Further Partial Differential Equations (2023)
Problem Sheet 1

Question description

- Question 1 is non-examinable material but may be of interest to those wanting to know the origin of the governing equations covered in the course.
- Question 2 is bookwork that is mostly covered in the lectures. This question will not be marked but will give an idea of the bookwork component of exam questions.
- Questions 3 and 4 will be marked.

Questions

1. **Flow on a vertical substrate**
In lectures we used the following equation to find solutions for the spreading of liquid on a vertical wall as shown in figure 1:

$$\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial}{\partial z}\left(h^3\right) = 0. \tag{1}$$

Here, h denotes the liquid thickness, z the vertical position, t time, g acceleration due to gravity and ρ and μ are respectively the density and viscosity of the fluid. All quantities are dimensional. In this question we will derive this equation.

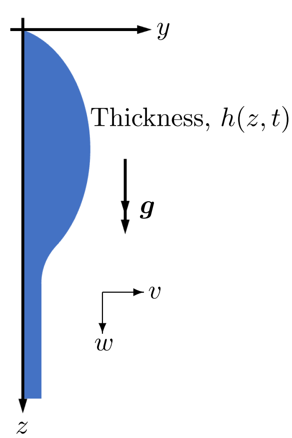

Figure 1: Schematic of liquid draining down a wall. The liquid profile is given by $h(z,t)$ at time t and vertical position z.

The Stokes equations describe the flow of viscous fluid and are given by

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2a}$$

$$-\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 0, \tag{2b}$$

$$-\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g = 0, \tag{2c}$$

where y and z are the coordinates normal and tangential to the surface and v and w are the respective velocities, p is the fluid pressure and g denotes acceleration due to gravity (see figure 1).

(a) Assume that the liquid layer is thin by scaling $y = \epsilon Y$ where $\epsilon \ll 1$ and $Y = O(1)$. In this case, we also expect the velocities in this direction to be small, so we also scale $v = \epsilon V$. The pressure in the liquid should be scaled as $p = P/\epsilon^2$. By introducing these scalings into the Stokes equations, (8), and considering the resulting system at leading order in ϵ, show that the system is governed by the equations

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3a}$$

$$\frac{\partial p}{\partial y} = 0, \tag{3b}$$

$$\frac{\partial p}{\partial z} - \rho g = \mu \frac{\partial^2 w}{\partial y^2}. \tag{3c}$$

(b) Explain the physical significance of each of the following boundary conditions:

$$w = 0 \quad \text{on} \quad y = 0, \tag{4a}$$

$$v = 0 \quad \text{on} \quad y = 0, \tag{4b}$$

$$\frac{\partial h}{\partial t} + w\frac{\partial h}{\partial z} = v \quad \text{on} \quad y = h, \tag{4c}$$

$$p = 0 \quad \text{on} \quad y = h, \tag{4d}$$

$$\frac{\partial w}{\partial y} = 0 \quad \text{on} \quad y = h. \tag{4e}$$

(c) Integrate (3a) over the thickness of the liquid and use (4) to show that the liquid thickness satisfies the following equation for mass conservation:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial z}(\overline{w}h) = 0, \tag{5}$$

where $\overline{w}$ is the average velocity parallel to the wall, defined by

$$\overline{w} = \frac{1}{h}\int_0^h w \, \mathrm{d}y. \tag{6}$$

(d) Use the remaining equations and boundary conditions to show that

$$w = -\frac{\rho g}{2\mu}(y^2 - 2yh) \tag{7}$$

and hence show that (1) governs the flow of liquid on a vertical substrate

Solution

(a) The Stokes equations are

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{8a}$$

$$-\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = 0, \tag{8b}$$

$$-\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g = 0. \tag{8c}$$

We scale $y = \epsilon Y$ where $\epsilon \ll 1$ to account for the thinness of the liquid. To retain the continuity equation we must scale $v = \epsilon V$ and the required leading-order balance for the pressure is $p = P/\epsilon^2$. Substituting this into (8) and retaining leading-order terms gives the system (3).

(b) Equation (4a) corresponds to no slip on the substrate; (4b) is no penetration; (4c) is the kinematic condition; (4d,e) are the dynamic conditions, representing respectively continuity in pressure at the interface (assuming zero atmospheric pressure without loss of generality) and no shear.

