Geometric Group Theory

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Some inspirational quotations

John von Neumann: "If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

Herman Weyl: "In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain."

Guillermo Moreno: "Groups, as men, will be known by their actions."

Resources

https://courses.maths.ox.ac.uk/course/view.php?id=747

- Lecture notes 2023.
- Problem Sheet 0.
- Four other Problem sheets.
- Broadening course mini-projects.
- Hand-out Notes with material from previous years.
- Books: J.P. Serre - Trees
  D. – Kapovich: Geometric Group Theory (Chapters 7,8,11).

Plan of the course

We will be studying countable infinite groups. Unless otherwise stated, all groups will be finitely generated.

We have two types of infinite groups:

- "Small": Infinite abelian $\subset$ Nilpotent $\subset$ Polycyclic $\subset$ Solvable (see MT Infinite Groups C2.4).
- "Large": Free groups $\subset$ Hyperbolic groups;
  Free groups $\subset$ Amalgamated products.

Methods used

(1) Geometric Approach: Make the group G act on an interesting metric space X. We can then deduce algebraic properties of G from

- the geometry of X
- the properties of the action of G on X

For example, X might be a Hilbert space (in particular $\mathbb{R}^n$) or a differentiable/Riemannian manifold. In this course we will be considering the case when X is a simplicial tree, and then towards the end of the course, when $X = \mathbb{H}^2_{\mathbb{R}}$ or $X = \mathbb{H}^n_{\mathbb{R}}$.

(2) Algorithmic Approach: We can design algorithms and construct Turing machines to solve algebraic questions. This works for groups described by finite data (i.e. finitely presented groups). Dehn (1912) formulated 3 fundamental problems:

I Word Problem

II Conjugacy Problem

III Isomorphism Problem

Methods used

Dehn solved the three problems for G acting on $\mathbb{H}^2$ by isometries, $G \leq \mathrm{Isom}(\mathbb{H}^2)$ discrete and $\mathbb{H}^2/G$ compact.

However, the problems are unsolvable in general (Novikov–Boone). The example of a group with unsolvable word problem that Novikov–Boone constructed acts on a tree.

(3) Approximation by finite groups: Idea is to take finite quotients G/N_k that become larger and larger. Ideally $\bigcap N_k = \{1\}$.

- Residually finite groups.

(4) Understanding of subgroups:

- What kind of subgroups?
- Can decompose G into "large building blocks"?

An example for "small" groups: finitely generated Abelian groups. There are analogous theories for "large" groups.

Methods used

The main trick of GGT is to move between:

Discrete world :

- Graphs

Toolkit: Combinatorics, discrete maths

Continuous world:

- Differentiable manifolds $\supseteq$ Riemannian manifolds

Toolkit: Calculus

Dehn used the geometry of a continuous space (the real hyperbolic plane) to solve a problem about a discrete group. This can be done backwards: use a group/discrete structure to solve a problem about a continuous space.

Generating sets

Given $S \subset G$ and $H \leq G$, TFAE

- H is the smallest subgroup of G containing S;
- $H = \bigcap_{S \subset K \leq G} K$;
- $H = \{s_1^{\pm 1} s_2^{\pm 1} \dots s_n^{\pm 1} \mid n \in \mathbb{N}, s_i \in S\} \cup \{\mathrm{id}\}$.

H is called the subgroup generated by S. We write $H = \langle S \rangle$.

- If $H = G$ then S is called a generating set.
- If S finite then G is called finitely generated.
- If $S = \{x\}$ then $\langle x \rangle$ is the cyclic subgroup generated by x.
- Rank of G= minimal number of generators.

Free groups

What is "the largest infinite group" generated by n elements?

Finite sets: A larger than $B \Leftrightarrow \mathrm{card}(A) \geq \mathrm{card}(B) \Leftrightarrow$ there exists $f : A \to B$ onto.

Infinite groups: We look for a group $G = \langle X \rangle$, $\mathrm{card}(X) = n$, such that for every group $H = \langle Y \rangle$, $\mathrm{card}(Y) = n$, a bijection $X \to Y$ extends to an onto group homomorphism.

Clearly cannot be done for any group G, e.g. if G is abelian then H would have to be abelian.

So G must be a group with no prescribed relation ("free").

Construction of a free group

$X \neq \emptyset$. Its elements = letters/symbols.

Take inverse letters/symbols $X^{-1} = \left\{ a^{-1} \mid a \in X \right\}$.

We call $\mathcal{A} = X \sqcup X^{-1}$ an alphabet.

A word w in $\mathcal{A}$ is a finite (possibly empty) string of letters in $\mathcal{A}$

$$a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_k}^{\epsilon_k},$$

where $a_i \in X, \epsilon_i = \pm 1$.

The length of w is k.

We will use the notation 1 for the empty word (the word with no letters). We say it has length 0.