Geometric Group Theory

Cornelia Druţu

University of Oxford

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A quotation

Leo Moser: “A mathematician named Klein

Thought the Möbius band was divine.

Said he: "If you glue

The edges of two,

You'll get a weird bottle like mine.”

Indeed, by glueing two Möbius bands one obtains a Klein bottle.

There is a second way of obtaining a Klein bottle, that relates to a construction we will introduce today: the HNN extension.

Amalgams and actions on trees

Theorem

$G = A *_H B$ acts on a tree T with fundamental domain an edge $[P, Q]$ such that $\mathrm{Stab}(P) = A$, $\mathrm{Stab}(Q) = B$, $\mathrm{Stab}([P, Q]) = H$.

Corollary

If $F \leq A *_H B$ is such that $F \cap gAg^{-1} = \{1\}$ and $F \cap gBg^{-1} = \{1\}$ for every $g \in G$, then F is free.

Proof.

F acts on the tree T with trivial vertex or edge stabilisers and so it acts freely on the tree T. □

Proposition

The kernel of the map $\varphi : A * B \to A \times B$ is free.

Amalgams and actions on trees

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The kernel of the map $\varphi : A * B \to A \times B$ is free.

Proof.

$F = \ker \varphi$ intersects gAg^{-1} and gBg^{-1} trivially. □

Corollary

If A, B finite, then $A * B$ is virtually free.

Amalgams and actions on trees

Theorem

Suppose $G \curvearrowright T$ with fundamental domain an edge $e = [P, Q]$. If $A = \mathrm{Stab}(P)$, $B = \mathrm{Stab}(Q)$, $H = \mathrm{Stab}(e)$ then $G = A *_H B$.

Proof

Since we have $\alpha_1 : A \to G$, $\beta_1 : B \to G$ agreeing on H, there exists some $\varphi : A *_H B \to G$, by the Universal Property of $A *_H B$.

Step 1: $G = \langle A, B \rangle$, that is, φ is onto.

For all $g \in G$, ge is joined to e by a unique path of length n (counting e and ge). We will prove that $g \in \langle A, B \rangle$ by induction on n. If $n = 1$ then $g \in H$.

Assume true for n, and let ge be joined to e by a path of length $n + 1$. Let $g'e$ be the previous edge on the path.

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Then either $gP = g'P$ or $gQ = g'Q$ and so $g^{-1}g' \in A \cup B$. Since $g' \in \langle A, B \rangle$ we are done.

Step 2: φ is injective.

Let $hs_1 \ldots s_n \in \ker \varphi$. We can prove by induction on n that $hs_1 \ldots s_n e$ can be joined to e by an edge path with no spikes of length $n+1$. Hence $hs_1 \ldots s_n \neq 1$ in G. □

HNN extensions

Definition

Suppose we have $A \subseteq G$ and $\theta : A \to G$ an injective homomorphism. The HNN extension of G on A with respect to θ is

$$\begin{aligned} G*_A &:= \langle G, t | tat^{-1} = \theta(a), \forall a \in A \rangle \\ &= G * \langle t \rangle \ / \ \langle\langle \{ tat^{-1}\theta(a)^{-1} : a \in A \} \rangle\rangle \end{aligned}$$

t is called the stable letter of the HNN extension.

The name comes from Graham Higman, Bernhard Neumann and Hanna Neumann.

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Remark

If $G = \langle S|R \rangle$ *then* $G*_A = \langle S \cup \{t\} | R \cup \{tat^{-1} = \theta(a) : a \in A\} \rangle$.

Examples

- *The Baumslag-Solitar groups* $BS(m, n) = \langle a, t | ta^m t^{-1} = a^n \rangle$, *where* $m, n \in \mathbb{Z}$.
- *When* $m = n = 1$, *we have* $\mathbb{Z}^2$ *(fundamental group of the torus.)*
- *When* $m = 1$, $n = -1$, *the fundamental group of the Klein bottle.*
- *When* $m = 1$ *(or* $n = 1$*) the group is solvable.*

Reduced words of HNN extensions

Suppose A_1 is a set of right coset representatives for A and A_2 is a set of right coset representatives for $\theta(A)$ such that $1 \in A_1 \cap A_2$.

A reduced word of $G*_A$ is some $(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, ..., t^{\epsilon_n}, g_n)$ such that

- $\epsilon = \pm 1$
- $g_0 \in G$
- $g_i \in A_1$ if $\epsilon_i = 1$, $g_i \in A_2$ if $\epsilon_i = -1$
- $g_i \neq 1$ if $\epsilon_{i+1} = -\epsilon_i$

A reduced element of $G*_A$ is an element of the form $g_0 t^{\epsilon_1} g_1 ... t^{\epsilon_n} g_n$.

Theorem

Each $g \in G*_A$ is represented by a unique reduced word.