(c) Integrating (3a) over the thickness of the liquid gives

$$v(h, z) + \int_0^h \frac{\partial w}{\partial z} \, \mathrm{d}y = 0 \tag{9}$$

$$\Rightarrow v(h, z) + \frac{\partial}{\partial z} \left(\int_0^h w \, \mathrm{d}y \right) - w(h, z) \frac{\partial h}{\partial z} = 0 \tag{10}$$

using Leibniz' rule. Applying the kinematic condition (4c) and the definition of the average velocity (6) gives the required result.

(d) Equation (3b) implies that $p = p(z, t)$. Substituting this into (3c), integrating and using (4a,e) gives the required result.

2. **Similarity solutions for the flow on a vertical substrate**

Consider the governing equation for flow on a vertical substrate:

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu}\frac{\partial}{\partial \hat{z}}\left(\hat{h}^3\right) = 0. \tag{11}$$

(a) By introducing the following non-dimensionalization,

$$\hat{z} = \hat{z}_0 z, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h, \tag{12}$$

show that, for an appropriate choice of $\hat{t}_0$, the resulting dimensionless equation is

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) = 0. \tag{13}$$

Explain any restrictions that must be placed on $\hat{z}_0$ and $\hat{h}_0$ to obtain this equation.

(b) By making the ansatz

$$h(z,t) = f(\eta) \qquad \text{where} \quad \eta = \frac{z}{t^\alpha}, \tag{14}$$

show that, for a particular value of α, the governing equation (13) reduces to an ordinary differential equation for $f(\eta)$.

(c) Solve the resulting ordinary differential equation to determine the solution for $f(\eta)$ and hence use this to state the dimensionless and dimensional solutions, $h(z,t)$ and $\hat{h}(\hat{z},\hat{t})$, respectively.

(d) By replacing derivatives $\partial y/\partial x$ with Y/X where X and Y denote the typical sizes of x and y respectively, find a scaling-law approximation for h from (11). Compare this result to the one that you found in part (c).

Solution

(a) Substitution of the non-dimensionalization (12) into (11) gives the required dimensionless version (16) if we choose

$$\hat{t}_0 = \frac{\mu \hat{z}_0}{\rho g \hat{h}_0^2}.$$

This result holds for any choice in $\hat{z}_0$ and $\hat{h}_0$. These could be provided by further information, such as the profile at a given time.

(b) Substituting the ansatz (14) into (13) gives

$$-\alpha \frac{z}{t} f' + \frac{1}{3}(f^3)' = 0,$$

which transforms into an ordinary differential equation (ODE) for $f(\eta)$ if we choose $\alpha = 1$. The resulting ODE is then

$$\left(f^2 - \alpha\eta\right) f' = 0. \tag{15}$$

(c) The solution to (15) is $f = \sqrt{\eta}$ (or the trivial solution, $f =$ constant), which corresponds to

$$h = \sqrt{z/t} \qquad\qquad \hat{h} = \left(\frac{\mu}{\rho g}\right)^{1/2} \left(\frac{\hat{z}}{\hat{t}}\right)^{1/2}$$

(or h and $\hat{h}$ constant).

(d) A scaling-law analysis in (11) gives

$$\frac{H}{T} \sim \frac{\rho g}{3\mu} \frac{H^3}{Z}$$
$$\Rightarrow H \sim \sqrt{3\frac{Z}{T}}.$$

Replacing Z and T with $\hat{z}$ and $\hat{t}$, respectively, gives the same solution as in part (c), except for a different prefactor (which contains an additional $\sqrt{3}$).

3. **An analytic solution for the flow on a vertical substrate**

(a) Use the method of characteristics to show that the solution to the dimensionless equation for the flow on a vertical substrate,

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) = 0. \tag{16}$$

subject to the initial condition $h(z,0) = h_0(z)$, is given by $h(z,t) = h_0(\xi(z,t))$, where $\xi(z,t)$ satisfies the implicit relation

$$h_0(\xi)^2 t + \xi = z. \tag{17}$$

(b) By expanding for small t by setting $t = \epsilon T$ where $\epsilon \ll 1$ and $T = O(1)$ show that, for an initial profile of the form $h_0(z) = \tanh(\alpha z)$ for $z > 0$, the early time behaviour is

$$h \sim \tanh(\alpha z - \alpha \tanh^2(\alpha z) t). \tag{18}$$

(c) By expanding for large time by setting $t = T/\epsilon$ where $\epsilon \ll 1$ and $T = O(1)$ show that the long time behaviour for large $z = O(1/\epsilon)$ is

$$h \sim \sqrt{\frac{z}{t}} \tag{19}$$

if we assume that $\xi = O(1)$.

