arise because the maximal development is extendible, but the strong cosmic censorship conjecture would exclude this. Hence it is expected that generically the singularity arises because the maximal development is singular. In fact a different singularity theorem, due by Hawking and Penrose, eliminates the assumption that spacetime is globally hyperbolic and still proves the existence of incomplete geodesics. So even if the maximal development is extendible then the Hawking–Penrose theorem implies that this extended spacetime must be geodesically incomplete, i.e. singular.

Hence there are very good reasons to believe that gravitational collapse leads to formation of a singularity. Note that these theorems tell us nothing about the nature of this singularity, we do not know that it must be a curvature singularity as occurs in spherically symmetric collapse.

9 Laws of black hole thermodynamics

In 1973 Bardeen, Carter and Hawking (BCH) wrote a paper, [3], in which they considered

The Four Laws of Black Hole Mechanics

J. M. Bardeen*

Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking

Institute of Astronomy, University of Cambridge, England

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Abstract. Expressions are derived for the mass of a stationary axisymmetric solution of the Einstein equations containing a black hole surrounded by matter and for the difference in mass between two neighboring such solutions. Two of the quantities which appear in these expressions, namely the area A of the event horizon and the "surface gravity" κ of the black hole, have a close analogy with entropy and temperature respectively. This analogy suggests the formulation of four laws of black hole mechanics which correspond to and in some ways transcend the four laws of thermodynamics.

Figure 19: The BCH paper.

stationary axisymmetric black holes. They found that black holes obeyed laws reminiscent of the laws of thermodynamics. At the time they thought it was just an analogy. There seem to be some glaring flaws in this analogy: since nothing can escape from a black hole the temperature must vanish, secondly, the entropy is dimensionless whereas the horizon area is a length squared, our final perceived flaw is that the area of every black hole is separately non-decreasing, whereas only the total entropy is non-decreasing in thermodynamics. The resolution to all these flaws lies in the incorporation of quantum theory, recall that going to a quantum theory was also the resolution for apparent paradoxes in thermodynamics, for example black body radiation. We will not study quantum gravity, this is an active area of research but present the classical laws and possibly some semi-classical analysis.

There are four laws of black hole thermodynamics which should be contrasted with the laws of thermodynamics:

Law	Thermodynamics	Black holes
0^{th}	The temperature T is constant throughout a system in thermal equilibrium.	The surface gravity κ is constant over the even horizon of a stationary black hole.
1^{st}	$\mathrm{d}E = T\mathrm{d}S + \sum_i \mu_i \mathrm{d}N_i$	$\mathrm{d}M = \frac{1}{8\pi}\kappa\mathrm{d}A + \Omega_H\mathrm{d}J + \Phi_H\mathrm{d}Q$
2^{nd}	$\mathrm{d}S \geq 0$	$\mathrm{d}A \geq 0$
3^{rd}	T cannot be reduced to zero by a finite number of operations.	κ cannot be reduced to zero by a finite number of operations.

It may seem strange to say that a black hole has a temperature since nothing can escape from a black hole and therefore they cannot radiate. This would also mean that they cannot have a physical entropy. Once quantum effects are taken into account it turns out that a black hole can have a temperature. Moreover, as pointed out by Jacob Bekenstein the second law of thermodynamics would be violated if black holes did not have an entropy. One could throw in arbitrary objects into the black hole which have a large entropy and thus lower the entropy of the exterior universe. In order to save the 2nd law of thermodynamics it is essential for a black hole to have an entropy and more over it must be proportional to the surface area of the horizon. Bekenstein's generalised second law states that

$$\mathrm{d}S_{\text{total}} = \mathrm{d}\big(S_{\text{extermal}} + S_{\text{BH}}\big) \geq 0. \tag{9.1}$$

In 1974 Hawking announced that black holes are hot and radiate just like any hot body with a temperature

$$T_H = \frac{\hbar\kappa}{2\pi k_B}\,, \tag{9.2}$$

from which it follows that a black holes has an entropy given by

$$S_{\text{BH}} = \frac{A}{4G_N\hbar}\,. \tag{9.3}$$

which is known as the *Bekenstein–Hawking* entropy.

In the remainder of the course our goal is to understand the laws of black hole thermodynamics as presented above.

9.1 Zeroth law of black hole mechanics

Proposition

Consider a null geodesic congruence that contains the generators of a Killing horizon $\mathcal{N}$. Then $\theta = \sigma = \omega = 0$ on $\mathcal{N}$.

Proof: We have already seen that $\omega = 0$ since the generators are hypersurface orthogonal. Let ξ be a Killing vector field normal to $\mathcal{N}$. On $\mathcal{N}$ we can write $\xi^\mu = hU^\mu$ where U^μ is tangent to the affinely parametrised generators of $\mathcal{N}$ and h is a function on $\mathcal{N}$. Le $\mathcal{N}$ be specified by an equation $f = 0$. Then we can write $U^\mu = h^- \xi^\mu + fV^\mu$ where V^μ is a smooth vector field. We can then calculate

$$B_{\mu\nu} = \nabla_\nu U_\mu = (\partial_\nu h^{-1})\xi_\mu + h^{-1}\nabla_\nu \xi_\mu + \partial_\nu f V_\mu + f\nabla_\nu V_\mu \,, \tag{9.4}$$

evaluating on $\mathcal{N}$ and using Killing's equation gives

$$B_{(\mu\nu)}\Big|_{\mathcal{N}} = \left[\xi_{(\mu}\partial_{\nu)} h^{-1} + V_{(\mu}\partial_{\nu)} f\right]\Bigg|_{\mathcal{N}} . \tag{9.5}$$

Since both ξ_μ and $\partial_\mu f$ are parallel to U_μ on $\mathcal{N}$ when we project onto $T_\perp$ both terms are eliminated and we have

$$\hat{B}_{\mu\nu}\Big|_{\mathcal{N}} = 0 \tag{9.6}$$

and thus $\theta = \sigma = 0$ on $\mathcal{N}$.

Theorem: *Zeroth law of black hole mechanics*

The surface gravity κ is constant on the future event horizon of a stationary black hole spacetime obeying the dominant energy condition.

Proof: Using Hawking's theorem we have that $\mathcal{H}^+$ is a Killing horizon with respect to some Killing vector ξ. We know that $\theta = 0$ along the generators of $\mathcal{H}^+$, and therefore $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = 0$ along these generators. Moreover we have just seen that on $\mathcal{H}^+$ $\sigma = \omega = 0$. Therefore Raychaudhuri's equation gives

$$0 = R_{\mu\nu}\xi^\mu\xi^\nu\Big|_{\mathcal{H}^+} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho{}_\rho)\xi^\mu\xi^\nu\Big|_{\mathcal{H}^+} = 8\pi T_{\mu\nu}\xi^\mu\xi^\nu\Big|_{\mathcal{H}^+} \tag{9.7}$$

where we have used Einstein's equation and that ξ is null on $\mathcal{H}^+$. This implies

$$J_\mu \xi^\mu\Big|_{\mathcal{H}^+} = 0 \,, \qquad \text{where} \quad J_\mu \equiv -T_{\mu\nu}\xi^\nu \,. \tag{9.8}$$

Since ξ is a future-directed causal vector field, then by the dominant energy condition, so is J_μ (unless it is zero). Thus J^μ is parallel to ξ^μ on $\mathcal{H}^+$ and consequently

$$0 = \xi_{[\mu}J_{\nu]}\Big|_{\mathcal{H}^+} = -\xi_{[\mu}T_{\nu]\rho}\xi^\rho\Big|_{\mathcal{H}^+} - \frac{1}{8\pi}\xi_{[\mu}R_{\nu]\rho}\xi^\rho\Big|_{\mathcal{H}^+} \,, \tag{9.9}$$

where we have used Einstein's equation in the final step. One problem sheet 4 you are asked to show that this is equivalent to

$$0 = \frac{1}{8\pi}\xi_{[\mu}\partial_{\nu]}\kappa \,. \tag{9.10}$$

Therefore $\partial_\nu \kappa$ is proportional to ξ_ν and therefore for any vector field t tangent to $\mathcal{H}^+$ it follows that $t^\mu \partial_\mu \kappa$. Therefore κ is constant on $\mathcal{H}^+$ provided $\mathcal{H}^+$ is connected.

Let us the identity we need for proving that the surface gravity is constant on the horizon. We must be very careful with the formulae we use for the surface gravity and acting on them with derivatives since some only hold on the horizon. Since $\xi^2\big|_{\mathcal{H}^+} = 0$ we have that $\nabla_\mu\big(\xi^2\big)$ is normal to the horizon and therefore there is a function κ on the horizon such that

$$\nabla_\mu(\xi^2) = -2\kappa\xi_\mu\,. \tag{9.11}$$

We may rewrite this as

$$\xi^\mu\nabla_\nu\xi_\mu = -\xi^\mu\nabla_\mu\xi_\nu = -\kappa\xi_\nu\,, \tag{9.12}$$

which is just the geodesic equation in a non-affine parametrisation. The above derivation of the expression for the surface gravity makes clear that it holds on the Killing surface. This means that applying derivatives to the above expression is somewhat subtle, we can only differentiate on the Killing surface and not normal to it. Instead observe that if $\epsilon_{\mu\nu\rho\sigma}$ is the 4d volume element then $\epsilon^{\mu\nu\rho\sigma}\xi_\sigma$ is tangent to the horizon since $\epsilon^{\mu\nu\rho\sigma}\xi_\sigma\xi_\rho = 0$. Therefore we may use this to project the differential operator onto the horizon by acting with $\epsilon^{\mu\nu\rho\sigma}\xi_\rho\nabla_\sigma$ and then this may be applied to any object defined on the horizon. Equivalently we may act with $\xi_{[\mu}\nabla_{\nu]}$ on any object. Now applying this to (9.12) we obtain

$$\begin{aligned}
\xi_{[\rho}\nabla_{\sigma]}(\kappa\xi_\nu) &= \xi_\nu\xi_{[\rho}\nabla_{\sigma]}\kappa + \kappa\xi_{[\rho}\nabla_{\sigma]}\xi_\nu \\
&= \xi_{[\rho}\nabla_{\sigma]}\big(\xi^\mu\nabla_\mu\xi_\nu\big) \\
&= \big(\xi_{[\rho}\nabla_{\sigma]}\xi^\mu\big)\big(\nabla_\mu\xi_\nu\big) + \chi^\mu\xi_{[\rho}\nabla_{\sigma]}\nabla_\mu\xi_\nu \\
&= \big(\xi_{[\rho}\nabla_{\sigma]}\xi^\mu\big)\big(\nabla_\mu\xi_\nu\big) + \chi^\mu R_{\mu\nu[\rho}{}^{\tau}\xi_{\sigma]}\xi_\tau
\end{aligned} \tag{9.13}$$

