SUMMARY OF C2.6 LECTURES (VERSION 2)

JAY SWAR

Corrections and feedback is welcome! Send to swar@maths.ox.ac.uk

This document is currently in a rough state, and will be gradually filled in. The main point is to give a list of what's been covered in lectures and references, and so I will aim to be updating it regularly. Feel free to ask for more clarity regarding mathematical content, what is examinable (or, what's easier for me to answer, what is strictly non-examinable), or anything else.

A remark on the course: It'll likely be to your advantage to go through the course first hunting for the big pictures, and then later chisel out the details. The theory of schemes is made much easier by time and by examples.

The lectures in week 8 have been moved to week 7; that is, there are 4 lectures in week 7: Monday, 2pm; Tuesday, 3pm; Wednesday, 2pm; and Thursday, 3pm. Locations will be confirmed on Monday.

0. Overview

There are two sets of lecture notes accompanying this course: Ritter's course notes from last year and Rössler's notes from some previous years. I expect almost all of the material in Ritter's notes will appear during the lectures or at least be directly adjacent. The stuff in Rössler's notes which does not appear in lectures, here, or in Ritter's notes will not be examinable, but might still be helpful (especially if you are more (co)homologically inclined). The material we cover which does not appear in either set of notes will appear here. So far (March 22), the course has looked like:

- Parts of sections 1 and 2 of Rössler's notes (especially concerning the adjunction between pushforwards and inverse images on sheaves, locally ringed spaces, $\mathcal{O}_{\operatorname{Spec} R}$ defining a sheaf, schemes as locally ringed spaces) should be helpful (you can skip the spectral sequence stuff).
- The representability criterion for a Set-presheaf on Sch (**not in either set of notes**, we went along section 26.15 of the Stacks project)
- Ritter's notes: Chapters 1, 2, 3, 4, 5, 6, 7, 8, 9 so far (not everything appeared directly in lectures). Important things to emphasize are affine-locality of various properties of schemes and of morphisms of schemes; induced reduced closed subschemes (not covered in depth during lectures, but section 5.6 in Ritter notes); section 5.1 of Ritter's notes on the existence of fiber products in Sch_S should be compared with a functor-of-points approach.
- The valuative criteria for universally-closedness/separatedness/properness (**not in either set of notes**, we went along sections 26.19-26.22 of the Stacks project). I also briefly talked about the (non-examinable) valuative criteria for smoothness/unramifiedness/étaleness.

- $\mathsf{Mod}_{\mathcal{O}_X}$, $\mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_X})$, the equivalence of categories between $\mathsf{Mod}_R \simeq \mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_R})$. Vector schemes over X, relative spec (we went along sections 27.2-27.6 of the Stacks project).
- We then talked about Čech cohomology and sheaf cohomology; Serre's criterion for affineness (we went along sections 30.2-30.3 of the Stacks project which is more general than the version in section 2 of R'ossler's notes which just considers Noetherian schemes); line bundles, ampleness (we did not mention this in the lectures!). Chapter 10 of Ritter may be helpful, e.g. for understanding line bundles on $\mathbb{P}^n$.

If I were to convert these into lecture notes, the sections would be: 1) basics of schemes as locally ringed spaces and as functors (ideally on CRing, but we focused on characterising them amongst $\mathsf{PSh}(\mathsf{Sch})$ rather than descending to $\mathsf{PSh}(\mathsf{Aff})$. The descent isn't too difficult.); 2) further properties of schemes and morphisms of schemes, valuative criteria; 3) properties of sheaves, $\mathcal{O}_X$-modules, and cohomology; and 4) bonus bits.

You should try not to feel unhealthily stressed about the course if you understand the above material and problem sheets. The problem sheets are intended to require considerably more new ideas and work than would be appropriate in an exam. I make lots of non-examinable remarks in this document which are meant to either be directly helpful or provide some sense of the upshots and shortcomings of scheme theory, especially in the context of modern mathematics[1]. Some of the best complementary/additional reading material is:

- Hartshorne's Algebraic Geometry
- Vakil's online notes, The Rising Sea
- Eisenbud and Harris' The Geometry of Schemes
- The Stacks project, especially Chapter 26, and parts of Chapters 28, 29, 30 as relevant.

The following lectures are fully "summarized":

- Lecture 8

0.1. I'll include here some nice results/exercises which didn't appear in lectures/problem sheets.

Exercise 1. Show that a scheme X is reduced iff the canonical morphism $\coprod_{x\in X} \operatorname{Spec}(\kappa(x)) \to X$ is an epimorphism (hence an isomorphism).

1. Lecture (Monday, Week 1, 2pm)

Why are schemes necessary? If you want to work with $\mathbb{Z}$ (or even $\mathbb{R}$), with nilpotent rings, or do coordinate-independent algebraic geometry, then you need schemes. Nilpotent rings arise naturally even when considering classical algebraic geometry (e.g. in non-transverse intersections such as $\operatorname{Spec}(\mathbb{C}[x,y]/(y-x^2))$ and $\operatorname{Spec}(\mathbb{C}[x,y]/y)$ intersecting in $\operatorname{Spec}(\mathbb{C}[x,y]) =: \mathbb{A}^2_{\mathbb{C}}$). For number theory, there's plenty of phenomena

[1]Many might call schemes the birth of *modern* algebraic geometry. But it's 2023, and "postmodern" has acquired too many connotations to be universally embraced (and initiates the unsavory terminology of post$\cdots$postmodern)... still, the theory of schemes is a sine qua non for (and in fact sufficient to approach!) a stunning proportion of modern research in algebraic geometry, arithmetic geometry, higher category theory, etc.

which a priori looks like it should be geometric. Schemes give a theory of geometry which is sufficiently general to explain much of this. For example, the Diophantine equation $E : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}$ (the punctured elliptic curve) is associated to $\operatorname{Spec}(\frac{\mathbb{Z}[x,y]}{y^2-(x^3+ax+b)})$. Note that the solutions to E valued in a commutative ring R are in a functorial bijection with $\operatorname{Hom}_{\mathsf{CRing}}(\frac{\mathbb{Z}[x,y]}{y^2-(x^3+ax+b)}, R)$. This is an instance of what's known as the functor of points perspective. The curve considered as over the complex numbers or as over a finite field (i.e. the $\mod p$ reduction of E) can then be uniformly considered as base-changes (fibers!) of $\operatorname{Spec}(\frac{\mathbb{Z}[x,y]}{y^2-(x^3+ax+b)})$ along $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{Z}$, respectively.

