Geometric Group Theory

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A quotation

William Thurston: "Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."

Residually finite groups

Definition

G is Hopf if every onto homomorphism $f : G \to G$ is an isomorphism.

Theorem

A finitely generated residually finite group is Hopf.

Corollary

If $F(X) = \langle A \rangle$ and $|A| = |X| = n < \infty$, then $F(X) \simeq F(A)$. (i.e. A freely generates $F(X)$ i.e. A is a free basis for $F(X)$).

Proof.

A bijection $X \to A$ extends to $X \to F(A)$ which extends to an onto homomorphism $F(X) \to F(A)$. By Universal Property, we have a second onto homomorphism, hence an onto hom. $F(X) \to F(A) \to F(X)$. Since $F(X)$ is Hopf, the latter hom. is an isomorphism, hence all are. □

Residually finite groups. Simple groups

Theorem

A finitely generated residually finite group is Hopf.

The assumption finitely generated cannot be dropped from the theorem.

Example

- *Consider X, Y countable.*
- *There exists $f : X \to Y$ onto and not injective.*
- *f extends uniquely to an onto group homomorphism $F(X) \to F(Y)$.*

At the other extreme, we have simple groups.

Definition

G is simple if the only normal subgroups are $\{1\}$ and G.

Simple groups

Example

$\mathbb{Z}/p\mathbb{Z}$, A_n, A_∞, $PSL(n, \mathbb{Q})$, *infinite f.g. due to Higman, Thompson, Olshanskii, Burger-Mozes.*

Theorem

A finitely presented simple group has solvable word problem.

Proof.

Let $w \in F(S)$. Since G is simple, if $w \neq 1$ in G then $G = \langle\langle w \rangle\rangle$ and hence $\langle\langle \{w\} \cup R \rangle\rangle = F(S)$.

Two procedures:

1. Enumerate $\langle\langle R \rangle\rangle$. Check if w appears.
2. Enumerate $\langle\langle \{w\} \cup R \rangle\rangle$. Check if every $s \in S$ appears.

□

Graphs and Cayley graphs

A main method of investigation is to endow an infinite group with a geometry compatible with its algebraic structure, i.e. invariant by multiplication. This can easily be done for finitely generated groups via Cayley graphs.

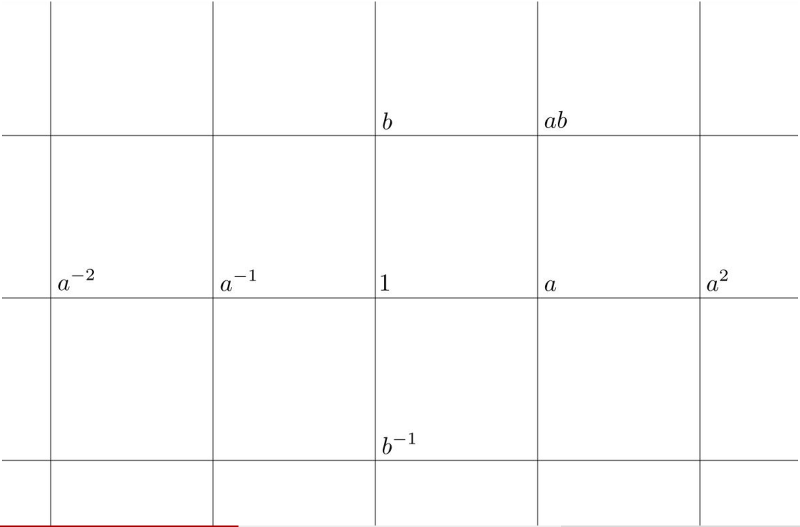
Given a countable group G and a subset S such that $S^{-1} = S$, the Cayley graph of G with respect to S, denoted $\Gamma(S, G)$, is a directed/oriented graph with

- set of vertices G;
- set of oriented edges $\{(g, gs) : g \in G, s \in S\}$;

We denote an edge $[g, gs]$. The underlying non-oriented graph is denoted $\hat{\Gamma}(S, G)$.