We may simplify the first term by using the condition that ξ is hypersurface orthogonal and hence satisfies $\xi_{[\mu}\nabla_\nu\xi_{\rho]} = 0$. We find

$$\begin{aligned}
\big(\xi_{[\rho}\nabla_{\sigma]}\xi^\mu\big)\big(\nabla_\mu\xi_\nu\big) &= -\frac{1}{2}\big(\xi^\mu\nabla_\rho\xi_\sigma\big)\nabla_\mu\xi_\nu \\
&= -\frac{1}{2}\kappa\xi_\nu\nabla_\rho\xi_\sigma \\
&= \kappa\xi_{[\rho}\nabla_{\sigma]}\xi_\nu
\end{aligned} \tag{9.14}$$

This cancels the second term of the first row of (9.14). We therefore have

$$\xi_\nu\xi_{[\rho}\nabla_{\sigma]}\kappa = \xi^\mu R_{\nu\mu[\sigma}{}^{\tau}\xi_{\rho]}\xi_\tau \tag{9.15}$$

Since ξ is hypersurface orthogonal we have

$$\xi_\rho\nabla_\mu\xi_\nu = -2\xi_{[\mu}\nabla_{\nu]}\xi_\rho\,, \tag{9.16}$$

and acting on this with with $\xi_{[\sigma}\nabla_{\tau]}$ we obtain

$$\big(\xi_{[\sigma}\nabla_{\tau]}\xi_\rho\big)\nabla_\mu\xi_\nu + \xi_\rho\xi_{[\sigma}\nabla_{\tau]}\nabla_\mu\xi_\nu = -2\big(\xi_{[\sigma}\nabla_{\tau]}\xi_{[\mu}\big)\nabla_{\nu]}\xi_\rho - 2\big(\xi_{[\sigma}\nabla_{\tau]}\nabla_{[\nu}\xi_{|\rho|}\big)\xi_{\mu]}\,. \tag{9.17}$$

Application of (9.16) results in

$$-\xi_\rho R_{\mu\nu[\tau}{}^\lambda\xi_{\sigma]}\xi_\lambda = 2\xi_{[\mu}R_{\nu]\rho\sigma}{}^\lambda\xi^\rho\xi_\lambda\,. \tag{9.18}$$

Contracting over the ρ and τ indices gives

$$-\xi_{[\mu}R_{\nu]}{}^\lambda\xi_\lambda\xi_\sigma = \xi_{[\mu}R_{\nu]\rho\sigma}{}^\lambda\xi^\rho\xi_\lambda\,, \tag{9.19}$$

with the right-hand-side being the expression we required above. We therefore find

$$\xi_{[\mu}\nabla_{\nu]}\kappa = -\xi_{[\mu}R_{\nu]}{}^\rho\xi_\rho\,. \tag{9.20}$$

Plugging this into the formulae above gives the required result.

9.2 First law

We have already seen a form of the first law when we considered the irreducible mass of the Kerr solution. We will give a somewhat heuristic argument here of the first law and then check it in more detail for the black holes we have studied previously consider the Killing vector associated to the Killing horizon, it takes the form $\xi = K + \Omega_H R$ where K generates time translations and R generates the axisymmetry. The corresponding charge is a combination of the mass and the angular momentum:

$$Q_\xi = -\frac{1}{8\pi}\int_{S^2_\infty}\star\mathrm{d}\xi = -\frac{1}{8\pi}\int_{S^2_\infty}\star\mathrm{d}K - \frac{\Omega_H}{8\pi}\int_{S^2_\infty}\star\mathrm{d}R = M - 2\Omega_H J\,. \tag{9.21}$$

We can also evaluate Q_ξ in another way. Let Σ be a spacelike hypersurface intersecting the horizon $\mathcal{H}^+$ on a two-sphere S^2_H which together with the two-sphere S^2_∞ at spatial infinity forms the boundary of Σ. Using Stoke's theorem we have:

$$\begin{aligned} Q_\xi &= -\frac{1}{8\pi}\int_{S^2_H}\star\mathrm{d}\xi - \frac{1}{8\pi}\int_\Sigma \mathrm{d}\star\mathrm{d}\xi \\ &= -\frac{1}{8\pi}\int_{S^2_H}\star\mathrm{d}\xi + 2\int_\Sigma\big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho{}_\rho\big)\xi^\nu\star\mathrm{d}x^\mu\,, \end{aligned} \tag{9.22}$$

where in the last step we used

$$\star\mathrm{d}\star\mathrm{d}X = 8\pi J\,, \qquad J = 2\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho{}_\rho\Big)X^\nu\mathrm{d}x^\nu \tag{9.23}$$

The integral over S^2_H may be regarded as the contribution from the black hole while the one over Σ is a combination of the mass and angular momentum of the matter and radiation outside the horizon. In order to treat the integral over S^2_H we observe that the volume form on S^2_H, can be written as

$$\mathrm{dvol}(S^2_H) = \star(n \wedge \xi)\,, \tag{9.24}$$

evaluated at the horizon. Here n^μ is another null vector normal to S^2_H, normalised so that $n^\mu \xi_\mu = -1$. Therefore

$$\begin{aligned}\int_{S^2_H} \star \mathrm{d}\xi &= \int_{S^2_H} \mathrm{dvol}(S^2_H)\Big(\star\left(n\wedge\xi\right)\Big)^{\mu\nu}(\star\mathrm{d}\xi)_{\mu\nu} \\ &= 2\int_{S^2_H} \mathrm{dvol}(S^2_H) n^\nu \xi^\mu \nabla_\mu \xi_\nu \\ &= -2\kappa \int_{S^2_H} \mathrm{dvol}(S^2_H) \\ &= -2\kappa A_H\,.\end{aligned} \tag{9.25}$$

Plugging this into (9.22) we arrive at

$$M = \frac{\kappa A_H}{4\pi} + 2\Omega_H J + 2\int_\Sigma \Big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho{}_\rho\Big)\xi^\nu \star \mathrm{d}x^\mu \tag{9.26}$$

If we are in pure GR, then $T_{\mu\nu} = 0$ and our spacetime is the Kerr black hole and the formula reads

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J\,. \tag{9.27}$$

This is Smarr's formula for the mass of a Kerr black hole. A formula for δM in the vacuum case can be obtained by varying (9.27)

$$\delta M = \frac{1}{4\pi}\big(A_H \delta\kappa + \kappa\delta A_H\big) + 2\big(J\delta\Omega_H + \Omega_H \delta J\big)\,. \tag{9.28}$$

An alternative computation gives

$$\delta M = -\frac{1}{4\pi} A_H \delta\kappa - 2J\delta\Omega_H\,. \tag{9.29}$$

Adding the two equations gives

$$\delta M = \frac{1}{8\pi}\kappa\delta A_H + \Omega_H \delta J\,. \tag{9.30}$$

In the case where there is an electric charge, we need to define the electric potential

$$\Phi_H = \xi^\mu A_\mu\Big|_{\mathcal{H}^+} - \xi^\mu A_\mu\Big|_\infty\,. \tag{9.31}$$

For asymptotically flat spacetimes we have that $A_\mu \to 0$ as we tend to ∞ and so the second term drops out. The 1^{st} law with electric charge is then

$$\delta M = \frac{1}{8\pi}\kappa\delta A_H + \Omega_H \delta J + \Phi_H \delta Q \tag{9.32}$$

Kerr–Newman Let us check this for the Kerr–Newman solution:

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{\Delta(r) - a^2\sin^2\theta}{\rho(r,\theta)^2}\mathrm{d}t^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta(r))}{\rho(r,\theta)^2}\mathrm{d}t\mathrm{d}\phi \\
&+ \frac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta(r)}{\rho(r,\theta)^2}\sin^2\theta\mathrm{d}\phi^2 + \frac{\rho(r,\theta)^2}{\Delta(r)}\mathrm{d}r^2 + \rho^2(r,\theta)\mathrm{d}\theta^2\,, \\
A &= -\frac{1}{\rho(r,\theta)^2}\Big(Qr(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)\Big)\,.
\end{aligned} \tag{9.33}$$

The functions are

$$\rho(r,\theta)^2 = r^2 + a^2\cos^2\theta\,, \quad \Delta(r) = r^2 - 2Mr + a^2 + e^2\,, \quad e^2 = Q^2 + P^2\,. \tag{9.34}$$

The Kerr–Newman solution is the unique stationary black hole solution of the Einstein–Maxwell theory.

Let us compute the quantities that we will need to check the relation. The outer Killing horizon is at $r = r_+$ with

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}\,. \tag{9.35}$$

First let us consider the horizon surface area. We fix an arbitrary time $t = t_0$ and look at the induced metric on the intersection $t = t_0$ and $r = r_+$, we find

$$\mathrm{d}s^2(H) = \gamma_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \rho(r_+,\theta)^2\mathrm{d}\theta^2 + \frac{r_+^2 + a^2}{\rho(r_+,\theta)^2}\sin^2\theta\mathrm{d}\phi^2\,. \tag{9.36}$$

The volume form is

$$\mathrm{dvol}(\gamma) = (r_+^2 + a^2)\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \tag{9.37}$$

and so the surface area is

$$A_H = \int_{\mathcal{H}^+}\mathrm{dvol}(\gamma) = \int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\mathrm{d}\theta(r_+^2 + a^2)\sin\theta = 4\pi(r_+^2 + a^2)\,. \tag{9.38}$$

Next let us consider the surface gravity. We first need to find the Killing vector which is null on the horizon and then to compute the surface gravity. Since the horizon is a Killing horizon we know that it must be of the form

$$\xi = K + \Omega_H R\,, \tag{9.39}$$

where K and R are the generators of time translations and the axis symmetry respectively. Note that since a Killing vector remains a Killing vector under a constant rescaling there is an arbitrariness in how we pick such a Killing vector. We normalise such that K has coefficient 1. Now this needs to have zero norm on the horizon. The norm is

$$\begin{aligned}\xi^2\Big|_{\mathcal{N}^+} &= \frac{a^2\sin^2\theta}{r_+^2+a^2\cos^2\theta} - \frac{2a\sin^2\theta(r_+^2+a^2)}{r_+^2+a^2\cos^2\theta}\Omega_H + \frac{(r_+^2+a^2)^2}{r_+^2+a^2\cos^2\theta}\sin^2\theta\Omega_H^2 \\ &= \frac{\sin^2\theta}{r_+^2+a^2\cos^2\theta}\Big(a^2 - 2a(r_+^2+a^2)\Omega_H + (r_+^2+a^2)^2\Omega_H^2\Big)\,,\end{aligned} \tag{9.40}$$

and for this to vanish we need

$$\Omega_H = \frac{a}{r_+^2+a^2}\,. \tag{9.41}$$

This is the angular velocity of the black hole.

