Geometric Group Theory

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Inspirational quotations

Henri Poincaré: "If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living."

Jean-le-Rond D'Alembert, to his students (quoted by Florian Cajori in "A history of mathematics"): "Allez en avant et la foi vous viendra."

"Keep going and faith will come later."

Corollary (unique root property)

If $g, h \in F(X)$ are such that $g^k = h^k$ for some k then $g = h$.

Question: Find a torsion-free group G for which there exist $g \neq h$ such that $g^k = h^k$ for some k.

Take $G = \langle g, h \mid g^k = h^k \rangle$.

It is an amalgamated product $G = A *_H B$, where $A = \langle g \rangle$, $B = \langle h \rangle$, and $H = \mathbb{Z} \simeq \langle g^k \rangle \simeq \langle h^k \rangle$.

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Exercise: If every pair of distinct elements have an equal power then $G = \mathrm{Tor}\, G$.

Example due to Olshanskii: There exist finitely generated, non-cyclic, torsion-free groups G where any two elements have equal powers, i.e., for any g, h there exist m, n such that $g^m = h^n$.

Tietze transformations

How do we recognise when two finite presentations give the same group?

There are two types of transformations.

(T1) Given $\langle S | R \rangle$ and $r \in \langle\langle R \rangle\rangle$, change the presentation to $\langle S | R \cup \{r\} \rangle$ (or do the inverse operation).

(T2) Given $\langle S | R \rangle$, a new symbol $a \notin S$ and $w \in F(S)$, change the presentation to $\langle S \cup \{a\} | R \cup \{a^{-1}w\} \rangle$ (or do the inverse operation).

Theorem

Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.

Proof: $(\Leftarrow)$ (T1) defines isomorphic groups because $\langle\langle R \rangle\rangle = \langle\langle R \cup \{r\} \rangle\rangle$.

Tietze transformations

Theorem

Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.

Proof continued: For (T2), consider the homomorphisms

$$\iota : F(S) \hookrightarrow F(S \cup \{a\}) \quad \text{(injection)}$$
$$f : F(S \cup \{a\}) \twoheadrightarrow F(S) \quad f(a) = w \quad \text{(surjection)}$$

Note that $f \circ \iota = \mathrm{id}_{F(S)}$. They induce homomorphisms

$$F(S) \xrightarrow{\bar{\iota}} F(S \cup \{a\})/\langle\langle a^{-1}w \rangle\rangle \xrightarrow{\bar{f}} F(S)$$

with $\bar{f} \circ \bar{\iota} = \mathrm{id}_{F(S)}$. $\bar{\iota}$ is onto, and hence $\bar{\iota}$ and $\bar{f}$ are isomorphisms. Since also $\bar{f}^{-1}(\langle\langle R \rangle\rangle) = \langle\langle R \cup \{a^{-1}w\} \rangle\rangle / \langle\langle a^{-1}w \rangle\rangle$ we have that $\bar{f}$ induces the desired isomorphism.

Tietze transformations

Theorem

Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.

Proof continued:

($\Rightarrow$) Let $G_1 = \langle S_1 | R_1 \rangle$, $G_2 = \langle S_2 | R_2 \rangle$. WLOG $S_1 \cap S_2 = \emptyset$.

There exist inverse isomorphisms $\phi : G_1 \to G_2$, $\psi : G_2 \to G_1$. $\forall s \in S_1$, choose $w_s \in F(S_2)$ representing $\phi(s)$ in G_2. $\forall t \in S_2$, choose $v_t \in F(S_1)$ representing $\psi(t)$ in G_1.

Take the two subsets of $F(S_1 \cup S_2)$:

$$U_1 = \{s^{-1} w_s : s \in S_1\} \qquad U_2 = \{t^{-1} v_t : t \in S_2\}$$

Claim: There exist finitely many Tietze transformations from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$.

Tietze transformations

Claim: There exist finitely many Tietze transformations from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$.

Proof of claim: Use finitely many (T2) to get from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$. There exists an isomorphism

$$\rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \to \langle S_1 | R_1 \rangle \quad \rho(s) = s, \forall s \in S_1 \quad \rho(t) = v_t, \forall t \in S_2$$

Then $\phi \circ \rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \to \langle S_2 | R_2 \rangle$ is an isomorphism such that $t \overset{\rho}{\mapsto} v_t \overset{\phi}{\mapsto} t$. Also, $\forall r \in R_2$

$$\phi \circ \rho(r) = r \equiv 1 \text{ in } \langle S_2 | R_2 \rangle \Rightarrow r \in \langle\langle R_1 \cup U_2 \rangle\rangle \Rightarrow R_2 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$$

Thus $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$ is related to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_2 \rangle$ by a sequence of (T1) transformations. Also, $\forall s \in S_1$

$$\phi \circ \rho(s) = w_s(t_1 ... t_k) \quad \phi \circ \rho(w_s) = \phi \circ \rho(w_s(t_1 ... t_k)) = w_s(t_1 ... t_k)$$

Hence, $s^{-1} w_s \in \langle\langle R_1 \cup U_2 \rangle\rangle$, which implies that $U_1 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$. So we can apply several (T1) to get $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$. □

Properties of finite presentability

Proposition

Let G be a group.

1. *G finitely presented does not imply that a subgroup is finitely presented or that a quotient is finitely presented.*
2. *If H is a finite index subgroup of G then G is finitely presented if and only if H is.*
3. *If $N \trianglelefteq G$ is finitely presented and G/N is finitely presented then G is finitely presented.*

A proof can be found in the notes.

List of algorithmic problems of M. Dehn

Word problem: Given a finite presentation $G = \langle S|R \rangle$ design an algorithm recognising when $w \in F(S)$ satisfies $w = 1_G$ in G.

Conjugacy problem: Given a finite presentation $G = \langle S|R \rangle$ design an algorithm recognising when $u, v \in F(S)$ represent conjugate elements in G.

Remark

The conjugacy problem implies the word problem.

Isomorphism problem: Given finite presentations $G_i = \langle S_i|R_i \rangle, i = 1, 2$, determine if $G_1 \simeq G_2$.

Triviality problem (a particular case of the isomorphism problem): Given a finite presentation $G = \langle S|R \rangle$ determine if $G \simeq \{1\}$.

Novikov, Boone, Rabin ['56]: All of the above are unsolvable.

Fridman ['60]: There exists a group with solvable word problem, but unsolvable conjugacy problem.