# Geometric Group Theory

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# Amalgams

**Theorem**

*Each $g \in G = A *_H B$ is represented by a unique reduced word.*

**Corollary**

*$i_A$ and $i_B$ are injective. Hence $A$ and $B$ can be seen as subgroups of $A *_H B$.*

**Corollary**

*If $(g_1, ..., g_n)$, $n \geq 2$, is such that $g_i \in A \cup B$, $g_i \notin H$, $\forall i \geq 2$, and $g_i$ alternate between $A$ and $B$, then $g_1 ... g_n \neq 1$ in $A *_H B$.*

**Proof.**

Use induction to show that it can be represented by a reduced word of length $n - 1$ if $g_1 \in H$ or of length $n$ if $g_1 \notin H$. $\square$

# Amalgams

**Theorem**

*Each $g \in G = A *_H B$ is represented by a unique reduced word.*

**Corollary**

*In $G$, $A \cap B = H$.*

**Definition**

The reduced word $(h, s_1, ..., s_n)$ and the reduced element $hs_1...s_n \in A *_H B$ are cyclically reduced if $n \geq 2$ and $s_1 s_n$ is reduced.

**Proposition**

- *Every $g \in A *_H B$ is conjugate either to a cyclically reduced element or to some $a \in A$ or to some $b \in B$.*
- *Every cyclically reduced element has infinite order.*

# Amalgams

## Proposition

1. *Every $g \in A *_H B$ is conjugate either to a cyclically reduced element or to some $a \in A$ or to some $b \in B$.*
2. *Every cyclically reduced word has infinite order.*

Proof: (1) : If $g = hs_1 ... s_n$ is not cyclically reduced, i.e. $s_1$, $s_n$ are both in $A$ or both in $B$, then $s_n g s_n^{-1}$ is represented by a word of length $n-1$.
Repeat until we have a cyclically reduced word or a word of length 1.

(2) : If $g$ is cyclically reduced of length $n \geq 2$ then $g^k$ has length $kn$, so $g^k \neq 1$. $\square$

## Corollary

*Given any finite subgroup $F \leq A *_H B$, $F$ must be contained in a conjugate $gAg^{-1}$ or $gBg^{-1}$.*

Proof: exercise.

# Amalgams and actions on trees

### Definition

- Suppose $G$ is a group acting on a graph $X$. We say that $G$ acts on $X$ without inversions if for every $g \in G$ and $[v, w] \in E(X)$ we have that $g([v, w]) \neq [w, v]$.
- A free action of $G$ on $X$ is an action that is free on the vertices and without inversions.

Suppose $G$ is a group acting without inversions on a tree $T$.

A subtree $S \subseteq T$ is a fundamental domain if it intersects the orbit $G \cdot v$ of every vertex $v$ of $T$, and it intersects the orbit of every edge exactly once.

### Theorem

*$G = A *_H B$ acts on a tree $T$ with fundamental domain an edge $[P, Q]$ such that* $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

# Amalgams and actions on trees

**Theorem**

*$G = A *_H B$ acts on a tree $T$ with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.*

Proof:

Let $V(T) = G/A \sqcup G/B$.

Edges are $(gA, gB)$, i.e. we join two left cosets of $A$ and $B$ if they have a common representative $g$. Given an edge what is the set of common representatives corresponding to it?

$$g_1 A = gA, \quad g_1 B = gB \iff g^{-1} g_1 \in A \cap B = H$$

So the set is exactly $gH$. We label the edge $(gA, gB)$ by $gH$ and the edge $(gB, gA)$ by $g\bar{H}$. Clearly $G$ acts transitively on the edges and there are two orbits of vertices.

$T$ is connected: For each edge $\{gA, gB\}$, $g = hs_1 \dots s_n$, we will prove it is connected by an edge path to $\{A, B\}$ by induction on $n$. Moreover, the length of the edge path (including $\{A, B\}$ and $\{gA, gB\}$) is $n+1$. The $n = 0$ case is obvious.

Induction: if $s_n \in A_1 \setminus \{1\}$ then

$$gA = \underbrace{hs_1 \dots s_{n-1}}_{g'} A$$

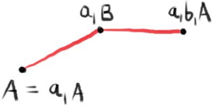
and $\{gA, gB\}$ shares a common endpoint with $\{g'A, g'B\}$.
Similarly, if $s_n \in B_1 \setminus \{1\}$ then $gB = hs_1 \dots s_{n-1} B$ and $\{gA, gB\}$ shares a common endpoint with $\{g'A, g'B\}$.

# Amalgams and actions on trees

$T$ is a tree: A path without spikes in $T$ of origin $A$ and even length $2n$ has vertices of the form:

$$A = a_1 A, a_1 B, a_1 b_1 A, \dots, a_1 b_1 \dots a_n b_n A$$

where $a_i \notin H$ and $b_i \notin H$.




An easy induction on $n$ shows that the reduced form of $a_1 b_1 \dots a_n b_n$ is $h a_1' b_1' \dots a_n' b_n'$: for $n = 1$ we have

$$a_1 b_1 = a_1 \underbrace{h b_1'}_{b_1' \neq 1 \text{ as } b_1 \notin H} = h' a_1' b_1' \quad \text{where} \quad a_1', b_1' \neq 1$$

# Amalgams and actions on trees

Likewise,

$$a_1 b_1 a_2 b_2 \ldots a_{n+1} b_{n+1} = a_1 b_1 h a_2' b_2' \ldots a_{n+1}' b_{n+1}' = h' a_1' b_1' \ldots a_{n+1}' b_{n+1}'$$

In particular we cannot have $a_1 b_1 \ldots a_n b_n A = A$ otherwise

$$\underbrace{h a_1' b_1' \ldots a_n' b_n'}_{\text{length } 2n} = \underbrace{h' a'}_{\text{length } 0 \text{ or } 1}$$

So there is no cycle through $A$ and so there is no cycle in $T$ (every cycle must contain one vertex in $G/A$ and so can be $G$-translated to a cycle through $A$). $\square$