We can now try to compute the surface gravity. In order to use the formula

$$\nabla_\mu(\xi^2) = -2\kappa\xi_\mu\,. \tag{9.42}$$

we need to use coordinates in which the horizon is not a coordinate singularity. Rather than changing coordinates we will instead use an alternative formula for the surface gravity

$$\kappa^2 = \lim_{r\to r_+}\frac{g^{\mu\nu}\partial_\nu(\xi^2)\partial_\mu(\xi^2)}{4\xi^2}\,. \tag{9.43}$$

After a slightly painful computation we find

$$\kappa = \frac{r_+-r_-}{2(r_+^2+a^2)}\,. \tag{9.44}$$

Next let us compute the electric potential. We have

$$\begin{aligned}\Phi_H = \xi^\mu A_\mu\Big|_{\mathcal{H}_+} &= \frac{Qr_+}{r_+^2+a^2\cos^2\theta}(1-\Omega_H a\sin^2\theta) \\ &= \frac{Qr_+}{r_+^2+a^2}\,.\end{aligned} \tag{9.45}$$

Finally let us remember that the electric charge is Q and the angular momentum is $J = aM$. Putting everything together we have

$$\begin{aligned}A_H &= 4\pi\Big(\big(M+\sqrt{M^2-a^2-Q^2}\big)^2 + a^2\Big) \\ &= 4\pi\Big(2M^2 - Q^2 + 2M\sqrt{M^2-Q^2-a^2}\Big)\,.\end{aligned} \tag{9.46}$$

Since M, Q and J are independent parameters this implies

$$\delta A = \frac{\partial A}{\partial M}\delta M + \frac{\partial A}{\partial Q}\delta Q + \frac{\partial A}{\partial J}\delta J\,. \tag{9.47}$$

After a some explicit computation (which you will do in problem sheet 4) and a little rearranging we find

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H \delta J + \Phi_H \delta Q\,. \tag{9.48}$$

We see that the proof is deceptively simple, all the hard work goes into proving the uniqueness theorems. You need to know that the black hole settles down to another Kerr–Newman black hole and not some other spacetime. It is worth noting that there exist proofs of the first law known as physical process proofs that do not assume this.

9.3 Second law

The second law states that in any physical process the *area of the event horizon can never decrease*. This is a very surprising feature of these complicated nonlinear PDEs which Hawking proved using just the Einstein equation, the weak energy condition and cosmic censorship.

Let us give a sketch of the proof. Consider the congruence of the horizon and take a cross sectional area A_H at some value of the affine parameter λ along the geodesics. Then the expansion θ satisfies

$$\frac{\mathrm{d}A_H}{\mathrm{d}\lambda} = \theta A_H\,. \tag{9.49}$$

If we imagine the theorem is violated so that the area decreases then we must have $\theta < 0$ somewhere on the event horizon. Since the generators are geodesics the evolution of the expansion is governed by Raychaudhuri's equation. Recall that if $\theta < 0$ and the null energy condition is satisfied then $\theta \to -\infty$ in finite λ. This causes a caustic, see figure 20. Since the points p and q are timelike separated, this contradicts the assumption that the null curves are the generators of an event horizon, as no two points on the event horizon can be timelike separated. Thus by contradiction the cross sectional area of an event horizon cannot decrease. Note that the proof assumes Einstein's equations, they are not used in an essential way.

Let us use the second law. Consider a Schwarzschild black hole of mass M. Can a black hole split into two black holes of smaller mass? It turns out that the second law forbids this. To see this let the masses of the new black holes be m_1 and m_2. Conservation of energy implies $M = m_1 + m_2$. The surface area of a Schwarzschild black hole is $A = 4\pi M^2$. We have that the entropy of the final state is $A_f = A_1 + A_2 = 4\pi(m_1^2 + m_2^2)$ and the entropy of the initial state is $A_i = 4\pi M^2 = 4\pi(m_1 + m_2)^2 = 4\pi(m_1^2 + m_2^2 + 2m_1 m_2)$. It is clear that $A_i > A_f$ and therefore this process violates the second law. Black holes cannot split in two!

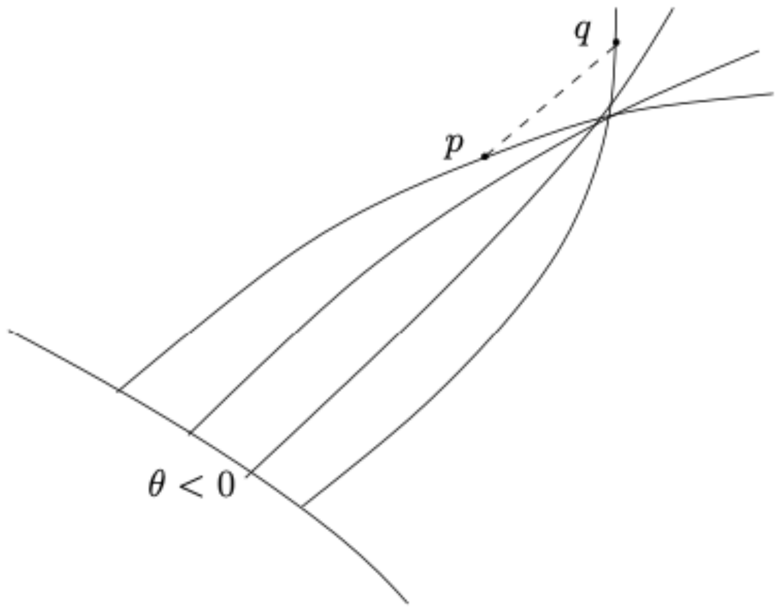

Figure 20: A family of null geodesics with $\theta < 0$ initially will form a caustic; the dotted curve connecting p and q lies within the local light cone, so these points are timelike separated.

9.4 Third law

Of all the laws this is on the least firm ground. When the surface gravity of a black hole vanishes it is called extremal. For the Kerr–Newman this condition corresponds to $M^2 = a^2 + Q^2 + P^2$. For Kerr and electrically charged Kerr black holes one can try to throw matter into the black hole and make it extremal. One finds that it gets harder and harder for the matter to make the black hole become closer to being an extremal black hole.

9.5 Why should black holes carry an entropy?

Two arguments supporting why black holes should have an entropy.

Black holes are formed from the collapse of matter which carries entropy. However the matter that has contributed to form a black hole is not visible from an observer watching from outside the event horizon. So the observer must conclude either that the entropy disappears in the formation and growth of black holes and thus that the second principle of thermodynamics is violated or that the black holes themselves carry entropy.[17]

[17] Famously Bekenstein's advisor Wheeler, asked "what happens if we throw a cup of tea into a black hole?"

In general relativity, black hole solutions are fully characterised by few conserved quantities such as the mass, the angular momentum and the electric charge. Black holes do not have hair. However there are many ways of forming a black hole with assigned values of these charges. From this perspective black holes are macroscopic thermodynamic objects with many microstates, corresponding to the different possible ways of forming the same macroscopic solution. Enumerating these microstates leads to an entropy.

10 Hawking temperature (non-examinable)

General relativity is not a complete theory. For one, the singularity theorem provides evidence that the theory is incomplete. More convincingly, GR is a classical theory while the world is fundamentally quantum mechanical. Trying to understand quantum gravity is one of the leading avenues of research in high energy theory. Though there has been much progress, a full understanding of quantum gravity remains elusive.

There are two parts to GR: spacetime curvature and its influence on matter and the dynamics. of the metric in response to a varying energy momentum tensor. Lacking a true theory of quantum gravity we may still use the first part, saying that the quantum mechanical matter propagates in a curved background which we will hold fixed. Rather than obeying some dynamical equations, we take the metric to be fixed.

To begin let us review some quantum mechanics and quantum field theory before defining quantum field theory in curved space.

10.1 Quantum mechanics

Quantum mechanics is profoundly different from classical mechanics, despite this both try to answer the same three fundamental questions.

- The state of the system is represented as an element of a *Hilbert space*. Mathematically a Hilbert space is just a complex vector space equipped with a complex-valued inner product with the property that taking the inner product of two states in the opposite order is equivalent to complex conjugation. We denote elements of the Hilbert space as $|\psi\rangle$ and elements of the dual space as $\langle\psi|$ so that the inner product of $|\psi_1\rangle$ and $|\psi_2\rangle$ is $\langle\psi_2|\psi_1\rangle$ and obeys

  $$\langle\psi_2|\psi_1\rangle^* = \langle\psi_1|\psi_2\rangle\,. \tag{10.1}$$

  In quantum mechanics the Hilbert space of interest are very often infinite-dimensional. For example, if a classical system is represented by coordinate x and momentum p, the

Hilbert space could be taken to consist of all square-integrable complex-valued functions of x, or equivalently all square-integrable complex valued functions of p but not both at once.

- Observables are represented by *self-adjoint operators* on the HIlbert space. An operator is Hermitian if
$$A^\dagger = A \,, \tag{10.2}$$
where
$$\langle \psi_2 | A\psi_1 \rangle = \langle A^\dagger \psi_2 | \psi_1 \rangle \,, \tag{10.3}$$
for all states $|\psi_1\rangle$, $|\psi_2\rangle$. Many operators will not be Hermitian, but observables should be real and this requires the operator to be Hermitian. In general such operators do not commute. This means that we cannot simultaneously specify the precise values of everything we might want to measure. There will be a maximally set of commuting observables which would represent all we can say about a system at once.