Definition 2. Ringed spaces. Locally Ringed Spaces.

Presheaves and Sheaves on $\mathsf{Op}(X)$ (the category of opens of a topological space X). Stalks.

A morphism $s : B \to E$ is a section of a map $\pi : E \to B$ iff $\pi \circ s = \mathrm{id}_B$. Sections of presheaves are sections of the induced projection map from the associated the 'etale space. Historically, the topological picture came first and sheaves served as generalizations (e.g. as coefficients in cohomology).

Definition 3. $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.

The closed subsets of $\operatorname{Spec} R$ are defined to be $V(I) := \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$. For any open U we can write $U = \operatorname{Spec} R - V(I) = \bigcup_{f\in I} D_f(R)$ (since a prime ideal $\mathfrak{p}$ does not contain I iff $\exists f \in I$ which is not in $\mathfrak{p}$).

Exercise 4. $\mathcal{O}_{\operatorname{Spec} R}(D_f(R)) := R[\frac{1}{f}]$ defines a sheaf.

By the sheaf property, it is thus sufficient to define the values on any basis of opens to know $\mathcal{O}_{\operatorname{Spec}(R)}(U)$ on any open $U \subseteq \operatorname{Spec} R$.

2. Lecture

Definition 5. Morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, of locally ringed spaces (maps on stalks are local morphisms of local rings). Schemes are a full subcategory of $\mathsf{LocRingSpaces}$, i.e. a morphism of schemes is just a morphism of locally ringed spaces.

Pushforward of presheaves along a morphism of topological spaces.

Natural Transformations

Definition 6. Kernel of a morphism of sheaves.

Lemma 7. *Let $\phi : F \to G$ be a morphism of sheaves on a topological spaces X. The presheaf $U \mapsto \ker(\phi_U : F(U) \to G(U))$ is a sheaf.*

Example 8. $U \mapsto \operatorname{coker}(\phi_U : F(U) \to G(U))$ is not necessarily a sheaf. We considered $\exp : \mathcal{O} \to \mathcal{O}^\times$ where $\mathcal{O}$ is the sheaf of holomorphic functions on $\mathbb{C}$ and $\mathcal{O}$ is the sheaf of nowhere-zero holomorphic functions on $\mathbb{C}$.

Definition 9. Sheafification. Universal property of sheafification.

Definition 10. Étale space associated to a presheaf. Let $\mathcal{F} \in \mathsf{PSh}(X)$ Let $\text{Ét}_{\mathcal{F}} := \coprod_{x\in X} \mathcal{F}_x$. We equip this space with the topology generated by opens $\{(s, U) \in \mathcal{F}_x\}_{x\in U}$ where $s \in \mathcal{F}(U)$, i.e. the subset of elements induced by sections over opens.

There is a natural projection map $\coprod_{x\in X}\mathcal{F}_x \to X$ which can be observed to be continuous.

Theorem 11. *The category of abelian-group-valued sheaves* $\mathsf{Ab}(X)$ *on a topological space* X *is an abelian category.*

Proof. Omitted. □

Exercise 12. A SES $0 \to F \to G \to H \to 0$ of Ab-sheaves is exact iff the induced sequences on stalks $0 \to F_x \to G_x \to H_x \to 0$ are exact for all $x \in X$.

Definition 13. The induced map on spectra of a ring homomorphism.

This induced map is continuous.

Example 14. Examples of affine schemes:
$\mathbb{F}[x_1, \dots, x_n]/I$ for $\mathbb{F}$ a field.
$\operatorname{Spec} K$ for K a field. Topologically, these are just points, but they have different structure sheaves for different K.
$\operatorname{Spec} \mathbb{Z}$

You should think about $\operatorname{Spec} \mathbb{Z}[x]$. Drawing this picture relies directly on the structure theory of algebraic integers and number fields.

3. Lecture (Monday, Week 2, 2pm)

Lemma 15. $\mathsf{CAlg}_{\mathbb{Z}}^{op} \to \mathsf{LocRingedSpaces}$ *is fully faithful.*

Proof. In Ritter's notes (chapter 1.13). □

The essential image consists of what we call affine schemes.

Lemma 16. *All schemes are colimits of affine schemes.*

Lemma 17. *The Yoneda lemma.*

One motivation: $\operatorname{Hom}_{\mathsf{CAlg}_{\mathbb{Z}}}(\frac{\mathbb{Z}[x,y]}{y^2-(x^3+ax+b)}, -)$ as a functor $\mathsf{CAlg}_{\mathbb{Z}} \to \mathsf{Set}$ allows one to think of a collection of points valued in different rings instead of a locally ringed space.

Another very important motivation: there turns out to be an algebro-geometric version of the theory of Lie groups (manifolds which possess a group structure defined using smooth maps, i.e. group objects in the category of manifolds). This is the theory of algebraic groups (and group schemes; a group scheme is a group object in the category of schemes; an algebraic group is a group scheme over a field). It is very helpful to talk about algebraic groups as functors. This is outside the remit for this course however.

Remark 18. An S-scheme G is a group scheme iff the Yoneda embedding h_G of G is valued in the category of groups (viewed as a subcategory of Set).

Definition 19. Site = category with a Grothendieck topology.
Grothendieck topology = generalization of open covers
Sheaves on sites

Remark 20. We will focus mostly on $\mathsf{Op}(X)$ for topological spaces X (in particular, for schemes, we focus on the Zariski topology). I introduce sites since they present a clean isolation of the structure on a category for which we can discuss sheaves. In particular, it's more natural to think about sheaves on CRing or on Sch_S or Top_X (i.e. the category of topological spaces with a continuous morphism to X) using sites. The site most used in algebraic geometry is arguably the étale site in which the covers (jointly-surjective collections of étale maps) are algebraic versions of (topological) covering maps. Such a notion gives one descriptions of fundamental groups of schemes. The fundamental group of $\operatorname{Spec} K$ for a field K turns out to be the absolute Galois group (the choice of base-point corresponds to a choice of seperable closure).

