Some inspirational quotations

George Polya: "Where should I start? Start from the statement of the problem. ... What can I do? Visualize the problem as a whole as clearly and as vividly as you can. ... What can I gain by doing so? You should understand the problem, familiarize yourself with it, impress its purpose on your mind."

Th. Bröcker and K. Jänich, "Introduction to differential topology" (p.25) "Having thus refreshed ourselves in the oasis of a proof, we now turn again into the desert of definitions."

Amalgams

Let A, B be groups with two isomorphic subgroups: i.e. there exist injective homomorphisms $\alpha : H \to A$, $\beta : H \to B$.

The amalgam of A and B over H is the "largest" group containing copies of A and B identified along H such that no other relation is imposed and such that it is generated by the copies of A and B.

We will define the amalgam by its universal property.

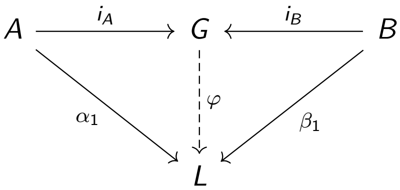
Amalgams

Notation: $\alpha(h) = h \in A$; $\beta(h) = \bar{h} \in B$.

Definition

G is the amalgamated product of A and B over H (written $G = A *_H B$) if

- there exist homomorphisms $i_A : A \to G$, $i_B : B \to G$ with $i_A(h) = i_B(\bar{h})$ for all $h \in H$;
- $\forall$ group L and $\forall$ homomorphisms $\alpha_1 : A \to L$, $\beta_1 : B \to L$ satisfying $\alpha_1(h) = \beta_1(\bar{h})$ for all $h \in H$, there exists a unique homomorphism $\varphi : G \to L$ such that $\alpha_1 = \varphi \circ i_A$ and $\beta_1 = \varphi \circ i_B$:



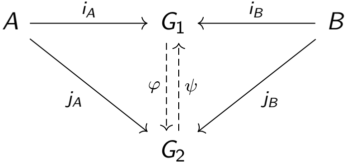
Amalgams

Remarks

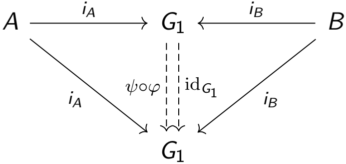
1. *The construction depends on the homomorphisms $\alpha : H \hookrightarrow A$, $\beta : H \hookrightarrow B$ but the notation is simplified.*
2. *It is not clear from the definition whether i_A and i_B are injective. However this turns out to be the case.*

Uniqueness of the amalgam

Uniqueness of the amalgam: Suppose G_1 and G_2 are both amalgams of A, B over H. Then we have a commutative diagram



This implies that $\mathrm{id}_{G_1} : G_1 \to G_1$ and $\psi \circ \varphi : G_1 \to G_1$ both make the following diagram commute



And so $\psi \circ \varphi = \mathrm{id}_{G_1}$ by uniqueness of the induced homomorphism. Similarly $\varphi \circ \psi = \mathrm{id}_{G_2}$.

Existence of the amalgam

Existence of the amalgam:

Let $A = \langle S_1 | R_1 \rangle$, $B = \langle S_2 | R_2 \rangle$. WLOG $S_1 \cap S_2 = \emptyset$. Then

$$A *_H B = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup \{h = \bar{h} : h \in H\} \rangle$$

Proof: Check that it satisfies the universal property (exercise).

Remarks

- *A and B generate $A *_H B$.*
- *i_A and i_B are injective.*

When $H = \{1\}$, the amalgam does not depend on α, β and it is called the free product of A and B, denoted by $A * B$.

Example

$F_2 = \mathbb{Z} * \mathbb{Z}$ *since if* $\mathbb{Z} = \langle a | \ \rangle$, $\mathbb{Z} = \langle b | \ \rangle$, *then* $\mathbb{Z} * \mathbb{Z} = \langle a, b | \ \rangle = F_2$.