- Evolution of hte system may be represented in one of two ways: as unitary evolution of the state vector in Hilbert space in the *Schrodinger picture*, or by keeping the state fixed and allowing observables to evolve according to equations of motion called the *Heisenberg picture.*

Consider a harmonic oscillator. This has Lagrangian
$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \,, \tag{10.4}$$
which has equation of motion
$$\ddot{x} + \omega^2 x = 0 \,. \tag{10.5}$$
In the Schrodinger picture, where states are represented by complex-valued wave functions that evolve with time, such as $\psi(x,t)$. The wave function is really the set of components of the state vector $|\psi\rangle$ expressed in the delta function position basis $|x\rangle$ so that $|\psi(t)\rangle = \int \mathrm{d}x \psi(x,t)|x\rangle$. Canonical quantisation consists of imposing the canonical commutation relation
$$[\hat{x}, \hat{p}] = \mathrm{i} \,, \tag{10.6}$$
on the coordinate operator $\hat{x}$ and its conjugate momentum $\hat{p}$. For states represented as wave functions depending on t and x, the operator $\hat{x}$ is simply multiplication by x, so the commutation relation can be implemented by fixing
$$\hat{p} = -\mathrm{i}\partial_x \,. \tag{10.7}$$

The Hamiltonian operator is

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}\omega^2 x^2\,, \tag{10.8}$$

and the equation of motion is the Schrodinger equation

$$\mathrm{i}\partial_t \psi = H\psi\,. \tag{10.9}$$

Since the Hamiltonian is time independent the solutions separate into functions of space and functions of time, $\psi(x,t) = f(t)g(x)$. The solutions then come in a discrete set labelled by an integer $n \geq 0$ and we find

$$\psi_n(x,t) = \mathrm{e}^{-\frac{\omega x^2}{2}} H_n(\sqrt{\omega}x)\mathrm{e}^{-\mathrm{i}E_n t}\,, \tag{10.10}$$

where H_n is a Hermite polynomial of degree n and

$$E_n = \Big(n + \frac{1}{2}\Big)\omega\,. \tag{10.11}$$

These states are all eigenfunctions of H and E_n is an energy eigenvalue. An arbitrary state of the oscillator will consist of a superposition of the energy eigenstates,

$$\psi(x,t) = \sum_n c_n \psi_n(x,t)\,, \tag{10.12}$$

for some set of appropriately normalised coefficients c_n.

Note that there is a discrete spectrum of energy eigenstates, this is a quantum property. There is a ground state of lowest energy plus a set of excited states labelled by their energy eigenvalue. The ground state has a nonvanishing energy

$$E_0 = \frac{1}{2}\omega\,, \tag{10.13}$$

which is sometimes called the zero-point energy. The classical system would have had zero energy representing a particle with $x = p = 0$. The quantum zero-point energy can be traced to the Heisenberg uncertainty principle, which forbids us from localizing a state simultaneously in both position and momentum. There is a consequently a minimum amount of jiggle in the oscillator leading to a non-zero ground state energy.

An alternative way to solve the simple harmonic oscillator is to introduce creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ defined by

$$\hat{a} = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + \mathrm{i}\hat{p})\,, \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} - \mathrm{i}\hat{p})\,. \tag{10.14}$$

From the commutation relations for $\hat{x}$ and $\hat{p}$ we find

$$[\hat{a}, \hat{a}^\dagger] = 1\,, \tag{10.15}$$

and the Hamiltonian becomes

$$H = \omega\Big(\hat{a}\hat{a}^\dagger + \frac{1}{2}\Big)\,. \tag{10.16}$$

The creation and annihilation operators satisfy

$$[H, \hat{a}] = -\omega\hat{a}\,, \qquad [H, \hat{a}^\dagger] = \omega\hat{a}^\dagger\,. \tag{10.17}$$

We define the number operator

$$\hat{n} = \hat{a}^\dagger\hat{a}\,. \tag{10.18}$$

Consider an eigenstate $|n\rangle$ of the number operator,

$$\hat{n}|n\rangle = n|n\rangle\,. \tag{10.19}$$

By playing with the commutation relations we have

$$\begin{aligned} \hat{n}\hat{a}^\dagger|n\rangle &= (n+1)\hat{a}^\dagger|n\rangle \\ \hat{n}\hat{a}|n\rangle &= (n-1)\hat{a}|n\rangle\,, \end{aligned} \tag{10.20}$$

thus when acting with $\hat{a}^\dagger$ on $|n\rangle$ we obtain another eigenstate of $\hat{n}$ with eigenvalue raised by one and $\hat{a}$ gives an eigenstate with eigenvalue lowered by 1. n takes integral values from 0 to ∞ and therefore there must be a vacuum state with

$$\hat{a}|0\rangle = |0\rangle\,. \tag{10.21}$$

By acting with $\hat{a}^\dagger$ we can construct all of the eigenstates

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,. \tag{10.22}$$

The basis states are taken to be tome independent so a physical system observing Schrödinger's equation will be described by a state

$$|\psi(t)\rangle = \sum_n c_n \mathrm{e}^{-\mathrm{i}E_n t}|n\rangle\,, \tag{10.23}$$

with c_n constant coefficients.

In order to transition more smoothly to quantum field theory it is useful to also have the Heisenberg picture in which the states are fixed and the operators evolve with time. Any state can be written formally as some fixed initial state acted on by a unitary time evolution operator

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle\,, \tag{10.24}$$

where

$$U(t) = \mathrm{e}^{-\mathrm{i}\int H \mathrm{d}t}\,. \tag{10.25}$$

The Schrödinger picture expression for the matrix element of a time-independent operator, A between two time-dependent states can be written in Heisenberg picture in terms of a time dependent operator $A(t)$ and time independent states as

$$\begin{aligned}\langle\psi_2(t)|A|\psi_1(t)\rangle &= \langle\psi_2(0)|U^\dagger(t)AU(t)|\psi_1(0)\rangle \\ &= \langle\psi_2|A(t)|\psi_1\rangle\,,\end{aligned} \tag{10.26}$$

with

$$A(t) = U^\dagger(t)AU(t)\,. \tag{10.27}$$

Such an operator satisfies the Heisenberg equation of motion:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \mathrm{i}[H, A(t)]\,, \tag{10.28}$$

which replaces the role of Schrödinger's equation in this picture.

10.2 Quantum field theory

Quantum field theory is a particular example of a quantum mechanical system in which we quantise a field (a function or tensor field defined on spacetime). Let us first consider a free scalar field in flat space. This has action

$$S = \int \mathrm{d}^n x\Big[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\Big] \equiv \int \mathrm{d}^n x\mathcal{L}\,. \tag{10.29}$$

The equation of motion is the Klein–Gordon equation,

$$\Box\phi - m^2\phi = 0\,. \tag{10.30}$$

To translate into a Hamiltonian picture one defines the conjugate momentum to be

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\,. \tag{10.31}$$

For the free scalar field this is

$$\pi = \dot{\phi}\,. \tag{10.32}$$

Of course since we are using time derivative we have assumed a particular inertial frame and therefore the Hamiltonian procedure necessarily violates manifest Lorentz invariance. With care however, the observables remain Lorentz invariant. The Hamiltoian is represented as the integral of a Hamiltonian density over the spatial directions directions. The Hamiltonian density is related to the Lagrangian by a Legendre transformation,

$$\begin{aligned} \mathcal{H}(\phi,\pi) &= \pi\dot{\phi} - \mathcal{L}(\phi,\partial_\mu\phi) \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\,, \end{aligned} \tag{10.33}$$

with $(\nabla\phi)^2 = \delta^{ij}\partial_i\phi\partial_j\phi$. In comparison to the harmonic oscillator the field $\phi(x)$ plays the role of the coordinate x and the momentum field $\pi(x)$ plays the role of p. Instead of a state being specified by two number (x,p) at some fixed time, the initial conditions are values of the field over all of the spatial directions at a fixed time.

Note that $\phi(x^\mu)$ is not a wave function; it is a dynamical variable generalising the single degree of freedom x in the case of the harmonic oscillator. We will use a Heisenberg picture of time evolution where we promote ϕ to an operator.

First we need to solve the classical theory. The solutions of the Klein–Gordon equation include the plane wave solution

$$\phi(x^\mu) = \phi_0 \mathrm{e}^{\mathrm{i}p_\mu x^\mu} = \phi_0 \mathrm{e}^{-\mathrm{i}p^0 t + \mathrm{i}\vec{p}\cdot\vec{x}}\,, \tag{10.34}$$

where the wave vector has components

$$p^\mu = (p^0, \vec{p})\,, \tag{10.35}$$

and the frequency must satisfy

$$(p^0)^2 = \vec{p}^{\,2} + m^2\,, \qquad p^0 > 0\,. \tag{10.36}$$

The latter condition is in order to consider the positive frequency modes only.