Example 21. Open covers on $\mathsf{Op}(X)$ define a Grothendieck topology on the category.

The following will be returned to after Lecture 8

Definition 22. Let $M \in \mathsf{Mod}_R$. We defined an associated sheaf $\tilde{M}$ on $\operatorname{Spec} R$.

We mentioned that these types of sheaves generalize vector bundles; in fact, we will see eventually that they produce an equivalence between Mod_R and the category of quasi-coherent $\mathsf{Mod}_{\mathcal{O}_{\operatorname{Spec} R}}$-sheaves on $\operatorname{Spec} R$.

4. Lecture

We begin proving the representability criterion. We mostly follow the route contained in the Stacks project, chapter 26.

Definition 23. A subfunctor $H \subseteq F \in \mathsf{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set})$ is representable by open immersions iff for all pairs (T, ξ) where T is a scheme and $\xi \in F(T)$, there exists an open subscheme $U_\xi \subseteq T$ such that a morphism $f : T' \to T$ factors through U_ξ iff $f^*\xi \in H(T')$.

Definition 24. There is a helpful equivalent definition: : a subfunctor $H_i \subseteq F$ is representable by an open immersion iff for all schemes T and all $\xi : h_T \to F$, there is an open subscheme $U_{i,\xi} \subseteq T$ such that $h_{U_{i,\xi}} \simeq H_i \times_F h_T$. That is, a subfunctor is representable by open immersions iff "schemes collectively view it in F as an open immersion" (the way a scheme T views a functor F is $\operatorname{Hom}_{\mathsf{PSh}(\mathsf{Sch})}(h_T, F)$, and the way T views H_i is $\operatorname{Hom}(h_T, H_i)$. Thus, the way T views $H_i \subseteq F$ along $\xi : h_T \to F$ is precisely $\operatorname{Hom}(h_T, H_i \times_F h_T)$. In particular, for a T-scheme S, we're asking for the S-points of $H_i \times_F h_T$ to be the S-points of an open immersion $U_{i,\xi}$ of T.

5. Lecture (Monday, Week 3, 2pm)

Results on Gluing (chapter 4 of Ritter, chapter 26.14 of the Stacks Project).
Finish proof of representability criterion:

Theorem 25. $F \in \mathsf{PSh}(\mathsf{Sch}) = \mathsf{Fun}(\mathsf{Sch}^{op}, \mathsf{Set})$ *is representable by a scheme iff* $F \in \mathsf{Sh}(\mathsf{Sch}, \tau^{Zariski})$ *and* F *admits a covering by subfunctors representable by open immersions.*

Remark 26. A moduli problem = A functor on $\mathsf{Sch}^{\mathrm{op}}$ which one would like to represent; in particular, let's view the functor as valued in Set, although often one wants a target with more structure (e.g. the category Grpd of groupoids; here, a groupoid is a category in which every morphism is an isomorphism. In contrast, a set S is a category with objects in correspondence with the elements of the set S and the *only* morphisms are the identity morphisms. In more terminology, a set is tautologically a *small discrete* category and we view groupoids as moving away from the discreteness (one might have in mind moving from considering the set of points $\mathrm{Hom}_{\mathsf{Top}}(*, T)$ of a topological space T to the set of paths $[0, 1] \to T$).

A representing object is called the moduli solution. The following explains the terminology for universal family: Suppose $(X, \xi : h_X \xrightarrow{\sim} F)$ is a universal family where F is a moduli problem (i.e. we have some interpretation for the T-points of F where T is a scheme). We view a T-point of F as a T-parametrized family of "whatever the points of F represent". In particular, $\phi \in F(T) \simeq \mathrm{Hom}_{\mathsf{Sch}}(T, X)$ under ξ, and so the T-parametrized family of F's "points" are a map of T into X. Thus, X is a universal as a target for T-parametrized family of F's "points".

Definition 27. Noetherian topological space.

Lemma 28. *Noetherian iff all opens are quasi-compact.*

We began discussing some properties of morphisms.

Lemma 29. *All morphisms of affine schemes are separated.*

6. Lecture

Properties of morphisms. Sections 3.6 and 5.4 of Ritter's notes are decent references (as well as parts of chapter 26 of the Stacks Project).

Definition 30. Affine, quasi-compact, locally of finite type, separated, universally closed, open/closed/locally closed immersions. Universally closed. Separated.

Lemma 31. *Defining things with respect to all affine open covers vs. a single affine open cover.*

Remark 32. $X \to *$ in Top is separated iff X is Hausdorff.

7. Lecture (Monday, Week 4, 2pm)

We covered further properties of separated and universally closed morphisms.

Lemma 33. *For any morphism of schemes $f : X \to Y$, $\Delta_f : X \to X \times_Y X$ is a locally closed immersion.*

Lemma 34. *Affine, quasi-compact, locally of finite type, separated, universally closed, being a closed/open/locally closed immersion are all stable under base-change.*

Lemma 35. *A scheme X is separated iff $\exists$ affine-open cover $\bigcup_{i\in I} U_i = X$ such that $U_i \cap U_j$ is affine and the induced morphisms $\mathcal{O}_X(U_i) \otimes \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective for all $i, j \in I$.*

Lemma 36. *The graph $\Gamma_f : X \to Y \times X$ is a base-change of Δ_Y.*

Remark 37. Topological description of (quasi-)compactness, Hausdorffness can be phrased categorically. Translating those definitions to the setting of schemes highlights the schemes which have similarly nice behavior. For example, in compact, Hausdorff spaces, we can define "limits" which have nice existence/uniqueness properties.

The algebro-geometric analogue will be clarified in the Valuative Criteria for separated/universally-closed/properness.

Definition 38. $f : X \to Y$ is quasi-separated iff $\Delta_f : X \to X \times_Y X$ is quasi-compact.

