Geometric Group Theory

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Word and conjugacy problems

Proposition

If the word problem or conjugacy problem is solvable for $G = \langle S | R \rangle$ then it is solvable for any finite $\langle X | Q \rangle = G$.

Proof.

WP: Given $w \in F(X)$ we run simultaneously 2 procedures:

1. List all elements in $\langle\langle Q \rangle\rangle$ (i.e. multiply conjugates $q_i^{w_i}, w_i \in F(X), q_i \in Q$ and transform into reduced word); check if w is among them. If yes, stop and conclude $w = 1$.
2.
   a. List all homomorphisms $\phi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$ (i.e. enumerate all $|X|$-tuples of words in $F(S)$, then check if each $q \in Q$, rewritten by changing $x \mapsto w_x$, becomes $\equiv 1$ in $F(S)/\langle\langle R \rangle\rangle$). This can be done since the WP for $\langle S | R \rangle$ is solvable.
   b. For each ϕ, check if $\phi(w) \neq 1$ in $F(S)/\langle\langle R \rangle\rangle$. If yes, stop and conclude $w \neq 1$.

Word and conjugacy problems

Proof continued: CP: Given $w, v \in F(X)$, run the following 2 procedures in parallel:

1. a. List all $gvg^{-1}w^{-1}$ in $F(X)$.
   b. Check if $gvg^{-1}w^{-1}$ is among the list of elements in $\langle\langle Q \rangle\rangle$. If yes, stop and conclude: "v, w conjugate".
2. a. List all homomorphisms $\phi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$.
   b. Check if $\phi(v), \phi(w)$ are not conjugate. If yes, stop and conclude: "v, w not conjugate".

□

Residually finite groups

Idea: Approximate by finite quotients. So we will need enough of those.

Lemma

TFAE

1. $$\bigcap_{H \leq_{f.i.} G} H = \{1\}$$
2. *For all non-trivial $g \in G$, there exists $\phi : G \to F$ finite such that $\phi(g) \neq 1$.*
3. *For all $\{g_1, ..., g_n\}$ distinct, there exists $\phi : G \to F$ such that $\phi(g_1), ..., \phi(g_n)$ are distinct. In other words, every finite chunk of the infinite Cayley table of G can be reproduced identically in the Cayley table of a finite quotient.*

Residually finite groups

Proof.

The proof is based on the fact that

$$\bigcap_{H \leq_{f.i.} G} H = \bigcap_{N \trianglelefteq_{f.i.} G} N$$

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are OK.

And for (1) $\Rightarrow$ (3): $\forall i \neq j$, take $N_{ij} \not\ni g_i g_j^{-1}$ and define

$$N = \bigcap_{i \neq j} N_{ij}$$

and then consider $\phi : G \to G/N$. □

Residually finite groups

Examples

1. *$GL(n, \mathbb{Z})$ is residually finite.* $\forall g \neq \mathrm{id}$:
   - a. *If $\exists i \neq j$ such that $|g_{ij}| \neq 0$, take $p > |g_{ij}|$ and reduce mod p.*
   - b. *If $\forall i \neq j$, $g_{ij} = 0$, then $\exists\ g_{ii} = -1$. Reduce mod 3: $g_{ii} = 2$.*
2. *Any finitely generated $G \leq SL(n, \mathbb{Q})$ (or $GL(n, \mathbb{Q})$) is RF.*
3. *$(\mathbb{Q}, +)$ is not RF (and nor is $SL(n, \mathbb{Q})$): if we have some $\phi : \mathbb{Q} \to F$, $\phi(0) = \mathrm{id}$, take $g = \phi(1)$, $n = |F|$ and then $g = \phi(1) = \phi(\frac{1}{n})^n = \mathrm{id}$.*

Theorem (Mal'cev)

Let R be a commutative ring with unity. $\forall$ finitely generated $G \leq GL(n, R)$, G is residually finite.

Residually finite groups

Proposition

1. *If G is RF and $H \leq G$ then H is RF.*
2. *If H is a finite index subgroup of G then H is RF if and only if G is RF.*
3. *If two groups G and H are RF then $G \times H$ is RF.*
4. *If $G = H \rtimes K$ where H is finitely generated RF, K is RF, then G is RF.*

Proposition

For all finite or countable X, $F(X)$ is residually finite.

Proof: We have that $F_2 \simeq \langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \rangle \leq GL(2, \mathbb{Z})$. And for all X finite or countable, $F(X) \leq F_2$. $\square$

Residually finite groups

Theorem

A finitely presented residually finite group has a solvable word problem.

Remark

Note that every finite group has a solvable word problem.

Proof.

Suppose $G = \langle S | R \rangle$. Take $w \in F(S)$. Run simultaneously two procedures:

1. List all elements in $\langle\langle R \rangle\rangle$ and check if w is among them.
2. List all homomorphisms $\phi : F(S)/\langle\langle R \rangle\rangle \to S_n$ and check if $f(w) \neq 1$.

□

Residually finite groups

Definition

G is Hopf if every onto homomorphism $f : G \to G$ is an isomorphism.

Example

Every finite group is Hopf.

Theorem

A finitely generated residually finite group is Hopf.

Residually finite groups

Theorem

A finitely generated residually finite group is Hopf.

Proof.

Assume there exists an onto homomorphism $f : G \to G$ that is not 1-to-1.

Take $g \in \ker f \setminus \{1\}$. There exists $\phi : G \to F$ with $\phi(g) \neq 1$. Construct a sequence

$$g = g_0,\ \ g_1 \in f^{-1}(g_0),\ \ g_2 \in f^{-1}(g_1),\ \ \ldots\ ,\ \ g_n \in f^{-1}(g_{n-1})$$

$\forall n$, $f^n(g_n) = g$ and $f^k(g_n) = 1$ for all $k > n$. Hence, for all $n > \ell$, $\phi \circ f^n(g_\ell) = 1$ and $\phi \circ f^n(g_n) \neq 1$. So the homomorphisms $\phi \circ f^n$ are pairwise distinct. But this contradicts $\mathrm{Hom}(G, F) \leq |F|^{|S|}$. □