We can write down the most general solution by constructing a complete orthonormal set of modes in terms of which any solution may be expressed. We need to first define an inner product on the space of solutions. To inner product is an integral over a constant time hypersurface Σ_t and is

$$(f,g) = \mathrm{i}\int_{\Sigma_t} (f^* \overleftrightarrow{\partial_t} g)\mathrm{d}^{n-1}x\,, \qquad f^*\overleftrightarrow{\partial_t} g = f^*\partial_t g - \partial_t f^* g\,. \tag{10.37}$$

By using Stoke's theorem and the equation of motion one can check that this is independent of the chosen hypersurface. Let us define

$$\psi_p = N_p \mathrm{e}^{\mathrm{i}p_\mu x^\mu}\,, \tag{10.38}$$

with $p^2 + m^2 = 0$. Then $\{\psi_p, \psi_p^*\}$ form a basis of solutions and any field configuration can be expanded as

$$\phi(x) = \int \mathrm{d}^3p \big(a_p \psi_p(x) + a_p^* \psi_p^*(x)\big) \,, \tag{10.39}$$

with a_p and a_p^* are complex constants. In order for the basis to be orthonormal we take

$$N_p = \frac{1}{\sqrt{2p^0}(2\pi)^{3/2}}, . \tag{10.40}$$

We quantise the theory by promoting ϕ and π to be operators and impose the standard commutation relations:

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = \mathrm{i}\delta^{(3)}(\vec{x}-\vec{y}) \,, \quad [\phi(t,\vec{x}), \phi(t,\vec{y})] = 0 \,, \quad [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0 \,. \tag{10.41}$$

This may then be translated into commutation relations for the a's, with

$$[a_p, a_q^\dagger] = \delta(\vec{p}-\vec{q}) \,, \quad [a_p, a_q] = 0 \,, \quad [a_p^\dagger, a_q^\dagger] = 0 \,. \tag{10.42}$$

We may then define a vacuum state by

$$a_p|0\rangle = 0 \,, \quad \forall p \,. \tag{10.43}$$

It may seem that the definition of the vacuum state depends on the initial choice of inertial frame, however this is not the case. Consider a different inertial frame $\tilde{x}^\mu$ related by a Lorentz transformation $\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$. In this new frame the positive frequency mode functions are

$$\tilde{\psi}_p = N_p \mathrm{e}^{\mathrm{i}p_\mu \tilde{x}^\mu} \,, \tag{10.44}$$

and the field expansion is

$$\phi(\tilde{x}) = \int \mathrm{d}^3p \Big(\tilde{a}_p \tilde{\psi}_p + \tilde{a}_p^\dagger \tilde{\psi}_p^*\Big) \,, \tag{10.45}$$

and in terms of these modes the new vacuum state satisfies $\tilde{a}_p|\tilde{0}\rangle = 0, \forall p$. We need to show that

$$a_p|0\rangle = 0 \ \ \forall p \quad \Rightarrow \quad \tilde{a}_p|0\rangle \ \ \forall p \,. \tag{10.46}$$

We have

$$\tilde{\psi}_p = \frac{1}{\sqrt{2p_0}(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}p_\mu \tilde{x}^\mu} = \left(\frac{\tilde{p}^0}{p^0}\right)^{1/2} \frac{1}{\sqrt{2\tilde{p}^0}(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}\tilde{p}_\mu x^\mu} = \left(\frac{\tilde{p}^0}{p^0}\right)^{1/2} \psi_{\tilde{p}} \,. \tag{10.47}$$

More over since we restrict to the orthochronous subgroup of the Lorentz group, i.e. $\Lambda^0{}_0 > 0$ we have $p^0 > 0 \Rightarrow \tilde{p}^0 > 0$. Therefore we have

$$a_p|0\rangle = 0 \ \ \forall p \quad \Rightarrow \quad \tilde{a}_p|0\rangle \ \ \forall p \,, \tag{10.48}$$

and the converse follows by symmetry and the vacuum state is independent of the choice of frame.

10.3 QFT in curved spacetime

We now want to consider what changes when we try to quantise a field theory on curved spacetime. We fix a background (M, g) and assume that it is globally hyperbolic. Recall that this means that the spacetime admits a Cauchy surface and from initial conditions on the Cauchy surface we can solve the equations of motion on all of spacetime. We perform minimal coupling of the theory so that $\eta^{\mu\nu} \to g^{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$. The Klein–Gordon equation becomes

$$\nabla^2\phi \equiv g^{\mu\nu}\nabla_\mu\partial_\nu\phi = m^2\phi\,, \tag{10.49}$$

while the inner product is modified to

$$(f_1, f_2) = \mathrm{i}\int_\Sigma \mathrm{d}^3x\sqrt{\gamma}n^\mu(f_1^*\partial_\mu f_2 - \partial_\mu f_1 f_2^*)\,, \tag{10.50}$$

with Σ a spacelike hypersurface and n^μ a unit normal vector and γ the determinant of the induced metric. Let the background admit a Killing vector, K, then on functions we have

$$[K, \nabla^2]f = 0\,. \tag{10.51}$$

Since ∇^2 and $\mathrm{i}K$ are both self-adjoint and commuting they admit a complete set of common eigenfunctions

$$\nabla^2 f = m^2 f\,, \quad \mathrm{i}K^\mu\partial_\mu f = \omega f\,. \tag{10.52}$$

If K is timelike we are entitled to call the eigenvalue the frequency. Indeed this is how it works in Minkowski space where $K = \partial_t$. If f is an eigenfunction with positive frequency ω then f^* is an eigenfunction of negative frequency $-\omega$. We can then without loss of generality expand our fields in terms of positive and negative frequency eigenfunctions of the Laplacian in a basis $\{\psi_i\}$ of positive frequency modes and $\{\psi_i^*\}$ of negative frequency modes. We expand our field as

$$\phi = \sum_i (a_i\psi_i + a_i^\dagger\psi_i^*)\,, \tag{10.53}$$

with

$$[a_i, a_{i'}^\dagger] = \delta_{ij}\,. \tag{10.54}$$

Consider a sandwich spacetime (M, g) made up of three regions, region B bottom, region C for centre and region T for top, and assume the Klein–Gordon equation holds throughout spacetime. Region B is stationary and admits a timelike Killing vector K^B, region C is not stationary and all sorts of dynamical processes might take place so long as it remains globally hyperbolic, and finally region T is once again stationary with a new timelike Killing vector

K^T. If we quantise in region B we pick a set of modes $\{f_i, f_i^*\}$ that satisfy $\mathrm{i}K^B f_i = \omega_i f_i$ with $\omega_i > 0$. On the other hand in region T we choose another set of modes $\{g_i, g_i^*\}$ that satisfy $\mathrm{i}K^T g_i = \tilde{\omega}_i g_i$ with $\tilde{\omega}_i > 0$. Note that even though the positive-frequency conditions are imposed using the Killing vectors in specific regions the modes extend throughout the whole of spacetime. In the two cases the respective expansion is then

$$\phi(x) = \sum_i \left(a_i f_i + a_i^\dagger f_i^* \right) = \sum_i \left(b_i g_i + b_i^\dagger g_i^* \right), \tag{10.55}$$

where the modes have been normalised with respect to the Klein–Gordon inner product so that the commutation relations are

$$[a_i, a_j^\dagger] = \delta_{ij}\,, \qquad [b_i, b_j^\dagger] = \delta_{ij}\,. \tag{10.56}$$

Since $\{f_i\}$ forms a basis we can also expand any function in terms of it, we have

$$g_i = \sum_i A_{ij} f_j + B_{ij} f_j^*\,. \tag{10.57}$$

The coefficients A_{ij} and B_{ij} are called the *Bogoliubov coefficients* and the transformation between the different bases is called a *Bogoliubov transformation.* Using the normalisation conditions it can be shown that they satisfy

$$\begin{aligned} \sum_k A_{ik} A_{jk}^* - B_{ik} B_{jk}^* &= \delta_{ij}\,, \\ \sum_k A_{ik} B_{jk} - B_{ik} A_{jk} &= 0\,. \end{aligned} \tag{10.58}$$

Or in matrix notation

$$AA^\dagger - BB^\dagger = 1\,, \qquad AB^T = BA^T\,. \tag{10.59}$$

We can also relate the different operator coefficients to each other

$$b_i = \sum_j A_{ij}^* a_j - B_{ij}^* a_j^\dagger\,. \tag{10.60}$$

The procedure above defines a vacuum state associated with the modes $\{f_i, f_i^*\}$ called the *in*-vacuum as the states satisfy $a_i|0\rangle_{in} = 0\ \forall i$. In a stationary reference frame in region B (i.e. an integral curve of K^B this will appear empty. What about in region T? What is the expected number of particles of the state $|0\rangle_{in}$ with momentum i. It is given by the expectation value

$$\langle N_i \rangle = {}_{in}\langle 0|b_i^\dagger b_i|0\rangle_{in} = \sum_j B_{ij} B_{ij}^* \qquad \text{no summation over i}\,. \tag{10.61}$$

If this is non-zero there is pair production. Alternatively one can see this as the in-vacuum and out-vacuum are different. Hence a changing spacetime geometry generically causes particle production.

10.4 Unruh effect

Even though we have made an effort above to understand QFT in curved space we will first consider a phenomenon that uses the above ideas but manifests in flat space. This is the Unruh effect, which states that an accelerating observer in the Minkowski vacuum will observe a thermal spectrum of particles.

The basic idea is very simple, observers with different notions of positive and negative frequency modes will disagree on the particle content of a given state. A uniformly accelerated observe in Minkowski moves along an orbit of a time-like Killing vector, however this is not the usual time-translation Killing vector. We can therefore expand the field in terms of modes appropriate for the accelerated observer and calculate the number operator in the ordinary Minkowski vacuum. We will find that this leads to a thermal spectrum of particles.

To simplify things as much as possible let us consider a massless scalar field in two dimensions. The wave equation is

$$\Box\phi = 0\,. \tag{10.62}$$

Before trying to quantise the theory consider a uniformly accelerating observer, we have seen this earlier in section 4.3, but let us review the details. In inertial coordinates the metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2\,. \tag{10.63}$$

An observer moving at a uniform acceleration of magnitude α follows the trajectory

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)\,, \qquad x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau)\,, \tag{10.64}$$

note that

$$x^2 = t^2 + \alpha^2\,. \tag{10.65}$$

We can choose new coordinates on two-dimensional Minkowski space that are adapted to uniformly accelerated motion as

$$t = \frac{1}{a}\mathrm{e}^{a\xi}\sinh(a\eta)\,, \qquad x = \frac{1}{a}\mathrm{e}^{a\xi}\cosh(a\eta)\,, \quad (x > |t|)\,. \tag{10.66}$$

The new coordinates have ranges

$$-\infty < \eta, \xi < \infty\,, \tag{10.67}$$

and cover the wedge $x > |t|$ *Rindler space* corresponds to the right wedge $x > |t|$ foliated by the worldlines of the accelerated observers and labelled by region I in figure 21. In these