Lemma 39. *Closed immersions are quasi-compact.*

Corollary 40. *Separated morphisms are quasi-separated.*

Definition 41. Let x, y be points of a topological space. x is a genericization of y (equiv. y is a specialization of x) iff $y \in \overline{\{x\}}$

Example 42. Let R be a local domain which is not a field. Then $\mathfrak{m}_R$ is a specialization of (0). (we have a "most special" and "most generic" point)

8. Lecture

The content of this lecture is not in Rössler or Ritter's notes and so we write out details. Source: this is more-or-less all well-documented in the Stacks Project, Chapter 26.

Moral: In topology, a compact space has "no punctures" and a Hausdorff space has "unique limits when they exist". Here is one completely non-examinable way of making this precise:

Fact. *A topological space X is:*

- *Hausdorff iff every ultrafilter converges to at most one point.*
- *Compact iff every ultrafilter converges to at least one point.*

Definition. A filter is: a non-empty family $\mathcal{U}$ of subsets of X s.t.

(1) $\emptyset \notin \mathcal{U}$,
   (a) $A \in \mathcal{U}$, $X \supseteq B \supseteq A \implies B \in \mathcal{U}$,
   (b) $A, B \in \mathcal{U} \implies A \cap B \in \mathcal{U}$,

Definition. A filter is an ultrafilter iff it is maximal amongst filters with respect to inclusion.

This generalizes the notion of "large" subsets of a space.

Definition. A filter $\mathcal{U}$ converges to a point x iff $\mathcal{N}_x \subseteq \mathcal{U}$ where $\mathcal{N}_x$ is the "neighborhood/principal filter of $x \in X$" defined as the collection of subsets of X containing x.

If $f : X \to Y$ is a function on sets and $\mathcal{U}$ is a filter on X, then $f_*\mathcal{U} := \{A \subseteq Y : f^{-1}(A) \in \mathcal{U}\}$ is a filter on Y (called the pushforward of $\mathcal{U}$ via f).

Example 43. (The motivating example for this aside) Let $f : \mathbb{N} \to X$ be a sequence in a topological space X. Let $\mathcal{I}$ denote the filter on $\mathbb{N}$ consisting of all infinite subsets. Then $\mathcal{N}_x \subseteq f_*\mathcal{I}$ iff $f(n) \to x$ as a sequence.

Example. Let $\mathcal{U}$ on $\mathbb{R}^n$ be the set of subsets with ∞ measure (using the standard measure).

There's no $x \in \mathbb{R}^n$ such that $\mathcal{N}_x \subseteq \mathcal{U}$. If we compactify $\mathbb{R}^n$ with a point p "at ∞", we have that $N_p \subseteq \hat{\mathcal{U}}$ (where the hat denotes an ultrafilter on $\mathbb{R}^n \cup \{p\}$ containing $\mathcal{U}$).

We will find that there is an algebro-geometric analogue of this in the form of the valuative criteria for

- universally closedness (cf. compactness)
- separatedness (cf. Hausdorff)

We first need some notion of converging to a point in algebro-geometric settings. Let $A \subseteq B$ be commutative rings (write the corresponding map of affine schemes as ι), $X \xrightarrow{f} S$ be a morphism of schemes. There is an equivalence between the set of commutative diagrams

$$\begin{array}{ccc} \operatorname{Spec} B & \xrightarrow{\beta} & X \\ \iota \downarrow & & \downarrow f \\ \operatorname{Spec} A & \xrightarrow[\alpha]{} & S \end{array}$$

and the set-theoretic limit $X(B) \times_{S(B)} S(A)$.

This is precisely: the B-points of X whose image in S can be genericized to A-points.

There is a natural map of sets $X(A) \to X(B) \times_{S(B)} S(A)$ which is not necessarily injective nor surjective in general. (β, α) are in the image of this map iff one can draw a diagonal map from $\operatorname{Spec} A$ to X in the diagram. When this occurs, one calls the diagonal map a *lift* of α along β. This can be viewed as a very large generalization of "converging to a point": recall that $f : X \to S$ is viewing X as a family of schemes $(f^{-1}\{s\})_{s \in S}$ indexed by the points of S. So a lift is saying that given a B-point of X whose image in S can be genericized to an A-point of S, then one can genericize the B-point of X to an A-point of X. Don't dwell too much on this until we talk about valuation rings.

Recall that being universally closed and being separated are both about certain maps being closed (either all base-changes or the diagonal map respectively).

Lemma 44. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed iff specializations lift along f.*

Proof. ($\Longrightarrow$) Suppose $f(x') = s'$ and $s \in \overline{\{s'\}}$. As f is closed, $f(\overline{\{x'\}}) \supseteq \overline{\{s'\}}$, and so there exists $x \in \overline{\{x'\}}$ such that $f(x) = s$.

($\Longleftarrow$) Let Z be a closed subset of X (viewed as a closed subscheme with the reduced induced structure sheaf). Z is closed, so $f|_Z$ is quasi-compact and specializations lift along $f|_Z : Z \to S$ since any specialization in X of a point in Z must be in Z, thus $f(Z)$ is stable under specializations.

Let $U = \operatorname{Spec} R$ be an affine open of S. As f is quasi-compact and U is affine, $f^{-1}(U)$ is quasi-compact (without loss of generality, we take it to be non-empty). We can thus take a finite affine open cover $f^{-1}(U) = \bigcup_{i \in I} \operatorname{Spec}(A_i)$ of R-algebras. The lemma below shows that since $f(Z) \cap U$ is closed under specializations in U we have that $U \cap f(Z) = \operatorname{im}(\operatorname{Spec}(\prod_{i \in I} A_i) \to \operatorname{Spec} R)$ is closed in U. As U was an arbitrary affine open, we that $f(Z)$ is closed. $\square$

Lemma 45. *Let $A \to B$ be a morphism of commutative rings. If im$(\operatorname{Spec} B \to \operatorname{Spec} A)$ is stable under specialization, then it is closed.*

Proof. Let $I := \ker(A \to B)$. SInce we can write $A \to B$ as $A \to A/I \to B$, then $\operatorname{im}(\operatorname{Spec} B \to \operatorname{Spec} A) \subseteq \operatorname{Spec}(A/I)$. Viewing the injection $A/I \hookrightarrow B$ as a morphism of

A/I-modules and using that localization is exact, we observe that $B_{\mathfrak{p}} := (A/I - \mathfrak{p})^{-1}B$ is a non-zero A/I-module and in fact a ring for any prime of A/I. In particular, if $\mathfrak{p}$ is a minimal prime, then $(A/I)_{\mathfrak{p}}$ has precisely one prime which must be the pre-image of any prime $\mathfrak{q}$ in $B_{\mathfrak{p}}$ (which exist as $B_{\mathfrak{p}}$ is a non-zero ring). Thus, there is a prime in B mapping to $\mathfrak{p}$.

