Geometric Group Theory

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Construction of a free group

Given n, we want to construct "the largest infinite group" generated by n elements.

This must be a group with no prescribed relation ("free").

Take $X \neq \emptyset$. Its elements = letters/symbols.

Take inverse letters/symbols $X^{-1} = \left\{a^{-1} \mid a \in X\right\}$.

We call $\mathcal{A} = X \sqcup X^{-1}$ an alphabet.

A word w in $\mathcal{A}$ is a finite (possibly empty) string of letters in $\mathcal{A}$

$$a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_k}^{\epsilon_k},$$

where $a_i \in X, \epsilon_i = \pm 1$. The length of w is k.

We will use the notation 1 for the empty word (the word with no letters).

We say it has length 0.

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A word w is reduced if it contains no pair of consecutive letters of the form aa^{-1} or $a^{-1}a$.

The reduction of a word w is the deletion of all pairs of consecutive letters of the form aa^{-1} or $a^{-1}a$.

An insertion is the opposite operation: insert one or several pairs of consecutive letters of the form aa^{-1} or $a^{-1}a$.

Denote by X^* the set of words in the alphabet $\mathcal{A} = X \sqcup X^{-1}$, empty word included.

Denote by $F(X)$ the set of reduced words in $\mathcal{A}$, empty word included.

We define an equivalence relation on X^* by $w \sim w'$ if w' can be obtained from w by a finite sequence of reductions and insertions.

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Proposition

$\forall w \in X^*$, there exists a unique $u \in F(X)$ such that $w \sim u$.

Proof

Existence: By induction on the length.

$$w = a_1 a_2 ... a_{n+1} \sim a_1 b_1 ... b_k \quad \text{if } a_1 b_1 \neq xx^{-1}, x^{-1}x$$
$$\sim b_2 ... b_k \quad \text{otherwise}$$

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Uniqueness: We prove that if $u, v \in F(X)$, $u \neq v$, then we cannot have $u \sim v$.

Argue by contradiction and assume we can. So there exists a sequence of reductions and insertions

$$w_0 = u \sim w_1 \sim w_2 \sim \ldots \sim w_{n-1} \sim w_n = v$$

Take a sequence with $\sum |w_i|$ minimal. As u and v are reduced, $w_0 \to w_1$ is an insertion and $w_{n-1} \to w_n$ is a reduction. Hence $|w_0| < |w_1|$ and $|w_{n-1}| > |w_n|$. So there exists some i such that $|w_{i-1}| < |w_i| > |w_{i+1}|$. Say,

$$w_{i-1} \to w_i \text{ is an insertion of } aa^{-1}, \quad a \in \mathcal{A}$$
$$w_i \to w_{i+1} \text{ is a deletion of } bb^{-1}, \quad b \in \mathcal{A}$$

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- If aa^{-1} and bb^{-1} are the same letters in w_i, then we can suppress w_i and take $w_{i-1} = w_{i+1}$.
- If aa^{-1} and bb^{-1} are disjoint in w_i then we change the order: first delete bb^{-1}, then insert aa^{-1}.
- If aa^{-1} and bb^{-1} have one letter in common in w_i, for example:

$$\begin{aligned} w_{i-1} &= [...xyz...] \\ w_i &= [...xaa^{-1}yz...] \quad , \quad y = a \\ w_{i+1} &= [...xaz...] \end{aligned}$$

then we can take $w_{i-1} = w_{i+1}$. All three are decreasing $\sum |w_i|$, a contradiction. □

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Definition

The free group over X is the set $F(X)$ endowed with the product $*$ defined by: $w * w'$ is the unique reduced word equivalent to the word ww'. The unit is the empty word.

Example

1. If $\#X = 1$ then $F(X) \simeq \mathbb{Z}$.
2. IF $\#X \geq 2$ then $F(X)$ not abelian.

Terminology: We say that a free non-abelian group is a group $F(X)$ with $\mathrm{card}(X) \geq 2$.

Examples of free groups in real life: the ping-pong lemma

Example

Take $r \in \mathbb{R}$, $r \geq 2$,

$$g_1 = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}.$$

$SL(2, \mathbb{R})$ acts by isometries on $\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$, *via*

$$g \cdot z = \frac{az+b}{cz+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

Statement: We have that $\langle g_1, g_2 \rangle \leq SL(2, \mathbb{R})$ is isomorphic to $F(\{g_1, g_2\})$.