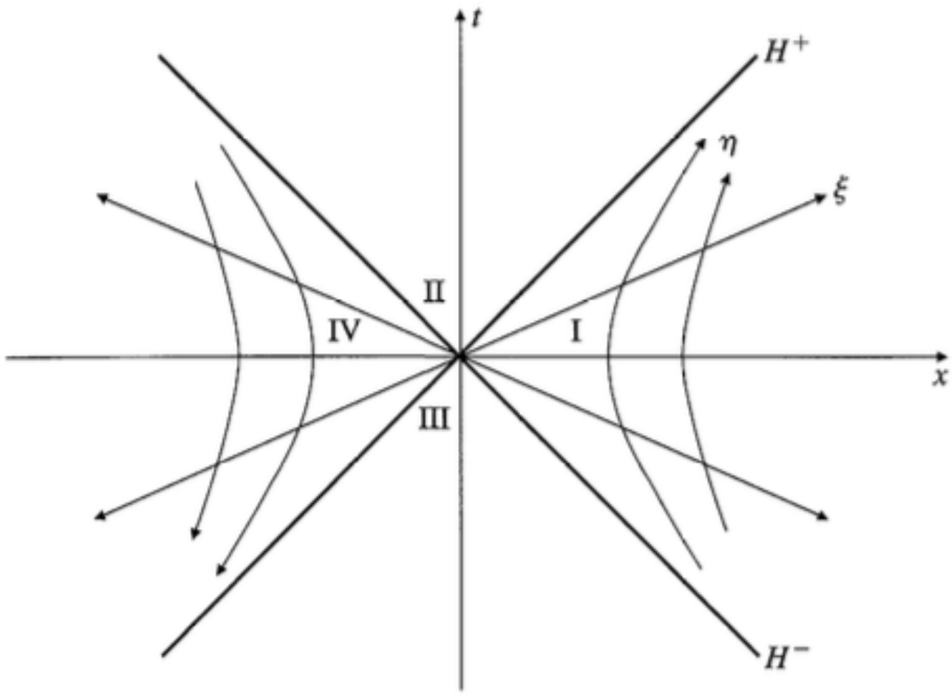

Figure 21: Minkowski spacetime in Rindler coordinates. Region I is the region accessible to an observer undergoing constant acceleration in the $+x$-direction. The coordinates (η, ξ) can be used in region I or region IV, where they point in the opposite direction. The vector filed ∂_η corresponds to the generator of Lorentz boosts and the horizons $H^\pm$ are Killing horizons for this vector field, which represent the boundaries of the past and future as witnessed by the Rindler observer.

coordinates the constant acceleration path is

$$\eta(\tau) = \frac{\alpha}{a}\tau\,, \qquad \xi(\tau) = \frac{1}{a}\log\frac{a}{\alpha}\,, \tag{10.68}$$

and we see that the proper time is proportional to η and the spatial constant ξ is constant. Then an observer with acceleration $\alpha = a$ moves along the path

$$\eta = \tau\,, \quad \xi = 0\,. \tag{10.69}$$

The metric in these coordinates takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{2a\xi}\big(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2\big)\,. \tag{10.70}$$

The null line $t = x$ labelled by H^+ is a future Cauchy horizon for any $\eta =$ constant spacelike hypersurface in region I. Similarly H^- is a past Cauchy horizon.

The metric is independent of η and therefore ∂_η is a Killing vector, however since this is Minkowski spacetime there are more of course. Indeed if we express ∂_η in the (t, x) coordinates we have

$$\partial_\eta = a(x\partial_t + t\partial_x)\,. \tag{10.71}$$

This is the Killing vector which generates a boost in the x-direction. It is clear that this Killing vector naturally extends throughout the spacetime. This extends naturally throughout the spacetime, in regions II and III it is spacelike while in region IV it is timelike but past-directed. The horizons are Killing horizons for ∂_η.

We can define coordinates (η, ξ) in region IV by flipping the signs in (10.66),

$$t = -\frac{1}{a}\mathrm{e}^{a\xi}\sinh(a\eta)\,, \qquad x = -\frac{1}{a}\mathrm{e}^{a\xi}\cosh(a\eta)\,, \quad (x < |t|)\,. \tag{10.72}$$

The sign guarantees that ∂_η and ∂_t point in opposite directions. Strictly speaking we cannot use the (η, ξ) simultaneously in regions I and IV since the ranges are the same in each region, we must explicitly indicate to which region the coordinate belongs to. We add labels to distinguish so that the metric takes the same form in both regions.

Along the surface $t = 0$ the Killing vector ∂_η is a hypersurface-orthogonal timelike Killing vector except for the single point $x = 0$ where it vanishes. We can therefore use it to define a set of positive and negative frequency modes on which we can build a Fock space for the scalar-field Hilbert space. The massless Klein–Gordon equation in Rindler coordinates takes the form

$$\Box\phi = \mathrm{e}^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0\,. \tag{10.73}$$

Therefore a normalised plane wave

$$g_k = \frac{1}{\sqrt{4\pi\omega}}\mathrm{e}^{-\mathrm{i}\omega\eta + \mathrm{i}k\xi}\,, \qquad \omega = |k|\,, \tag{10.74}$$

solves the equation and has positive frequency with respect to ∂_η since

$$\mathcal{L}_{\partial_\eta} g_k = -\mathrm{i}\omega g_k\,. \tag{10.75}$$

However this is only true in region I since we need our modes to be positive frequency with respect to a future directed Killing vector, in region IV the relevant Killing vector is $\partial_{-\eta} = -\partial_\eta$. To remove this problem of defining the modes we introduce two sets of modes one with support in region I and one with support in region IV:

$$g_k^{(1)} = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega\eta + \mathrm{i}k\xi} & I \\ 0 & IV \end{cases}$$
$$g_k^{(2)} = \begin{cases} 0 & I \\ \frac{1}{\sqrt{4\pi\omega}} \mathrm{e}^{\mathrm{i}\omega\eta + \mathrm{i}k\xi} & IV \end{cases} \tag{10.76}$$

with $\omega = |k|$ in each region. These then define the positive frequency with respect to the relevant future directed timelike Killing vector. The two sets with their conjugates form a complete set of modes for any solution to the wave equation throughout the spacetime. Both sets are non-vanishing in regions II and III however this is obscured by the choice of (η, ξ) coordinates. Denoting the associated annihilation and creation operators as $b_k^{(i)}$ and $b_k^{(i)\dagger}$, we can write

$$\phi = \int \mathrm{d}k \Big(b_k^{(1)} g_k^{(1)} + b_k^{(1)\dagger} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)\dagger} g_k^{(2)*} \Big) \, . \tag{10.77}$$

This gives an alternative expansion to the original Minkowski modes:

$$\phi = \int \mathrm{d}k \big(a_k f_k + a_k^\dagger f_k^* \big) \, . \tag{10.78}$$

The inner product of the Rindler modes gives

$$(g_{k_1}^{(i)}, g_{k_2}^{(j)}) = \delta^{ij} \delta(k_1 - k_2) \, , \tag{10.79}$$

and similarly for the conjugate modes. There are two sets of modes, Minkowski and Rindler, that we can expand the solution of the Klein–Gordon equation in. Although the Hilbert spaces are the same the Fock spaces are different, in particular the definition of the vacuum. The Minkowski vacuum $|0_M\rangle$ satisfies

$$a_k |0_M\rangle = 0 \, , \tag{10.80}$$

while the Rindler vacuum satisfies

$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0 \, . \tag{10.81}$$

We see that because an individual Rindler mode cannot be written in terms of positive frequency Minkowski modes, the Rindler annihilation modes are a superposition of both the Minkowski creation and annihilation operators.

A Rindler observer will be static with respect to orbits of the boost Killing vector ∂_η. Such an observer in region I will describe particles in terms of the Rindler modes $g_k^{(1)}$ and will observer a state in the Rindler vacuum to be devoid of particles, a state $b_k^{(1)\dagger}|0_R\rangle$ to contain a single particle of frequency $\omega = |k|$ and so forth. Conversely a Rindler observer travelling through the Minkowski vacuum state will detect a background of particles, even though to the inertial observer the vacuum is completely empty.

We would like to know what kind of particles does the Rindler observer detect? We know how to answer this, we need to compute the Bogolubov coefficients relating the Minkowski modes to the Rindler modes, and then use this to compute the expectation values. Unruh found a shortcut to this somewhat tedious computation. His idea was to find a set of modes that share the same vacuum as the Minkowski modes but for which the overlap with the Rindler modes is more direct. We start with the Rindler modes and extend them to all of spacetime, and then express the extension in terms of the original Rindler modes.

We have

$$\begin{aligned} \mathrm{e}^{-a(\eta-\xi)} &= \begin{cases} a(x-t) & I \\ a(t-x) & IV \end{cases} \\ \mathrm{e}^{a(\eta+\xi)} &= \begin{cases} a(t+x) & I \\ -a(t+x) & IV \end{cases} \end{aligned} \tag{10.82}$$

We can express the spacetime dependence of a mode $g_k^{(1)}$ with $k > 0$ in terms of the Minkowski coordinates in region I as

$$\sqrt{4\pi\omega}\, g_k^{(1)} = a^{\mathrm{i}\omega/a}(x-t)^{\mathrm{i}\omega/a}\,. \tag{10.83}$$

The analytic continuation of this throughout all of spacetime is then obvious, we just use this final expression for all (t, x). We want to express the result in terms of the Rindler modes everywhere and so we need to bring the $g_k^{(2)}$ modes into the game. We have

$$\sqrt{4\pi\omega}\, g_k^{(2)} = a^{-\mathrm{i}\omega/a}(-t-x)^{-\mathrm{i}\omega/a}\,. \tag{10.84}$$

This doesn't match the behaviour of our analytically extended mode, however if we take the complex conjugate and reverse the wave number we find

$$\sqrt{4\pi\omega}\, g_{-k}^{(2)*} = a^{\mathrm{i}\omega/a}\mathrm{e}^{\pi\omega/a}(-t+x)^{\mathrm{i}\omega/a}\,, \tag{10.85}$$

and therefore

$$\sqrt{4\pi\omega}\left(g_k^{(1)} + \mathrm{e}^{-\pi\omega/a} g_{-k}^{(2)*}\right) = a^{\mathrm{i}\omega/a}(-t+x)^{\mathrm{i}\omega/a}\,. \tag{10.86}$$

An identical result holds for the $k < 0$ modes. The properly normalised mode is

$$h_k^{(1)} = \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left(\mathrm{e}^{\pi\omega/(2a)} g_k^{(1)} + \mathrm{e}^{-\pi\omega/(2a)} g_{-k}^{(2)*}\right). \tag{10.87}$$

This is the appropriate analytic extension of the $g_k^{(1)}$ modes, the extension of the $g_k^{(2)}$ modes is

$$h_k^{(2)} = \frac{1}{\sqrt{2\sinh\frac{\pi\omega}{a}}}\left(\mathrm{e}^{\pi\omega/(2a)}g_k^{(2)} + \mathrm{e}^{-\pi\omega/(2a)}g_{-k}^{(1)*}\right). \tag{10.88}$$