Since $\operatorname{im}(\operatorname{Spec} B \to \operatorname{Spec} A)$ is stable under specialization, it must then be equal to $V(I) = \operatorname{Spec}(A/I) \subseteq \operatorname{Spec} A$ and is thus closed. □

So lifting specializations should be precisely what informs being universally closed and being separated. It turns out understanding specializations is equivalent to understanding appropriate A-points of our scheme of interest where A is a *valuation ring.* This is made precise by the following lemma:

Lemma 46. *Let S be a scheme and $s' \rightsquigarrow s$ a specialization of points in S. Then given any field extension $K/\kappa(s')$ of the residue field of s', there exists a morphism from a valuation ring* $\operatorname{Spec} A \to S$ *such that the generic point η of* $\operatorname{Spec} A$ *maps to s' and the special point σ of* $\operatorname{Spec} A$ *maps to s. Further, $\kappa(\eta)/\kappa(s')$ is isomorphic to $K/\kappa(s')$.*

Proof. We will prove this after some dialogue. □

Definition 47. A valuation ring R is a commutative unital ring satisfying the following equivalent conditions:

1. A local domain maximal with respect to domination amongst local subrings of its fraction field.
   - (a) Let A, B be local subrings of a field K. We say A dominates B iff $A \supseteq B$ and $\mathfrak{m}_A \cap B = \mathfrak{m}_B$ (i.e. the inclusion $i : B \to A$ is a local map).
2. One can write $R \subseteq K$ where K is a field, such that for all $x \in K^\times$, either x or x^{-1} is in R.
3. $\exists$ a valuation $v : K \to \Gamma \cup \{\infty\}$ such that $R = \{x \in K : v(x) \geq 0\}$
   - (a) A valuation on K is a map $v : K \to \Gamma \cup \{\infty\}$ where Γ is a totally ordered abelian group (called the value group) satisfying:
     - (i) $v(a) = \infty$ iff $a = 0$
     - (ii) $v(ab) = v(a) + v(b)$
     - (iii) $v(a + b) \geq \min(v(a), v(b))$ with equality iff $v(a) \neq v(b)$.
4. A local domain such that all finitely-generated ideals are principal.

Remark 48. A Noetherian valuation ring is thus a PID. In fact, a valuation ring is Noetherian iff it is a *discrete* valuation ring (i.e. $\Gamma \simeq (\mathbb{Z}, +)$) or a field.

A discrete valuation ring is a non-field local PID (thus has Krull dimension 1, is integrally-closed, and much more). They should be thought of as local rings of points on curves.

Remark 49. It is *not* the case that a valuation ring necessarily has Krull dimension ≤ 1. In fact, the dimension of the valuation ring is equal to the rank of its value group. Here is an innocuous example of a valuation ring of dimension 2: Let $v : k(x, y)^\times \to \mathbb{Z}^2$ be defined by $v(\sum_{i,j} a_{i,j} x^i y^j) := \min\{(i, j)\}$ where $\mathbb{Z}^2$ is ordered lexicographically with $\deg_x > \deg_y$. Then the valuation ring is $R_v := \{f \in k(x, y) : v(f) \geq 0\} = k[y, \{\frac{x}{y^i}\}_{i \in \mathbb{Z}_{\geq 0}}]_{(y)}$.

Remark 50. A Prüfer domain is an integral domain whose localization at any prime ideal is a valuation ring.

Equivalently, a ring is a Prüfer domain iff every finitely-generated projective module is torsion-free.

The Noetherian Prüfer domains are precisely the Dedekind domains, so Prüfer domains are the precise non-Noetherian generalizations demonstrating the same local behavior.

Given a non-field local domain A, the closed point $\mathfrak{m}_A$ of $\operatorname{Spec} A$ is called *the* special point. For any domain, (0) is the unique generic point.

Let's consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\beta} & X \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow[\alpha]{} & S \end{array} .$$

One can draw intuition from the Noetherian setting: suppose A is not a field. Then $\operatorname{Spec} A$ is a local one-dimensional ring, cf. an infinitesimal disk arount a point of a curve. $\operatorname{Spec} K$ is obtained by omitting the point from the infinitesimal disk, i.e. $\operatorname{Spec} K = \operatorname{Spec} A - V(\mathfrak{m}_A)$. So K-point of a scheme T is a punctured infinitesimal disk of T, and saying that the K-point factors through an A-point is saying that the puncture of the punctured infinitesimal disk of T can be filled in. Thus, asking for a lift of α along β is equivalent to: asking if one has a punctured infinitesimal disk β of X whose image $f\beta$ in S can be filled in $f\beta = \alpha\iota$, then can one fill in the puncture of β in X?

For non-Noetherian A, we have roughly the same intuition, just the infinitesimal disk is not quite around a point of a curve as it possesses intermediary infinitesimal neighborhoods between the closed point and the entire space. The puncture is removes all these intermediary infinitesimal neighborhoods of the closed point (i.e. $\operatorname{Spec} K$ is the localization at the generic point).

We can now practically guess what the valuative criteria should be: we should always have lifts (i.e. $X(A) \to X(K) \times_{S(K)} S(A)$ is surjective) when f is universally closed (cf. compactness in topology), and lifts should be unique (i.e. $X(A) \to X(K) \times_{S(K)} S(A)$ is injective) when f is separated (cf. Hausdorffness in topology). To make this an iff the quasi-compactness conditions pop-up as in Lemma 44.