Amalgams

We would like to describe the elements of $A *_H B$ by words.

Simplified notation: we identify H with $\alpha(H)$ and $\beta(H)$, and we identify A with $i_A(A)$, B with $i_B(B)$.

Let A_1 be a set of right coset representatives of H in A, and similarly let B_1 be a set of right coset representatives of H in B, such that $1 \in A_1$, $1 \in B_1$.

Definition

A reduced word of the amalgam $A *_H B$ is a word of the form $(h, s_1, ..., s_n)$, $h \in H$, $s_i \in A_1 \cup B_1$, $s_i \neq 1$, s_i alternating from A_1 to B_1. We associate to this the element $hs_1...s_n$ of $A *_H B$. The length of the reduced word is n.

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Amalgams

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Proof: For all $g \in G$, we can write $g = a_1 b_1 ... a_m b_m$ for some $a_i \in A$, $b_i \in B$.

We claim that g can be represented by a reduced word $(h, s_1, ..., s_n)$.

$m = 1$: $g = a_1 b_1 = a_1 \bar{h} b' = \underbrace{a_1 h}_{\in A} b' = h' a' b'$ where $a' \in A_1$, $b' \in B_1$.

Inductive step: exercise.

Uniqueness: Let X be the set of all reduced words. We will define an action of G on X, i.e. a group homomorphism

$$G \to Symm(X) = Bij(X)$$

Amalgams

By the universal property, it suffices to define $\alpha_1 : A \to Symm(X)$, $\beta_1 : B \to Symm(X)$ such that $\alpha_1(h) = \beta_1(\bar{h})$.

Definition of α_1:

Case 1: $a = h_0 \in H$:

$$h_0 \cdot (h, s_1, \ldots, s_n) = (h_0 h, s_1, \ldots, s_n)$$

Case 2: $a \in A \setminus H$.

2.a: $s_1 \in B$. $\forall h \in H$, write $ah = h'a'$ where $a' \in A_1$, $a' \neq 1$.

$$a \cdot (h, s_1, \ldots, s_n) = (h', a', s_1, \ldots, s_n)$$

2.b: $s_1 \in A$, $s_2 \in B$. $\forall h \in H$, write $ahs_1 = h'a'$, $a' \in A_1$.

$$\begin{aligned} a \cdot (h, s_1, \ldots, s_n) &= (h', a', s_2, \ldots, s_n) \quad \text{if } a' \neq 1 \\ &= (h', s_2, \ldots, s_n) \qquad \text{if } a' = 1 \end{aligned}$$

Amalgams

This defines a map $\sigma_a : X \to X$.

Exercise: Check that $\sigma_{a_1 a_2} = \sigma_{a_1} \circ \sigma_{a_2}$.

Therefore $\sigma_a \circ \sigma_{a^{-1}} = \mathrm{id}$ and so σ_a is a bijection. So we have defined $\alpha_1 : A \to Symm(X)$.

Likewise we can define $\beta_1 : B \to Symm(X)$. Therefore there exists some $\varphi : A *_H B \to Symm(X)$.

Exercise: $\forall g \in G$, if $g = hs_1 ... s_n$ is a reduced word then

$$\varphi(g)(1) = (h, s_1, ..., s_n)$$

And so the reduced word is unique. □

Amalgams

Theorem

Each $g \in G = A *_H B$ is represented by a unique reduced word.

Corollary

i_A and i_B are injective. Hence A, B can be seen as subgroups of $A *_H B$.

Corollary

If $(g_1, ..., g_n)$, $n \geq 2$, is such that $g_i \in A \cup B$, $g_i \notin H$, $\forall i \geq 2$, and g_i alternate between A and B, then $g_1 ... g_n \neq 1$ in $A *_H B$.

Proof.

Use induction to show that it can be represented by a reduced word of length $n-1$ if $g_1 \in H$ or of length n if $g_1 \notin H$. □