Examples of Cayley graphs

1. $\mathbb{Z}^2$ with $S = \{(\pm 1, 0), (0, \pm 1)\}$

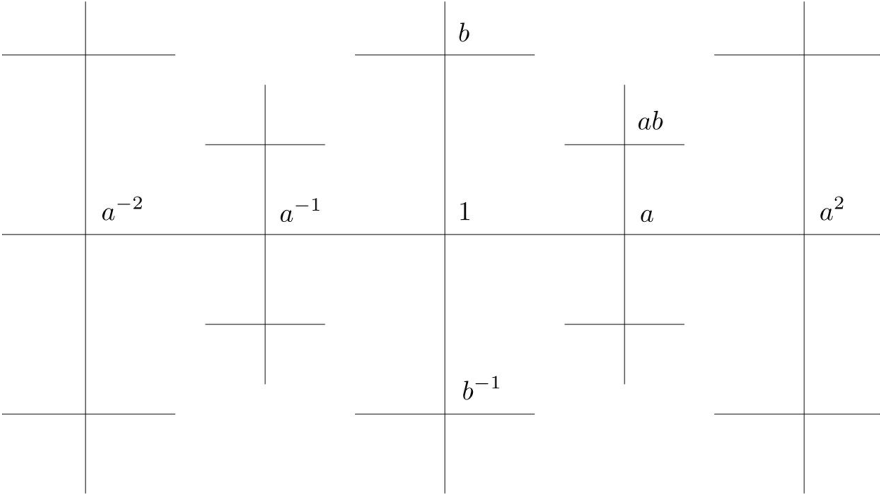


Examples of Cayley graphs

2. $\mathbb{Z}^2$ with $S = \{(\pm 1, 0), \pm(1,1)\}$

Examples of Cayley graphs

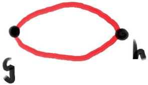
3. $F_2 = F(\{a, b\})$ with $S = \{a^{\pm 1}, b^{\pm 1}\}$



Particular features of Cayley graphs

1. No monogons (edges of the form $[g, g]$) if $1 \notin S$.

2. No digons if, when $s = s^{-1}$, we do not list both s and s^{-1} in S (i.e. no repetitions in S).



   In other words, this is a simplicial graph.

3. $\Gamma(S, G)$ is connected (i.e. any two vertices can be connected by an edge path) if and only if $G = \langle S \rangle$.
4.
   a. $\Gamma(S, G)$ is regular: the valency/degree of every vertex is $|S|$.
   b. $\Gamma(S, G)$ is moreover locally finite if and only if $|S| < \infty$.

Particular features of Cayley graphs

5. If $\Gamma(S, G)$ is infinite then it always contains a bi-infinite geodesic.

6. $\Gamma(S, G)$ always contains a cycle (in fact plenty) with one exception: $\Gamma(S, G)$ does not contain a cycle (i.e. it is a simplicial tree) $\iff$ $S = X \sqcup X^{-1}$ and $G = F(X)$.

Cayley Graphs

From now on assume that S is a finite generating set (with no repetitions), $1 \notin S$, $S = S^{-1}$. We endow $\Gamma(S, G)$ with a metric d_S as usual:

- each edge has length 1;
- $d_S(x, g)$ is the length of a shortest path from x to g.

Proposition

The action of G on its Cayley graph is an action by isometries. The action is free on the vertices. It is free on the whole Cayley graph if and only if no $s \in S$ is of order 2.

Proof.

We have a map

$$G \to \mathrm{Isom}(\Gamma(S, G)) \quad g \mapsto L_g$$

where $L_g \in \mathrm{Isom}(\Gamma(S, G))$ extends $L_g : G \to G$, $L_g(x) = gx$ to edges. □

Cayley Graphs

Definition

The restriction of d_S to $G \times G$ is called the word metric.

Exercises

- $|g|_S := d_S(1, g)$ *is the minimum length of a word w in S such that $g =_G w$.*
- $d_S(g, h)$ *is the minimum length of a word w in S such that $gw =_G h$.*

Proposition

If $G = \langle S \rangle = \langle \bar{S} \rangle$ then d_S and $d_{\bar{S}}$ are bi-Lipschitz equivalent. That is, there exists $L > 0$ such that

$$\frac{1}{L} d_S(g, h) \leq d_{\bar{S}}(g, h) \leq L d_S(g, h)$$

for every $g, h \in G$.

Cayley Graphs

A simplicial tree is a connected graph with no monogons, digons or cycles.

Theorem

$\hat{\Gamma}(S, G)$ a simplicial tree on which G acts freely $\iff S = X \sqcup X^{-1}$, $G = F(X)$.

Actions on simplicial trees

Theorem

$\hat{\Gamma}(S, G)$ a simplicial tree on which G acts freely $\iff$ $S = X \sqcup X^{-1}$, $G = F(X)$.

Proof.

$(\Leftarrow)$: A cycle would correspond to a reduced word $w = 1$ in $F(X)$. □