One can check that these are correctly normalised. We can now expand in these modes as

$$\phi = \int \mathrm{d}k\left(c_k^{(1)}h_k^{(1)} + c_k^{(1)\dagger}h_k^{(1)*} + c_k^{(2)}h_k^{(2)} + c_k^{(2)\dagger}h_k^{(2)*}\right). \tag{10.89}$$

The modes $h_k^{(i)}$ can be expressed purely in terms of positive frequency Minkowski modes f_k and therefore they share the same vacuum state $|0_M\rangle$ so that

$$c_k^{(i)}|0_M\rangle = 0\,. \tag{10.90}$$

In the Minkowski vacuum an observer in region I will observe particles defined by the operators $b_k^{(1)}$; the expected number of such particle of frequency ω is

$$\begin{aligned}\langle 0_M|n_R^{(1)}(k)|0_M\rangle &= \langle 0_M|b_k^{(1)\dagger}b_k^{(1)}|0_M\rangle \\ &= \frac{1}{2\sinh\frac{\pi\omega}{a}}\langle 0_M|\mathrm{e}^{-\pi\omega/a}c_{-k}^{(1)}c_{-k}^{(1)\dagger}|0_M\rangle \\ &= \frac{1}{\mathrm{e}^{2\pi\omega/a}-1}\delta(0)\,.\end{aligned} \tag{10.91}$$

Planck's law describes the spectral density of electromagnetic radiation emitted by a black body in thermal equilibrium at a give temperature T. It says that the spectral radiance of a body for frequency ω at temperature T is given by

$$B(\omega,T) = \frac{\hbar\omega^3}{4\pi^2c^2}\frac{1}{\mathrm{e}^{\hbar\omega/(K_BT)}-1}\,. \tag{10.92}$$

We conclude that an observer moving with uniform acceleration through the Minkowski vacuum observes a thermal spectrum of particles. (There is more to saying this is a thermal spectrum than just the above, one needs to check that there are no hidden correlations in the observed particles, this has indeed been shown and therefore the radiation detected by a Rindler observer is truly thermal.)

The temperature $T = \frac{a}{2\pi}$ is what would be measured by an observer moving along the path $\xi = 0$, which feels the acceleration $a = \alpha$. Any other path with $\xi =$ constant feels an acceleration

$$\alpha = a\mathrm{e}^{-a\xi}\,, \tag{10.93}$$

and thus should measure thermal radiation with a temperature $T = \frac{\alpha}{2\pi}$. As $\xi \to \infty$ the temperature goes to 0, which is consistent with the fact that near ∞ the Rindler observer is nearly inertial.

The Unruh effect tells us that an accelerated observer will detect particles in the Minkowski vacuum state. An inertial observer would say that the same state is completely empty, the expectation value of the energy momentum tensor $\langle T_{\mu\nu} \rangle = 0$. If there is no energy momentum how can the Rindler observer detect particles? If the Rindler observer is to detect background particles, they must carry a detector. This must be coupled to the particle being detected. However if a detector is being maintained at constant acceleration, energy is not conserved. From the point of view of the Minkowski observer the Rindler detector emits as well as absorbs particles, once the coupling is introduced the possibility of emission is unavoidable. When the detector registers a particle the inertial observer would say that it had emitted a particle and felt a radiation-reaction force in response. Ultimately the energy needed to excite the Rindler detector does not come from the background energy momentum tensor but from the energy we put into the detector to keep it accelerating.

10.5 Hawking temperature

We may now use a very quick argument following the above to conclude that a black hole has a temperature. Consider a static observer at radius $r_1 > R_S$ outside the Schwarzschild black hole. Such an observer moves along orbits of the time-like Killing vector $K = \partial_t$. The red-shift factor is given by

$$V = \sqrt{1 - \frac{2G_N M}{r}}\,, \tag{10.94}$$

and the magnitude of the acceleration is given by

$$a = \frac{G_N M}{r\sqrt{r - 2G_N M}}\,. \tag{10.95}$$

For observed close to the event horizon $r_1 - 2G_N M \ll 2G_N M$ this acceleration becomes very large compared to the scale set by the Schwarzschild radius

$$a_1 \gg \frac{1}{2G_N M}\,. \tag{10.96}$$

Let us assume that the quantum state of some scalar field ϕ looks like the Minkowski vacuum as seen by a freely falling observer near the black hole. The static observer looks just like a constant acceleration observer in flat spacetime and will detect Unruh radiation at a temperature $T_1 = a_2/(2\pi)$.

Now consider a static observer at infinity. The radiation will propagate to infinity with an appropriate red-shift factor. We find

$$T_\infty = \frac{V_1}{V_\infty}\frac{a}{2\pi}\,. \tag{10.97}$$

At infinity we have $V_\infty = 1$ so the observed temperature is

$$T_\infty = \lim_{r_1 \to 2G_N M} \frac{V_1 a_1}{2\pi} = \frac{\kappa}{2\pi} \,. \tag{10.98}$$

This is the Hawking effect and the radiation is known as Hawking radiation.

We can be more rigorous in the derivation of the Hawking temperature. Consider a spacetime that corresponds to a spherically symmetric collapsing star which forms a black hole, recall that the Penrose diagram is given in 5. This is a curved spacetime which is globally hyperbolic, for instance $\mathscr{I}^-$ is a Cauchy surface. Even though the Schwarzschild black hole solution is a static spacetime the collapsing star is not, and involves complicated dynamics. However the spacetime is approximately stationary in the far asymptotic past (near $\mathscr{I}^+$) and the far asymptotic future (near $\mathscr{I}^+$). We can therefore perform second quantisation with respect to stationary observers near $\mathscr{I}^-$ which give us "in"-modes and the "in"-vacuum and also a second quantisation associated with stationary observers at $\mathscr{I}^+$ leading to the "out"-vacuum. We have a sandwich spacetime and we can ask will observes in the far future see particles in the *in*-vacuum.

The field expansion defining the *in*-vacuum can be constructed by specifying a complete set of positive frequency modes on $\mathscr{I}^-$. For the quantisation in the far future $\mathscr{I}^+$ is not a Cauchy surface for the spacetime, one must take $\mathscr{I}^+ \cup \mathcal{H}^+$. We may therefore quantise the field in the far future by specifying a complete set on it. There are three sets of modes:

$$\begin{aligned} &f_i : \text{positive frequency on } \mathscr{I}^- \\ &g_i : \text{positive frequency on } \mathscr{I}^+ \text{ and zero on } \mathcal{H}^+ \\ &h_i : \text{positive frequency on } \mathcal{H}^+ \text{ and zero on } \mathscr{I}^+ \end{aligned} \tag{10.99}$$

Strictly speaking there is no timelike Killing vector on $\mathcal{H}$ so the term positive frequency is somewhat misleading, however the choice of modes h_i does not affect the outcome of the calculation. We can choose an arbitrary set and call them positive frequency modes and attach them to annihilation operators in the field expansion, we only require that the set $\{g, h\}$ give a basis of modes. We can therefore expand

$$\phi(x) = \sum_i a_i f_i(x) + \text{h.c.} = \sum_I b_I g_I(x) + \sum_\alpha c_\alpha h_\alpha(x) + \text{h.c.}\,. \tag{10.100}$$

The Bogoliubov coefficients in the expansion satisfy

$$g_i = \sum_j \left(A_{ij} f_j + B_{ij} f_j^* \right) . \tag{10.101}$$

We now want to look at the analytic solutions of the Klein–Gordon equation in the Schwarzschild black hole background. This is hard. Instead we can ask if we impose boundary condition to the solution at $\mathscr{I}^+$ and investigate what its corresponding form must be on $\mathscr{I}^-$. This amounts to tracing back in time the solution from $\mathscr{I}^+$ to $\mathscr{I}^-$.

The metric of the Schwarzschild black hole spacetime with coordinates (t, r_*, θ, ϕ) reads

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r_*^2\right) + r^2 \mathrm{d}s^2(S^2)\,. \tag{10.102}$$

We will also use the light-cone coordinates $u = t - r_*$ and $v = t + r_*$. We can find the Klein–Gordon equation for the field $\phi(t, r_*, \theta, \phi)$. Expanding in spherical harmonics

$$\phi(t, r_*, \theta, \phi) = \chi_l(t, r_*) Y_{lm}(\theta, \phi)\,, \tag{10.103}$$

we find

$$\left[\partial_t^2 - \partial_{r_*}^2 + V_l(r_*)\right]\chi_l = 0\,, \tag{10.104}$$

where

$$V_l(r_*) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]. \tag{10.105}$$

We set

$$\chi_l(t, r_*) = \mathrm{e}^{-\mathrm{i}\omega t} R_{l\omega}(r_*)\,, \tag{10.106}$$

so that

$$(\partial_{r_*}^2 + \omega^2) R_{\omega l} = V_l R_{\omega l}\,. \tag{10.107}$$

We can get some intuition by looking at the potential. Both near the horizon $\mathcal{H}^+$ $(r_* \to -\infty)$ and near $\mathscr{I}^\pm$ $(r_* \to \infty)$ the potential tends to zero. It takes the for of a potential barrier. If we consider how any solution to the above evolves in time, it will be partly transmitted and partly reflected as it comes in from $r_* = \infty$.

Near $\mathscr{I}^\pm$ the solutions are just plane waves. We define outgoing and ingoing as those which correspond to r_* increasing or decreasing with time. We define the *early modes*

$$\begin{aligned} f_{lm\omega+} &= \frac{1}{\sqrt{2\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega u} \frac{Y_{lm}}{r}\,, \quad \text{outgoing} \\ f_{lm\omega-} &= \frac{1}{\sqrt{2\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega v} \frac{Y_{lm}}{r}\,, \quad \text{ingoing} \end{aligned} \tag{10.108}$$

at $\mathscr{I}^-$ and *late modes*

$$\begin{aligned} g_{lm\omega+} &= \frac{1}{\sqrt{2\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega u} \frac{Y_{lm}}{r}\,, \quad \text{outgoing} \\ g_{lm\omega-} &= \frac{1}{\sqrt{2\pi\omega}} \mathrm{e}^{-\mathrm{i}\omega v} \frac{Y_{lm}}{r}\,, \quad \text{ingoing} \end{aligned} \tag{10.109}$$

at $\mathscr{I}^+$. We will be interested mainly in ingoing early modes and outgoing late modes, so we will use the shorthand notation:

$$f_\omega \sim f_{lm\omega -}\,, \qquad g_\omega \sim g_{lm\omega +}\,. \tag{10.110}$$

We need to express g_ω in terms of $f_{\omega'}$ and $f^*_{\omega'}$ on $\mathscr{I}^-$. First note that plane waves such as g_ω are in fact completely delocalised since they have support everywhere on $\mathscr{I}^+$.

