Further Partial Differential Equations (2023)
Problem Sheet 2

1. **Possible similarity solutions**
   Consider the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + x^{\alpha} f^{\beta}\right), \tag{1}$$

subject to the boundary conditions

$$f \to 0 \qquad \text{as} \quad x \to \pm\infty. \tag{2}$$

Suppose that $\beta > \alpha \geq 0$ and that f is suitably well behaved so that $xf \to 0$ as $x \to \pm\infty$.

(a) Show that

$$\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}x \tag{3}$$

is a constant.

(b) Using (1)–(3), show that a similarity solution of the form $f = t^a g(\eta)$ with $\eta = x/t^b$ exists for this system provided

$$\alpha = \beta - 2 \tag{4}$$

and given values of a and b.

(c) Show that g satisfies the ordinary differential equation

$$\frac{1}{2}\eta g + g' + \eta^{\beta-2} g^{\beta} = 0. \tag{5}$$

(d) In the case when $\beta = 2$, show that we can write (5) in the form

$$\left(\frac{\eta}{2g} + \frac{g'}{g^2}\right)\mathrm{e}^{-\eta^2/4} = -\mathrm{e}^{-\eta^2/4} \tag{6}$$

and so by recognizing an exact differential on the left-hand side, show that the solution is

$$g = \frac{\mathrm{e}^{-\eta^2/4}}{\sqrt{\pi}\,(\coth(G/2) + \operatorname{erf}(\eta/2))}, \tag{7}$$

where

$$G = \int_{-\infty}^{\infty} g(\eta)\,\mathrm{d}\eta \tag{8}$$

is a constant.

Solution

(a) Integration of (1) and use of (2) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}x = 0$$

and so

$$\int_{-\infty}^{\infty} f(x,t)\,\mathrm{d}x$$

is a constant.

(b) Try $f(x,t) = t^a g(\eta)$ with $\eta = x/t^b$. Equation (3) indicates that we must choose $a = -b$. Substituting the ansatz $f(x,t) = t^a g(\eta)$ with $\eta = x/t^b$ into (1) we obtain the ordinary differential equation

$$-bg - b\eta g' = t^{1-2b}g'' + \alpha x^{\alpha-1}t^{b+1-\beta b}\left(g^\beta + \beta\eta g^{\beta-1}g'\right) \tag{9}$$

and so we must choose

$$b = \frac{1}{2}, \qquad \alpha = \beta - 2 \tag{10}$$

for the equation to be in similarity variables.

(c) These choices can be imposed to equation (9) and then we may integrate this equation once. The constant of integration is determined to be zero by applying the conditions as $\eta \to \pm\infty$ and the regularity conditions given in the question assumptions that imply that ηg and $\eta^{\beta-2}g^\beta$ both tend to 0 as $\eta \to \pm\infty$. This leads to the required (5).

(d) Simple rearrangement and multiplication of both sides by $\mathrm{e}^{-\eta^2/4}$ gives (6). This can be written as

$$\frac{\partial}{\partial\eta}\left(\frac{\mathrm{e}^{-\eta^2/4}}{g}\right) = \mathrm{e}^{-\eta^2/4}. \tag{11}$$

Integration gives

$$g = \frac{\mathrm{e}^{-\eta^2/4}}{A + \sqrt{\pi}\mathrm{erf}(\eta/2)}; \tag{12}$$

A is determined by the integral constraint:

$$\int_{-\infty}^{\infty} g\,\mathrm{d}\eta = G, \text{ say}, \tag{13}$$

which may be evaluated and rearranged to give

$$A = \sqrt{\pi}\coth\left(G/2\right). \tag{14}$$

The result then follows.

2. **Outwardly radial spreading in a porous medium**

Consider the radial spreading of a fixed volume of liquid in a porous medium. The height $\hat{h}$ of the liquid is governed by the equation

$$\phi \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{1}{\hat{r}} \frac{\partial}{\partial \hat{r}} \left(\hat{r} \hat{h} \hat{Q} \right) = 0, \qquad \hat{Q} = -\frac{\rho g k}{\mu} \frac{\partial \hat{h}}{\partial \hat{r}} \tag{15}$$

where $\hat{r}$ and $\hat{t}$ denote respectively the radial coordinate and time and $\hat{Q}$ is the flux; ρ denotes the density of the fluid, g acceleration due to gravity, k the permeability, ϕ the porosity and μ the fluid viscosity.

