Chapter 5

(C) Hailstone numbers

Given any positive integer a, let us consider the sequence $(x_n)_{n\in\mathbb{N}}$, with initial term $x_0 = a$, and for $n \geq 1$,

$$x_n = \begin{cases} x_{n-1}/2 & \text{if } x_{n-1} \text{ is even} \\ 3x_{n-1}+1 & \text{otherwise.} \end{cases} \tag{5.1}$$

5.1 Exercise (C1)

Write a `function seq = sequence(a)`, that computes the elements of the sequence $(x_n)_{n\in\mathbb{N}}$ initiated at $x_0 =$`a`, where `a` may be any positive integer. Of course, we don't ask you to compute the whole (infinite) sequence, you will stop as soon as the last element found has already appeared earlier in the sequence. The output `seq` will thus be the list of sequence elements $x_0, x_1, \ldots, x_{N_0}$, where N_0 is the smallest positive integer such that there exists $N < N_0$ with $x_N = x_{N_0}$). *Hint: you may find the Matlab function* `ismember` *useful. In case you choose to test your function over different values of the input* `a` *at this stage, we suggest you not to consider values of* `a` *that are too large, otherwise your script might run for much longer that you may expect...*

5.2 Exercise (C2)

Run your function `sequence(a)` for `a` $\in \{14, 39, 40, 62, 92\}$ and plot the elements of the sequence returned by your function for the different above-mentioned choices of `a` (i.e., for each value of `a`, create a new figure). Do you see on this graph why the elements of the sequences (5.1) are sometimes referred to as *hailstone numbers*? (If not, a simple search on the web for *hailstone numbers sequence* should satisfy your curiosity).

5.3 Exercise (C3)

Create now a new figure, and plot for `a` $= 1, 2, \ldots, 50$ the maximum element contained in the sequence returned by your function created in Exercise (C1). The labels of the axes of your figure should thus be `a` for the x-axis, and `max element` for the y-axis. Use a logarithmic scale for the y-axis (see the command `semilogy`).

5.4 Exercise (C4)

We will focus now on the last elements of the sequence returned by your function written in Exercise (C1). For $a \in \{14, 39, 40, 62, 92\}$, display the value of the sequence x_n, restricted to the interval $n \in [N, N_0]$, where N and N_0 are defined in Exercise (C1). What do you observe?

5.5 Exercise (C5)

The Collatz conjecture says that the sequence (5.1) eventually reaches the value 1 for any positive integer a. This conjecture has been verified experimentally to be true for (very large) values of a. Verify numerically the conjecture for all $a = 1, \ldots, 10000$. *You may want to write here a* `function seq = sequence2(a)`*, with the appropriate stopping criterion in the generation of the sequence...*