We want to trace the solution of the late modes back in time in terms of the early modes. As the wave travels inwards from $\mathscr{I}^+$ toward decreasing values of r_*, it will encounter the potential barrier. One part of the wave, $g^{(r)}_\omega$ will be reflected and end up on $\mathscr{I}^-$ with the same frequency ω. This will correspond to a term of the form $A_{\omega\omega'} \propto \delta(\omega - \omega')$ in the expansion in (10.101). The remaining part $g^{(t)}_\omega$ will be transmitted through the barrier and will enter the collapsing matter. In that region the precise geometry of spacetime is unknown. However since we are interested in a packet peaked at late times and at some finite frequency ω_0 we know that the packet will be peaked at a very high frequency as it enters the collapsing matter due to the gravitational blueshift. This allows us to assume that the packet will obey the *geometric optics approximation* which means that g_ω takes the form $A(x)\mathrm{e}^{\mathrm{i}S(x)}$ where $A(x)$ is slowly varying compared to S. Substituting into the Klein–Gordon equation we find $\nabla_\mu S \nabla^\mu S = 0$, which means that surfaces of constant phase are null. Given a wave we can trace its surfaces of constant phase back in time by following null geodesics.

Consider tracing back the wave along a particular null geodesic γ which starts off at some $u = u_0$ at $\mathscr{I}^+$ and hits $\mathscr{I}^-$ at $v = v_0$. Denote by γ_H a null generator of the horizon $\mathcal{H}^+$ which has been extended into the past until it hits $\mathscr{I}^-$ at some value of v. We may set this value to $v = 0$ without loss of generality since the spacetime is invariant under shifts $v \to v + c$. We therefore have $v_0 < 0$ for the geodesic γ. Let n be a connecting vector between the two curves and fix its normalisation by requiring $n \cdot \xi = -1$ with ξ the generator of the Killing horizon $\mathcal{H}^+$. Near the horizon the Kruskal coordinate $U = -\mathrm{e}^{-\kappa u}$ is an affine distance along n and we can use it to measure the distance between γ and γ_H. In order to find the form of the wave at $\mathscr{I}^-$ we need to understand how the affine distance along the connecting vector n will change by the time γ reaches $\mathscr{I}^-$. At $\mathscr{I}^-$ the cooridnate v is an affine parameter aong the null geodesic integral curves of n. If $U_0 = 0$ then the affine distance is zero at $\mathscr{I}^-$. Hence we can expand the affine distance between γ and γ_H at $\mathscr{I}^-$ in powers of U_0: $v = cU_0 + \mathcal{O}(U_0^2)$ for some constant $c > 0$. Using $u = -\kappa^{-1}\log(-U) = -\kappa^{-1}\log(-cv)$ we can conclude that if a mode takes the form $g_\omega \sim \mathrm{e}^{-\mathrm{i}\omega u}$ on $\mathcal{I}^+$, the transmitted part $g_\omega^{(t)}$ on $\mathscr{I}^-$ will take the form

$$g_\omega^{(t)} \sim \begin{cases} \mathrm{e}^{\mathrm{i}\omega/\kappa \log(-v)} & \text{for} \quad v < 0 \\ 0 & \text{for} \quad v > 0 \end{cases} \tag{10.111}$$

up to a constant phase. This is exactly analogous to the Rindler modes in the previous section with $\kappa \leftrightarrow a$. We have $A_{\omega\omega'} = \mathrm{e}^{-\pi\omega/\kappa} B_{\omega\omega'}$ and therefore

$$\langle N_\omega \rangle \propto \frac{1}{\mathrm{e}^{\hbar\omega/(k_B T)} - 1}\,, \tag{10.112}$$

where the *Hawking temperature* is given by

$$T = \frac{\hbar\kappa}{2\pi k_B}\,. \tag{10.113}$$

Since the temperature is inversely proportional to the mass, the black hole hets up as it evaporates.

10.6 Black hole evaporation

If a black hole has a temperature it must evaporate. This leads to a serious problem with unitarity. We can compute the rate of mass loss due to the Hawking radiation. Stefan's law for the rate of energy loss by a blackbody:

$$\frac{\mathrm{d}E}{\mathrm{d}t} \sim -\alpha A T^4\,, \tag{10.114}$$

Plugging in $E = M$ and $A \propto M^2$ and $T \propto M^{-1}$ we have

$$\frac{\mathrm{d}M}{\mathrm{d}T} \propto -\frac{1}{M^2}\,, \tag{10.115}$$

and hence the black hole evaporates away completely in a time

$$\tau \sim \frac{G_N^2}{\hbar c^4} M^3\,, \tag{10.116}$$

note that the calculation of Hawking radiation assumed no backreaction, M was taken to be constant. This is good when $\frac{\mathrm{d}M}{\mathrm{d}t} \ll M$ but fails in the final stages of evaporation.

Consider a black hole which forms from collapsing matter and then evaporates away completely, leaving just thermal radiation. It should be possible to arrange that the collapsing matter is in a definite quantum state $|\psi\rangle$, the associated density matrix would be the one of a pure state, $\rho = |\psi\rangle\langle\psi|$. When the black hole is formed the Hilbert space naturally splits into the tensor product of Hilbert spaces, one with support in the interior of the black hole and the other with support on the exterior of the black hole: $\mathcal{H} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$. An outside

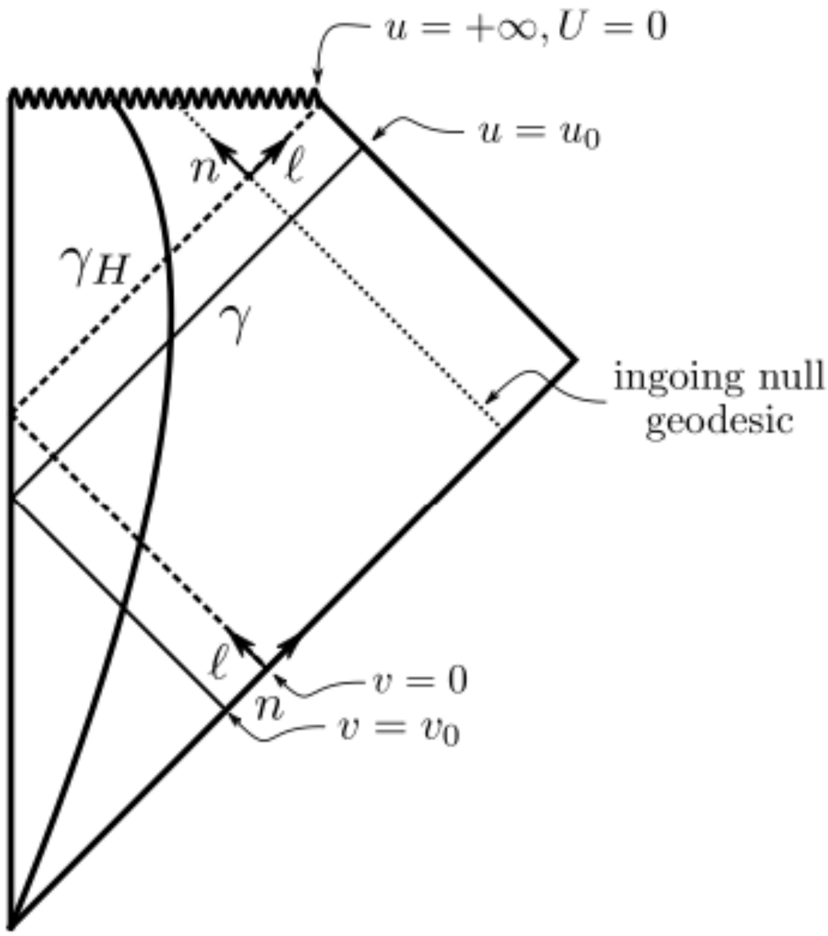

Figure 22: The evolution of the modes.

observer does not have access to $\mathcal{H}_{\text{in}}$ so their description of the black hole state is necessarily incomplete. They will describe the state outside the horizon as a reduced density matrix obtained by tracing over $\mathcal{H}_{\text{in}}$: $\rho_{\text{out}} = \text{tr}_{\text{in}}\rho$.

Since it described by a non-trivial density matrix the outside state is mixed. This is consistent with the fact that it contains thermal radiation, so there is no issue so far. The external state is entangled with the interior and the reduced density matrix is just a way in which the outside observer parametrises their ignorance of the interior. If we assume that the black hole has completely evaporated nothing is left in the interior and the exterior reduced density matrix will describe the full state, which is therefore a mixed state. However evolution from a pure state to a mixed state is forbidden by unitarity in quantum mechanics.

This is the black hole information paradox. It is important to emphasise the difference between thermal radiation produced in ordinary processes which do not violate unitarity. If we burn a printed copy of these lecture notes, thermal radiation is produced, however the process is unitary and in principle one could reconstruct all the information contained in the notes by studying the radiation and ashes. The early radiation is entangled with excitations inside the burning body, however the excitations inside the burning body can still transmit information to the radiation emitted later on which will thus contain non-trivial information. On the other hand, throwing the notes into a black hole, the information appears to be really lost once the black hole has fully evaporated because the final radiation is exactly thermal. The internal excitations are shielded by the horizon and by causality cannot influence the later outgoing radiation.

Nearly half a century after Hawking formulated the black hole information paradox it is still and open and active area of research. Our analysis has been in a funny hybrid theory of quantum field theory coupled to classical general relativity. General relativity predicts a singularity at the centre of a black hole, this is a regime where quantum effects will dramatically alter our classical expectations. **We need a quantum theory of gravity.**

References

[1] S. M. Carroll, *Spacetime and Geometry*. Cambridge University Press, 7, 2019.

[2] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, 10.7208/chicago/9780226870373.001.0001.

[3] J. M. Bardeen, B. Carter and S. W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161–170.