(a) Write down the equation that expresses conservation of mass.

(b) By choosing suitable non-dimensionalization show that the system may be reduced to one that contains no physical parameters.

(c) By finding the appropriate form of the similarity solution, show that the problem can be reduced to solving the following ordinary differential equation system,

$$(\eta f f')' + \frac{1}{4}\eta^2 f' + \frac{1}{2}\eta f = 0, \tag{16}$$

$$\int_0^{\eta_f} \eta f(\eta)\, \mathrm{d}\eta = 1, \tag{17}$$

$$f'(0) = 0, \tag{18}$$

$$f(\eta_f) = 0, \tag{19}$$

where you should define the functions $\eta = \eta(r,t)$, $\eta_f = \eta_f(r,t)$ and $f = f(h,t)$.

(d) By rescaling $s = \eta/\eta_f$ and $g = f/\eta_f^2$ find the ordinary differential equation that is satisfied by g and a condition for η_f in terms of g.

(e) By performing a local analysis show that the conditions at the front are

$$g(1) = 0, \qquad g'(1) = -\frac{1}{4}. \tag{20}$$

(f) Hence show that the solution is given by $g(s) = (1 - s^2)/8$, $\eta_f \approx 2$.

(g) Based on the results of this analysis, is this a similarity solution of the first or second kind? What physical feature of the problem indicates that it is a similarity solution of this kind?

(a) Conservation of mass is expressed via the equation

$$2\pi \int_0^{\hat{r}_f} \hat{r}\hat{h}(\hat{r},\hat{t})\,\mathrm{d}\hat{r} = \hat{V}, \tag{21}$$

where $\hat{V}$ is the volume of liquid. We non-dimensionalize via the following scalings

$$\hat{r} = \hat{r}_0 r, \quad \hat{r}_f = \hat{r}_0 r_f, \quad \hat{t} = \left(\frac{2\pi r_0^4 \mu\phi}{\rho g k \hat{V}}\right) t, \quad \hat{h} = \left(\frac{\hat{V}}{2\pi \hat{r}_0^2}\right) h, \quad \hat{Q} = \left(\frac{\rho g k \hat{V}}{2\pi\mu \hat{r}_0^3}\right) Q. \tag{22}$$

Here, the radial scale $\hat{r}_0$ is arbitrary. The governing equations and mass conservation become

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rhQ) = 0, \tag{23}$$

$$Q = -\frac{\partial h}{\partial r}, \tag{24}$$

$$\int_0^{r_f} rh\,\mathrm{d}r = 1. \tag{25}$$

(b) Try $h = t^\alpha f(\eta)$ with $\eta = r/t^\beta$. Substituting into the governing equations and volume constraint show that we require $\alpha = -1/2$ and $\beta = 1/4$. This leads to the required system (16) and (17) while the conditions (18) and (19) represent respectively symmetry at the centre at the behaviour at the front.

(c) Applying the scalings given leads to the new system

$$(sgg')' + \frac{1}{4}s^2 g' + \frac{1}{2}sg = 0, \tag{26}$$

$$\eta_f = \left(\int_0^1 sg(s)\,\mathrm{d}s\right)^{-1/4}. \tag{27}$$

(d) We perform a local analysis by scaling $s = 1 + \delta\xi$, $g = \delta G$ to obtain

$$(G(\xi)G'(\xi))' + \frac{1}{4}G'(\xi) = 0 \tag{28}$$

$$\Rightarrow G'(\xi) = -\frac{1}{4}, \tag{29}$$

which in terms of the original variables gives the required result $g'(1) = -1/4$.

(e) It is straightforward to substitute the given form for g and show that this is a solution. Alternatively, one could try $g(s) = a + bs^2$ and find the required values for a and b. The value of η_f is obtained numerically.

(f) This is a similarity solution of the first kind as the form of the similarity solution was determined by substituting into the governing equation. The problem has no natural lengthscale, which is an indicator that a similarity solution exists.

3. **Inwardly radial spreading in a porous medium**

Consider again the radial spreading of a fixed volume of liquid in a porous medium as described by equation (15). Suppose that the liquid is now confined in a cylindrical container of radius $\hat{r}_0$ and the liquid occupies a region $\hat{r}_f(\hat{t}) \leq \hat{r} \leq \hat{r}_0$ where $\hat{r}_f$ moves inwardly with time.