Theorem 51. *(The valuative criteria of universally-closedness and separatedness) Let $f : X \to S$ be a morphism of schemes.*

f is universally closed iff f is quasi-compact and $X(A) \to X(K) \times_{S(K)} S(A)$ is surjective for all valuation rings A with fraction field K.

f is separated iff f is quasi-separated (i.e. Δ_f is quasi-compact) and $X(A) \to X(K) \times_{S(K)} S(A)$ is injective for all valuation rings A with fraction field K.

Before we prove Lemma 46, we note the following:

Lemma 52. *Let s, s' be points of a scheme S. We have $s \in \overline{\{s'\}}$ iff $s' \in$ im$(\operatorname{Spec} \mathcal{O}_{S,s} \to S)$.*

Proof. ($\Longleftarrow$) This is trivial since continuous maps preserve specialization and every point of $\operatorname{Spec} \mathcal{O}_{S,s}$ has the unique closed point (corresponding to s) in its closure, so every point in the image genercizes s.

($\Longrightarrow$) Pick an affine open neighborhood $\operatorname{Spec} R$ of s. Note since $s \in \overline{\{s'\}} = \bigcap_{V \ni s', \text{closed subsets of } S} V$, we have that $s' \in \operatorname{Spec} R$. Since s' genericizes s, we have containment on the corresponding primes: $\mathfrak{p}' \subseteq \mathfrak{p}$, so s' is in the image of $\operatorname{Spec}(\mathcal{O}_{S,s}) = \operatorname{Spec}(R_{\mathfrak{p}}) \to \operatorname{Spec} R \subseteq S$. ☐

Lemma 53. *Let A be a local subring of K. There exists a valuation ring V with fraction field K dominating A.*

Proof. This is trivial if $\operatorname{FracField}(R) = K$. Suppose $t \in K - \operatorname{FracField}(R)$. If t is transcendental over A, then $A \subsetneq A[t]_{(t,\mathfrak{m}_A)} \subseteq K$ is a chain of local rings and $A[t]_{(t,\mathfrak{m}_A)}$ dominates A. If t is algebraic over A, then at is integral over A for some $a \in A$, and $A[ta]$ is finite over A and so by Nakayama's lemma, $\mathfrak{m}_A A[ta] \subsetneq A[ta]$ and so there exists a maximal ideal $\mathfrak{q}$ of $A[ta]$ containing $\mathfrak{m}_A A[ta]$. Now $\mathfrak{q} \cap A$ is a proper ideal of A containing $\mathfrak{m}_A$ and so must be equal to $\mathfrak{m}_A$ and thus $A \subsetneq A[ta]_{\mathfrak{q}} \subseteq K$ is a chain of local rings and $A[ta]_{\mathfrak{q}}$ dominates A. We want a maximal element.

Now let Σ denote the set of local subrings of K dominating A and partially ordered by domination, and note that every chain $\{A_i\}_{i\in I}$ in Σ has an upper bound $\bigcup_{i\in I} A_i$ (i.e. this is a local subring dominating A and dominating each A_i), hence Zorn's lemma implies there exists a maximal element V. $\operatorname{FracField}(V) = K$ since otherwise we could constrict a strictly greater local subring of K dominating A and V. ☐

Proof. (of Lemma 46) Let $s' \rightsquigarrow s$ be a specialization of points in S and $K/\kappa(s')$ be a field extension. Thus, by lemma 52, we have a map of rings from $\mathcal{O}_{S,s} \to \kappa(s') \to K$ where the first map is the natural map from $\mathcal{O}_{S,s} \to (\mathcal{O}_{S,s}/\mathfrak{p}') \hookrightarrow \operatorname{FracField}((\mathcal{O}_{S,s}/\mathfrak{p}')) =: \kappa(s')$ and the second map is the given inclusion of fields. By lemma 53, there exists a valuation ring $A \subseteq K$ with fraction field K and dominating the image of $\mathcal{O}_{S,s} \to K$. The map $\mathcal{O}_{S,s} \to A$ induces the desired map $\operatorname{Spec} A \to S$. ☐

9. Lecture (Monday, Week 5, 2pm)

We finish the proof of the Valuative Criteria.

Lemma 54. *Let $f : X \to S$ be a morphism of schemes. TFAE:*

1. *Specializations lift along any base-change of f.*
2. *The f satisfies the existence condition of the valuative criterion.*

Proof. We let A denote a vaulation ring with fraction field K.

($1 \Longrightarrow 2$) Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K\mathrm{ad}[r]^{\beta} & & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} A & \xrightarrow{\alpha} & S \end{array}$$

. We consider the base-change $X_A := X \times_{S,f,\alpha} \operatorname{Spec} A \to \operatorname{Spec} A$. Let $x' := \operatorname{im}(\operatorname{Spec} K \to X_A)$ be the point induced by β; note thus $\kappa(x') \subseteq K$. By assumption, there exists a point of X_A specializing x' which maps to the closed point, say denoted x, of $\operatorname{Spec} A$. We thus have local ring map $A \to \mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x} \to \kappa(x')$ which compose to yield $A \to \mathcal{O}_{X,x} \to \kappa(x') \to K$ which is the canonical injection $A \to K$. $A \to \mathcal{O}_{X,x}$ is local, and so $\mathrm{im}(\mathcal{O}_{X,x} \to K)$ dominates A and thus is equal to A since A is a valuation ring. Thus, we obtain a morphism $\mathcal{O}_{X,x} \to A$ lifting α.

($2 \implies 1$) Note that the existence condition of the valuative criterion must automatically hold for any base-change $X_T \to T$, so it suffices to verify that specializations lift along f. If $s \in \overline{s'}$, then we can apply Lemma 46 to view s as an A-point and s' as a K-point for a valuative ring A with fraction field K, and then this lifts by the valuative crtierion. □

Definition 55. The equalizer of two morphisms $a, b : X \to Y$ in a category $\mathcal{C}$ is the limit of a and b in $\mathcal{C}$. Equivalently, the equalizer is the pull-back of the diagonal map $Y \overset{(\mathrm{id}_Y, \mathrm{id}_Y)}{\to} Y \times Y$ along $X \overset{(a,b)}{\to} Y \times Y$ (assuming $\mathcal{C}$ has binary products and pull-backs).