(d) Comment on how the result (19) compares with the similarity solution found in lectures for the flow of liquid on a vertical surface and the implications of this result on the use of the similarity solution.

(e) Show that if we also assume that $\xi = O(1/\epsilon)$ in (c) then the solutions are travelling waves of the form $h_0(z - t)$.

Solution

(a) Write $h(z,t) = h(z(\xi,\eta), t(\xi,\eta)) = h(\xi,\eta)$. Then

$$\frac{\partial h}{\partial \eta} = \frac{\partial h}{\partial z}\frac{\partial z}{\partial \eta} + \frac{\partial h}{\partial t}\frac{\partial t}{\partial \eta}, \tag{20}$$

using the chain rule. Expand the derivative to write (16) as

$$\frac{\partial h}{\partial t} + h^2\frac{\partial h}{\partial z} = 0. \tag{21}$$

Comparing (20) with (21) where we have expanded out the derivative, we can set

$$\frac{\partial z}{\partial \eta} = h^2, \qquad \frac{\partial t}{\partial \eta} = 1, \qquad \frac{\partial h}{\partial \eta} = 0, \tag{22a–c}$$

subject to the initial data

$$z(\xi,0) = \xi, \qquad t(\xi,0) = 0, \qquad h(\xi,0) = h_0(\xi). \tag{23a–c}$$

Integration of (22b,c) and application of (23b,c) gives

$$t = \eta, \qquad\qquad h = h_0(\xi). \tag{24}$$

Integration of (22a) and application of (23a) then gives

$$z = h_0(\xi)^2\eta + \xi. \tag{25}$$

The result then follows.

(b) Setting $t = \epsilon T$ where $\epsilon \ll 1$ and writing $\xi = z + \epsilon\zeta$ gives

$$\zeta = -\tanh^2(\alpha z)T. \tag{26}$$

Substituting this into $h(z,t) = h_0(\xi(z,t))$ gives the required result.

(c) Writing $t = T/\epsilon$ where $\epsilon \ll 1$ and substituting into (17) we obtain

$$\frac{\epsilon}{T}z = \tanh^2(\alpha\xi) + \epsilon\xi. \tag{27}$$

To obtain a leading-order balance, we must scale $z = \epsilon Z$ where $Z = O(1)$. (This means that the deformations stretch out far as t becomes large.) This then gives

$$\frac{Z}{T} = \tanh^2(\alpha\xi) \tag{28}$$

to leading order. Noting that the right-hand side of this expression is simply h^2 and writing the left-hand side in terms of the original variables gives the required result.

(d) The expression (19) is identical to the similarity solution obtained in lectures, showing that the similarity solution replicates the long-time behaviour.

(e) Substituting $t = T/\epsilon$, $z = Z/\epsilon$ and $\xi = \zeta/\epsilon$ into (25) and expanding the tanh term for large argument gives $Z = T - \zeta$, or in original variables $z = t - \xi$. Substituting this into $h(z,t) = h_0(\xi(z,t))$ then gives the required result.

4. **Spreading of oil in a frying pan: a radial gravity current**
In lectures we looked at the two-dimensional spreading of a liquid. In this question we will consider radial spreading. The height of the liquid, $\hat{h}$ in terms of the radial coordinate $\hat{r}$ and time $\hat{t}$ is given by the equation

$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta\rho g}{3\mu\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\hat{h}^3\frac{\partial \hat{h}}{\partial \hat{r}}\right) = 0, \tag{29}$$

where $\Delta\rho$ is the difference in density between the liquid and the surrounding air, g denotes acceleration due to gravity and μ is the viscosity of the liquid.

(a) Explain the physical significance of the expression

$$2\pi\int_0^{\hat{r}_f(\hat{t})} \hat{r}\hat{h}(\hat{r},\hat{t})\,\mathrm{d}\hat{r} = \hat{V}, \tag{30}$$

and the quantity $\hat{V}$, where $\hat{r}_f$ is the position of the liquid front.