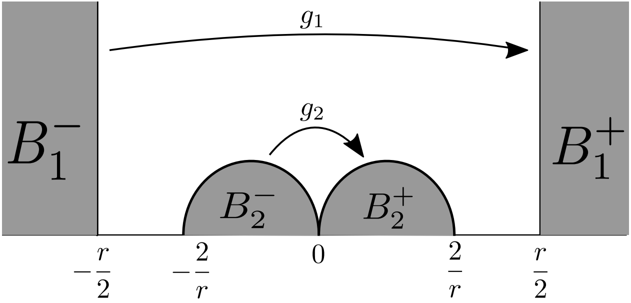
Why $\langle g_1, g_2 \rangle$ is free

Statement: We have that $\langle g_1, g_2 \rangle \leq SL(2, \mathbb{R})$ is isomorphic to $F(\{g_1, g_2\})$

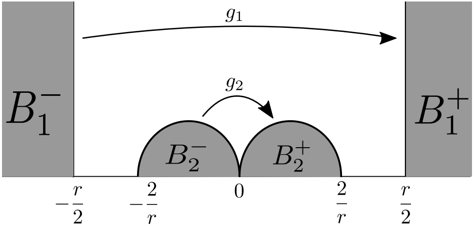
Proof

$g_1(z) = z + r, \; g_2(z) = \frac{z}{rz+1}$.

$I(z) = -\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z$, $g_2 = I \circ g_1^{-1} \circ I^{-1}$.



Why $\langle g_1, g_2 \rangle$ is free



$g_1(\mathbb{H}^2 \setminus B_1^-) \subset B_1^+$, $g_1^{-1}(\mathbb{H}^2 \setminus B_1^+) \subset B_1^-$.
g_2 does the same for $B_2^- = I(B_1^+)$, $B_2^+ = I(B_1^-)$.
Let $w = a_1 ... a_k$, $a_i \in \{g_1^\pm, g_2^\pm\}$, w non-trivial in $F(\{g_1^\pm, g_2^\pm\})$.

- Test 1: $\forall z \in \mathbb{H}^2 \setminus \bigcup_{i=1}^2 B_i^\pm$, $w(z) \in \bigcup_{i=1}^2 B_i^\pm$. And so $w(z) \neq z$.
- Test 2: Take $M = B_1^- \cup B_1^+$, $N = B_2^- \cup B_2^+$. Then $M \cap N = \emptyset$ and $g_1^n(N) \subset M$, $g_2^n(M) \subset N$. Then use a general ping-pong result (see Ex.3, Sheet 1). □

Free groups are the largest

Proposition (Universal property of free groups)

Let X be a set and let G be a group. A map $\varphi : X \to G$ has a unique extension

$$\Phi : F(X) \to G$$

that is a group homomorphism.

Proof

- φ can be extended to a map on $X \cup X^{-1}$ by $\varphi(a^{-1}) = \varphi(a)^{-1}$.
- For every reduced word $w = a_1 \cdots a_n$ in $F = F(X)$ define

$$\Phi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n).$$

- Set $\Phi(1_F) := 1_G$, the identity element of G.

Free groups are the largest

Uniqueness:
Let $\Psi : F(X) \to G$ be a homomorphism such that $\Psi(x) = \varphi(x)$ for every $x \in X$.

Then for every reduced word $w = a_1 \cdots a_n$ in $F(X)$,

$$\Psi(w) = \Psi(a_1) \cdots \Psi(a_n) = \varphi(a_1) \cdots \varphi(a_n) = \Phi(w).$$

□

Terminology: If $\varphi(X) = Y$ is such that Φ is an injective homomorphism, $\Phi(F(X)) = H$, we say that $Y \subset G$ generates a free subgroup or that Y freely generates H.

Example
$\{g_1, g_2\}$ freely generate $\langle g_1, g_2 \rangle \leq SL(2, \mathbb{R})$.

Free groups are the largest

Corollary

Every group $G = \langle X \rangle$, $\#X = n$, is a quotient of a free group $F(X)$.

Proof.

$X \hookrightarrow G$ extends to $\Phi : F(X) \to G$. Since $X \subset Im(\Phi)$, we have that

$$G \leq Im(\Phi) \leq G$$

and so Φ is onto. □

Rank of a free group

Corollary

Consider two groups G and H, $G = \langle X \rangle$.

1. *Every homomorphism $\phi : G \to H$ is uniquely determined by $\phi\big|_X : X \to H$. In particular there are at most $|H|^{|X|}$ homomorphisms.*
2. *If moreover $G = F(X)$ then there are exactly $|H|^{|X|}$ homomorphisms.*

Proof.

1. The map $\mathrm{Hom}(G, H) \to \mathrm{Map}(X, H), \phi \mapsto \phi\big|_X$ is injective. NB. It is not in general onto.
2. For $G = F(X)$ it is onto (by the universal property).

□