Lemma 56. *Suppose we have morphisms of S-schemes $a, b : X \to Y$. Then the equalizer* $\mathrm{Eq}(a, b)$ *(in Sch_S) is a locally closed subscheme.* $\mathrm{Eq}(a, b)$ *is a closed subscheme when $Y \to S$ is separated.*

Proof. The equalizer $\mathrm{Eq}(a, b)$ is the base-change of $\Delta_{Y \to S} : Y \to Y \times_S Y$ along $(a, b) : X \to Y \times_S Y$. □

Lemma 57. *If $f : X \to S$ is separated, then f satisfies the uniqueness condition of the valuative criterion.*

Proof. Suppose there are two $a, b \in X(A)$ mapping to a diagram $(\beta, \alpha) \in X(K) \times_{S(K)} S(A)$. The equalizer $\mathrm{Eq}(a, b)$ is a closed subscheme of $\operatorname{Spec} A$ as f is separated and contains the image of $\operatorname{Spec} K$ which is the generic point of $\operatorname{Spec} A$. Hence, $\mathrm{Eq}(a, b) = \operatorname{Spec} A$, i.e. $a = b$. □

Lemma 58. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-separated and satisfies the uniqueness condition of the valuative criterion, then f is separated.*

Proof. We show that $\Delta_f : X \to X \times_S X$ is universally-closed. It is quasi-compact (since f is quasi-separated), so we check the existence condition of the valuative criterion for Δ_f.

Given any commuative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta_f} \\ \operatorname{Spec} A & \xrightarrow[\alpha]{} & X \times_S X \end{array}$$

, note that α is precisely the data of two maps $a, b : \operatorname{Spec} A \to X$ for which $f(a) = f(b)$. Since f satisfies the uniqueness condition, this implies that $a = b$, so we have a lift, namely $\alpha = \Delta_f(a)$. □

As an application, we provided another proof that $\mathbb{P}^1$ is proper.

10. Lecture (Wednesday, Week 5, 2pm)

We covered the basic properties of types of $\mathcal{O}_X$-modules, e.g. locally free, coherent. A good reference is Ritter's chapter 6 and 7.

11. Lecture (Monday, Week 6, 2pm)

We continued covering the basic properties of $\mathcal{O}_X$-modules and functors on $\mathcal{O}_X$-modules. We showed that pullbacks preserve quasi-coherence. A good reference is Ritter's chapter 6 and 7.

We began proving:

Theorem 59. $\mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_R}) := \mathsf{QCoh}(\operatorname{Spec} R) \simeq \mathsf{Mod}_R$.

12. Lecture (Wednesday, Week 6, 2pm)

Today, we finished the proof of $\mathsf{QCoh}(\mathsf{Mod}_{\tilde{R}}) \simeq \mathsf{Mod}_R$. The main missing ingredient was the gluing lemma which is the topic of section 7.5 of Ritter's notes. I also talked about algebraic objects in categories.

The upshot is that one can characterize algebras as certain sets with maps which satisfy certain properties (in the class, we went through the example of groups as sets G with a multiplication $m : G \times G \to G$, inverse $i : G \to G$, and identity map $e : \{*\} \to G$ which satisfy certain properties (i.e. diagrams reflecting associativity and the properties of inverses/identity elements). Thus one can define algebraic objects in categories and these objects are preserved under equivalences of categories.

Definition 60. For example, an (commutative) R-algebra is characterized as a (commutative) monoid object in the module category over R, i.e. $\mathsf{CAlg}(\mathsf{Mod}_R) = \mathsf{CAlg}_R$.

Denote the (commutative) algebra objects of an abelian category $\mathcal{C}$ (this can be done more generally) by $\mathsf{CAlg}(\mathcal{C})$. A sheaf of algebras is a sheaf of (modules with an algebra structure) and is equivalently an algebra object in the category of sheaves of modules. That is, and $\mathsf{CAlg}(\mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_R})) = \mathsf{QCoh}(\mathsf{CAlg}_{\mathcal{O}_R})$ where by $\mathsf{QCoh}(\mathsf{CAlg}_{\mathcal{O}_R})$, I mean sheaves of $\mathcal{O}_R$-algebras which are quasi-coherent as $\mathcal{O}_R$-modules. Trivially, every commutative R-algebra is quasi-coherent as an R-module. Thus, we have

Corollary 61. $\mathsf{QCoh}(\mathsf{CAlg}_{\mathcal{O}_R}) = \mathsf{CAlg}(\mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_R})) \simeq \mathsf{CAlg}(\mathsf{Mod}_R) = \mathsf{CAlg}_R = \mathsf{QCoh}(\mathsf{CAlg}_R)$.

Because affine schemes over $\operatorname{Spec} R$ correspond antiequivalently to commutative algebras over R, we thus we have the following:

Theorem 62. *There is an (anti)equivalence between* $\mathsf{QCoh}(\mathsf{CAlg}_\mathsf{R}) = \mathsf{CAlg}_R \simeq \mathsf{AffSch}_{\operatorname{Spec} R}$.

This should be compared to the following (which can be proved from the exercises on the relative spectrum on sheet 3):

Theorem 63. *There is an (anti)equivalence* $\operatorname{Spec}_S : \mathsf{QCoh}(\mathsf{CAlg}_{\mathcal{O}_S}) \simeq \mathsf{AffSch}_S$.

Proof. Omitted for now, but this is just a globalization of things you've done. □

We also have the symmetric algebra functor $\operatorname{Sym}_R : \mathsf{Mod}_R \to \mathsf{CAlg}_R$. Or globally, $\operatorname{Sym}_{\mathcal{O}_S} : \mathsf{Mod}_{\mathcal{O}_S} \to \mathsf{CAlg}_{\mathcal{O}_S}$.