(a) Write down the equation that expresses conservation of mass in this case and comment on how it differs from that in question 2.

(b) By using the results of question 2, show that the system may be reduced to one that contains no physical parameters.

(c) Let t_c denote the time at which the central dry hole closes. Define

$$\tau = t_c - t, \qquad h = \frac{r^2}{\tau}\bar{h}(r,\tau), \qquad Q = \frac{r}{\tau}\bar{Q}(r,\tau) \tag{30}$$

and show that in terms of these new variables the system may be written as

$$2\bar{h} + \bar{Q} + r\frac{\partial \bar{h}}{\partial r} = 0, \tag{31}$$

$$\tau\frac{\partial \bar{h}}{\partial \tau} - \bar{h} - 4\bar{h}\bar{Q} - r\frac{\partial}{\partial r}\left(\bar{h}\bar{Q}\right) = 0. \tag{32}$$

(d) Now suppose that $\bar{h} = \bar{h}(\eta)$, $\bar{Q} = \bar{Q}(\eta)$ where $\eta = r/\tau^\alpha$ is a similarity variable, for some α. Find the equations that are satisfied by $\bar{h}$ and $\bar{Q}$.

(e) Show that the system can be written in the form

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\bar{h}} = \frac{\bar{h} + 4\bar{h}\bar{Q} - \alpha(\bar{Q} + 2\bar{h}) - \bar{Q}(\bar{Q} + 2\bar{h})}{\bar{h}(\bar{Q} + 2\bar{h})}. \tag{33}$$

(f) Based on the results of this analysis, is this solution a similarity solution of the first or second kind? What physical feature of this problem indicates that it is a similarity solution of this kind?

Solution

(a) Conservation of mass is expressed via the equation

$$2\pi \int_{\hat{r}_f}^{\hat{r}_0} \hat{r}\hat{h}(\hat{r},\hat{t})\,\mathrm{d}\hat{r} = \hat{V}, \tag{34}$$

where $\hat{V}$ is the volume of liquid. This differs from the result in question 2(a) since now we have an additional length scale in the problem: the radius of the cylinder.

(b) We non-dimensionalize in exactly the same way as in question 2(b) except now the radial scaling $\hat{r}_0$ that was arbitrary in problem 1 is now chosen to be the radius of the cylinder. The governing equations and mass conservation then become

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rhQ) = 0, \tag{35}$$

$$Q = -\frac{\partial h}{\partial r}, \tag{36}$$

$$\int_{r_f}^{1} rh\,\mathrm{d}r = 1. \tag{37}$$

(c) This is a straightforward substitution.

(d) Making the substitution proposed gives

$$2\bar{h} + \bar{Q} + \eta\bar{h}' = 0, \tag{38}$$

$$-\alpha\eta\bar{h}' - \bar{h} - 4\bar{h}\bar{Q} - \eta\left(\bar{h}\bar{Q}\right)' = 0. \tag{39}$$

(e) Solving for $\bar{Q}'$ and $\bar{h}'$ and dividing one by the other gives the required result.

(f) This is a similarity solution of the second kind as we have not yet found the value of the scaling. This comes from the application of boundary conditions. Note that in the original formulation, we have a natural lengthscale, which is the radius of the container, $\hat{r}_0$. This is what prohibits a similarity solution of the first kind. However, with the introduction of the variable $\eta = r/\tau^{\alpha}$, since $\tau \to 0$ as the hole closes up, $\eta \to \infty$ and so we lose the lengthscale, enabling a similarity solution. However, the price we pay is that η_f cannot be found from our similarity analysis: this would need to be determined by comparing our similarity solution with the actual numerical (or experimental) solution at a particular point in time.

We find three fixed points of the system (33): $(\bar{h},\bar{Q}) = (0,0)$, $(0,-\alpha)$ and $(1/8,-1/4)$. The first point is the trivial solution. The second fixed point corresponds to the moving front and the third point corresponds to the fluid arriving at the origin. If we apply the first and second conditions then this provides two boundary conditions for the first-order system, which forms an eigenvalue problem. The eigenvalue, δ is found to be $\alpha \approx 0.856$.