(b) Non-dimensionalize the system (29) and (30) using suitable scalings.

(c) Use a scaling argument to show that

$$r_f \sim t^{1/8} \qquad\qquad h \sim t^{-1/4}, \tag{31}$$

where the lack of hats denotes dimensionless quantities.

(d) By setting $\eta = r/t^{1/8}$ and $h = t^{-1/4}f(\eta)$ derive an ordinary differential equation for f.

(e) By defining the scaled coordinate $z = \eta/\eta_f$ and $f(\eta) = \alpha g(z)$ for an appropriate choice in α that you should determine, show that g satisfies

$$\left(zg^3g'\right)' + \frac{1}{8}z^2g' + \frac{1}{4}zg = 0, \tag{32}$$

where primes denote differentiation, and the position of the moving front is given by

$$\eta_f = \left(\int_0^1 zg(z)\,\mathrm{d}z\right)^{-3/8}. \tag{33}$$

(f) Consider the behaviour near the propagating front by setting $z = 1 - \epsilon\xi$ and $g = \delta G$ where $\epsilon, \delta \ll 1$. Find an appropriate relationship between ϵ and δ that provides a leading-order balance and use this to show that the behaviour near the front is given by

$$g \sim \left(\frac{3}{8}\right)^{1/3}(1-z)^{1/3}. \tag{34}$$

Solution

(a) Equation (30) corresponds to mass conservation. There is a finite amount of liquid in the frying pan, of volume $\hat{V}$.

(b) We non-dimensionalize using

$$\hat{r} = \hat{r}_0 r, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h, \tag{35}$$

and choose

$$\hat{t}_0 = \frac{24\pi^3 \mu \hat{r}_0^8}{\Delta\rho g \hat{V}^3} \qquad \hat{h}_0 = \frac{\hat{V}}{2\pi \hat{r}_0^2}. \tag{36}$$

There is no natural length scale so $\hat{r}_0$ remains arbitrary. This could be chosen in practice using, for instance, the initial conditions.

(c) Using a scaling argument in (29) and (30) gives respectively the relationships

$$\frac{H}{T} \sim \frac{H^4}{R^2}, \qquad R^2 H \sim 1. \tag{37}$$

Rearranging gives

$$R \sim T^{1/8}, \qquad H \sim T^{-1/4}. \tag{38}$$

as required.

(d) Defining $\eta = r/t^{1/8}$ and using the chain rule gives

$$\frac{\partial}{\partial t} = -\frac{1}{8}\frac{r}{t^{9/8}}\frac{\mathrm{d}}{\mathrm{d}\eta}, \qquad \frac{\partial}{\partial r} = \frac{1}{t^{1/8}}\frac{\partial}{\partial \eta}. \tag{39}$$

Defining $h = t^{1/4} f(\eta)$ and substituting into (29) and (30) gives

$$\frac{1}{\eta}\left(\eta f^3 f'\right)' + \frac{1}{8}\eta f' + \frac{1}{4} f = 0, \tag{40}$$

$$\int_0^{\eta_f} \eta f \,\mathrm{d}\eta = 1. \tag{41}$$

Defining $z = \eta/\eta_f$ and $f(\eta) = \alpha g(\eta/\eta_f)$ gives in (40) and (41),

$$\left(z g^3 g'\right)' + \frac{1}{8} g' + \frac{1}{4} g = 0, \tag{42}$$

$$\eta_f = \left(\int_0^1 z g \,\mathrm{d}z\right)^{-3/8} \tag{43}$$

if we choose $\alpha = \eta_f^{2/3}$.

(e) Substituting $z = 1 - \epsilon\xi$ and $g = \delta G$ into (42) and seeking a leading-order balance indicates that we must choose $\delta = \epsilon^{1/3}$. This results in the leading-order equation

$$\left(G^3 G'\right)' - \frac{1}{8} G' = 0. \tag{44}$$

Integrating this equation twice and applying the boundary condition that $G(0) = 0$ (and we also require $G(0)^3 G'0) = 0$, which imposes a constraint on how steeply G approaches 0 at the moving front), gives the required result.