We have

$$\begin{array}{ccc} \mathsf{VectSch}_S^{\mathrm{op}} & \xleftarrow{\operatorname{Spec}_S \operatorname{Sym}_{\mathcal{O}_S}} & \mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_S}) \\ {\scriptstyle incl.}\downarrow & & \downarrow{\scriptstyle \operatorname{Sym}_{\mathcal{O}_S}} \\ \mathsf{AffSch}_S^{\mathrm{op}} & \xleftarrow[\operatorname{Spec}_S]{} & \mathsf{CAlg}(\mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_S})) \end{array}.$$

We characterize the vector schemes over S coming from locally free $\mathcal{O}_S$-modules as the vector bundles over S (to align with Ritter's terminology – sometimes, people will take the vector bundles to be the class of Affine schemes over S associated to coherent $\mathcal{O}_S$-modules via $\operatorname{Spec}_S \operatorname{Sym}_{\mathcal{O}_S}$ and sometimes people (including EGA!) call $\mathsf{VectSch}_S$ the category of vector bundles).

Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_S$-module and $\mathcal{A}$ be a quasicoherent sheaf of $\mathcal{O}_S$-algebras. We have that $\operatorname{Hom}_{\mathsf{Mod}_{\mathcal{O}_S}}(\mathcal{F}, \mathcal{A}) \simeq \operatorname{Hom}_{\mathsf{CAlg}_{\mathcal{O}_S}}(\operatorname{Sym}_{\mathcal{O}_S}\mathcal{F}, \mathcal{A})$. Thus, the $\operatorname{Spec}_S \mathcal{A}$ points of $\operatorname{Spec}_S \operatorname{Sym}_{\mathcal{O}_S}\mathcal{F}$ inherit a $\mathcal{O}_S$-module structure (what might be tempted to call them module bundles, but that is extremely non-mainstream)! There exist various characterizations of the quasicoherent sheaves of $\mathcal{O}_S$-algebras which can be obtained by $\operatorname{Sym}_{\mathcal{O}_S}$, but we won't focus on this. A simple observation is that an affine morphism $\pi : V \to S$ is a vector scheme over S iff $\pi_*\mathcal{O}_V$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra of the form $\bigoplus_{n\geq 0} \operatorname{Sym}^n_{\mathcal{O}_S}(\mathcal{F})$ where $\mathcal{F}$ is some quasi-coherent $\mathcal{O}_S$-module.

13. Lecture (Monday, Week 7, 2pm)

In the next few lectures, we show that affineness of X can be characterized by the cohomology of quasicoherent sheaves. We go a different route than in Ritter's or Rössler's notes to avoid restricting to Noetherian schemes. The difficulty (which would be avoided for Noetherian schemes) which we don't prove is that $\mathsf{QCoh}(\mathsf{Mod}_{\mathcal{O}_X})$ has enough injectives.

14. Lecture (Tuesday, Week 7, 3pm)

We showed that:

Theorem 64. *If X is a quasi-compact scheme and $H^i(X, \mathcal{I}) = 0$ for all quasi-coherent sheaves of ideals $\mathcal{I}$, then X is affine.*

We then showed several lemmas which will be helpful for the converse. The main reference is the first sections of chapter 30 on the Stacks project.

15. Lecture (Wednesday, Week 7, 2pm)

We continued the stream of lemmas to show that $H^i(X, \mathcal{F}) = 0$ when X is affine for all quasi-coherent sheaves $\mathcal{F}$.

16. Lecture (Thursday, Week 7, 3pm)

We finished the proof that $H^i(X, \mathcal{F}) = 0$ when X is affine for all quasi-coherent sheaves $\mathcal{F}$.

We also proved that sheaf cohomology and Čech cohomology agree for quasi-compact, separated schemes.

We briefly discussed the interpretation of $H^1(X, \mathcal{O}_X^\times)$ as the moduli space of line bundles up to isomorphism on X. This is expanded on in Ritter's notes.

We also introduced smoothness. The following is non-examinable, but is meant to help get a bigger picture of the value of schemes.

Definition 65. A morphism $f : X \to S$ is formally smooth iff for all first order thickenings $T \to T'$, we have that $X(T') \to X(T) \times_{S(T)} S(T')$ is surjective.

f is formally unramified iff for all first order thickenings $T \to T'$, we have that $X(T') \to X(T) \times_{S(T)} S(T')$ is injective.

f is formally étale iff for all first order thickenings $T \to T'$, we have that $X(T') \to X(T) \times_{S(T)} S(T')$ is bijective.

Definition 66. Say $f : X \to S$ is smooth at a point $x \in X$ iff there exists an affine neighborhood $\operatorname{Spec}(R)$ of $f(x)$ and an affine neighborhood $\operatorname{Spec} A \ni \mathfrak{p}_x = x$ mapping to $\operatorname{Spec} R$ such that $A \simeq \frac{R[x_1,\ldots,x_n]}{(f_1,\ldots f_m)}$ with $n \geq m$ such that $\det((\frac{\partial f_i}{\partial x_j})) \notin \mathfrak{p}_x$, i.e. $(\frac{\partial f_i}{\partial x_j})$ maps to an invertible matrix on the fiber at x.

Say $f : X \to S$ is unramified at a point $x \in X$ iff there exists an affine neighborhood $\operatorname{Spec}(R)$ of $f(x)$ and an affine neighborhood $\operatorname{Spec} A \ni \mathfrak{p}_x = x$ mapping to $\operatorname{Spec} R$ such that $\Omega^1_{A/R} = 0$ and A is of finite type over R.

Again f is étale iff f is smooth and unramified.

Lemma 67. *A morphism $f : X \to S$ is smooth/unramified/étale iff f is locally of finite presentation and formally smooth/unramified/étale.*

Lemma 68. *Let $f : X \to S$ be a smooth morphism of schemes. Then $\Omega^1_{X/S}$ is a locally free of finite type sheaf over X and $\operatorname{rank}_{\mathcal{O}_{X,x}} \Omega_{X/S,x} = \dim_{\kappa(x)} X_{f(x)}$.*

Proof. This follows from the local computation. □

We then stated a comparison theorem between de Rham cohomology, singular cohomology, and étale cohomology. Then we stated the Weil conjectures (or rather Weil-Dwork-Grothendieck-Deligne theorems).