Geometric Group Theory

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Examples of Cayley graphs

Integer Heisenberg group:

$$H_3(\mathbb{Z}) := \langle x, y, z \mid [x, z] = 1, [y, z] = 1, [x, y] = z \rangle.$$

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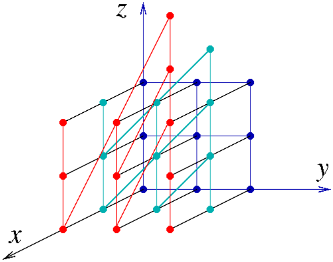


Actions on simplicial trees

Theorem

$\hat{\Gamma}(S, G)$ a simplicial tree on which G acts freely $\iff S = X \sqcup X^{-1}$, $G = F(X)$.

Proof.

Oriented paths in $\Gamma(S, G)$ without spikes correspond to pairs (g, w), w a reduced word in S.
$(\Leftarrow)$: A cycle would correspond to a reduced word $w = 1$ in $F(X)$.
$(\Rightarrow)$: G acts freely $\implies \forall s \in S, |\{s, s^{-1}\}| = 2$. For every such pair, pick one and together let these form X. X generates G and so there exists an onto homomorphism $\varphi : F(X) \to G$. Suppose $w \in F(X)$, $w \in \ker \varphi$. Since w is reduced as a word in X, it is also reduced as a word in S. So if $w \neq w_\emptyset$ then w gives a cycle in $\hat{\Gamma}(S, G)$. So $\ker \varphi = \{w_\emptyset\}$. □

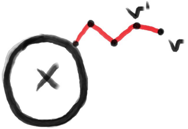
General theorem: G is free if and only if G acts freely by isometries on a simplicial tree T. The $(\Rightarrow)$ direction is given by the theorem above.

Actions on simplicial trees

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Proof: Take X maximal such that X intersects each orbit $G \cdot v$ in at most one point (X exists by Zorn's lemma). Assume there exists some v such that $Gv \cap X = \emptyset$. Take v at minimal distance from X. If $d(v, X) = 1$, then we can add it to X - contradiction. So assume $d(v, X) \geq 2$.



By minimality, $gv' \in X$ for some $g \in G$. Therefore $d(gv, X) = 1$ and so we can add gv to X - contradiction. □

Actions on simplicial trees

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.

Theorem

G is free if and only if G acts freely by isometries on a simplicial tree T.

Actions on simplicial trees

Proof.

$(\Leftarrow)$: Choose an orientation E^+ on the edges of T that is G-invariant. Let

$$S = \{g \in G : \exists e \in E^+, o(e) \in X, t(e) \in g(X)\}$$

We will prove that $G = F(S)$.

If $gX \cap X \neq \emptyset$ then there exists $v \in X$ such that $gv = v$ and so $g = 1$ by the freeness of the action. Hence if $g_1 \neq g_2$ then $g_1 X \cap g_2 X = \emptyset$.

$\{g_1, g_2\}$ is an edge of $\hat{\Gamma}(S \cup S^{-1}, G)$ if and only if there exists an edge of T with one endpoint in $g_1 X$ and the other in $g_2 X$.

$\hat{\Gamma}(S \cup S^{-1}, G)$ is connected because T is. It is simplicial because it is a Cayley graph. And if $\hat{\Gamma}(S \cup S^{-1}, G)$ contains a cycle then so does T. So $G = F(S)$. □

Actions on simplicial trees

Theorem

G is free if and only if G acts freely by isometries on a simplicial tree T.

Corollary

Subgroups of free groups are free.

In order to study groups having actions on simplicial trees that are not free, we need the notion of amalgam.

Amalgams

Let A, B be groups with two isomorphic subgroups: i.e. there exist injective homomorphisms $\alpha : H \to A$, $\beta : H \to B$.

The amalgam of A and B over H is the "largest" group containing copies of A and B identified along H such that no other relation is imposed and such that it is generated by the copies of A and B.

We will define the amalgam by its universal property.