C4.4: Hyperbolic Equations

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Course Weight: 1.00 unit(s)
Level: M-level
Method of assessment: Written examination
Course Term: Hilary Term 2023
Time & Place:

Lectures: 12:00 Mondays (L2) & Wednesdays (L6)
Tutorial Sessions: 9:30am–11:00am Tuesdays (Weeks 3, 5, 7)
(31 January, 14 & 28 February)

Recommended Prerequisites:

A good background in Multivariate Calculus and Lebesgue Integration is expected (e.g. as covered in the Oxford Prelims & Part A Integration). It would be useful to know some basic Functional Analysis and Distribution Theory; however, this is not strictly necessary as the presentation will be self-contained.

Course Overview:

We introduce analytical and geometric approaches to hyperbolic equations, by discussing model problems from transport equations, wave equations, and conservation laws. These approaches have been applied and extended extensively in recent research and lie at the heart of the Theory of Hyperbolic PDEs.

Learning Outcomes

You will learn the rigorous treatment of hyperbolic equations through analytical and geometric approaches as
as an introduction to the Theory of Hyperbolic PDEs.

You will see some model problems/methods for hyperbolic equations.

Course Synopsis:

1. **Transport equations and nonlinear first order equations:** Method of characteristics, formation of singularities
2. **Introduction to nonlinear hyperbolic conservation laws:** Discontinuous solutions, Rankine-Hugoniot relation, Lax entropy condition, shock waves, rarefaction waves, Riemann problem, entropy solutions, Lax-Oleinik formula, uniqueness.
3. **Linear wave equations:** The solution of Cauchy problem, energy estimates, finite speed of propagation, domain of determination, light cone and null frames, hyperbolic rotation and Lorentz vector fields, Sobolev inequalities, Klainerman inequality.
4. **Nonlinear wave equations:** local well-posedness, weak solutions

If time permits, we might also discuss parabolic approximation (viscosity method), compactness methods, Littlewood-Paley theory, and harmonic analysis techniques for hyperbolic equations/systems (off syllabus - not required for exam)

Reading List:

We refer to [1], [2, Chapters 2,3,5,7,11,12], and [3] for detailed exposition.

1. Alinhac, S.: Hyperbolic Partial Differential Equations,
   Springer-Verlag: New York, 2009.

2. Evans, L.: Partial Differential Equations. Second edition.
   Graduate Studies in Mathematics, 19.
   American Mathematical Society, 2010.

3. John, F.: Partial Differential Equations. Fourth edition.
   Applied Mathematical Sciences, 1.
   Springer-Verlag: New York, 1982

Please note that e-book versions of many books in the reading lists can be found on SOLO and ORLO.

III. Linear Wave Equations

We consider the Cauchy problem of linear wave equation

$$\begin{cases} u_{tt} - \triangle u = f(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{41}$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian operator on $\mathbb{R}^n$.

- A function $u \in C^2(\mathbb{R}^n \times [0,\infty))$ satisfying (41) is called a classical solution of (41).
- We prove the uniqueness result by deriving energy estimate and establish the existence result of classical solutions by deriving the solution formulae.

1. Uniqueness

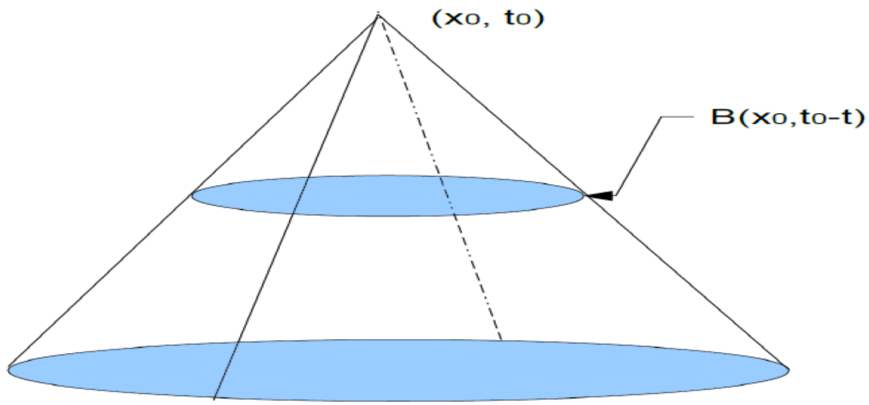
- We show that the Cauchy problem (41) has at most one classical solution.
- We establish uniqueness result by proving a general result, the so-called finite speed propagation property.
- Consider the homogeneous wave equation

$$\Box u := \partial_t^2 u - \triangle u = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty). \tag{42}$$

For any fixed $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, we introduce

$$C_{x_0, t_0} := \{(x, t) : 0 \le t \le t_0 \text{ and } |x - x_0| \le t_0 - t\}$$

which is called the backward light cone with vertex (x_0, t_0).



The following result says that any "disturbance" originating outside $B_{t_0}(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq t_0\}$ at $t = 0$ has no effect on the solution within C_{x_0,t_0}.

Theorem 22 (finite speed of propagation)

Let u be a C^2 solution of (42) in C_{x_0,t_0}. If $u(x,0) \equiv u_t(x,0) \equiv 0$ for $x \in B_{t_0}(x_0)$, then $u \equiv 0$ in C_{x_0,t_0}.

Proof. Consider for $0 \le t \le t_0$ the function

$$E(t) := \int_{B_{t_0-t}(x_0)} \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right) dx$$
$$= \int_0^{t_0-t} \int_{\partial B_\tau(x_0)} \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right) d\sigma(x) d\tau.$$

We have

$$\frac{d}{dt}E(t) = 2\int_{B_{t_0-t}(x_0)} \left(u_t(x,t)u_{tt}(x,t) + \nabla u(x,t) \cdot \nabla u_t(x,t)\right) dx$$
$$- \int_{\partial B_{t_0-t}(x_0)} \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right) d\sigma(x).$$

Since $\nabla u \cdot \nabla u_t = \operatorname{div}(u_t \nabla u) - u_t \triangle u$, we have

$$\frac{d}{dt}E(t) = 2\int_{B_{t_0-t}(x_0)} u_t \Box u dx + 2\int_{B_{t_0-t}(x_0)} \operatorname{div}(u_t \nabla u) dx - \int_{\partial B_{t_0-t}(x_0)} \left(|u_t|^2 + |\nabla u|^2\right) d\sigma.$$

Using $\Box u = 0$ and the divergence theorem we have

$$\frac{d}{dt}E(t) = 2\int_{\partial B_{t_0-t}(x_0)} u_t \nabla u \cdot \nu d\sigma - \int_{\partial B_{t_0-t}(x_0)} \left(|u_t|^2 + |\nabla u|^2\right) d\sigma,$$

where ν denotes the outward unit normal to $\partial B_{t_0-t}(x_0)$. We have

$$2|u_t \nabla u \cdot \nu| \leq 2|u_t||\nabla u| \leq |u_t|^2 + |\nabla u|^2.$$

Consequently $\frac{d}{dt}E(t) \leq 0$ which implies that

$$E(t) \leq E(0), \quad 0 \leq t \leq t_0.$$

Since $u(\cdot, 0) \equiv u_t(\cdot, 0) \equiv 0$ on $B_{t_0}(x_0)$, we have $E(0) = 0$. Thus $E(t) \equiv 0$ for $0 \leq t \leq t_0$. Therefore

$$u_t = \nabla u = 0 \quad \text{in } C_{x_0, t_0}.$$

So $u = constant$ in C_{x_0,t_0}. Since $u(x, 0) = 0$ for $x \in B_{t_0}(x_0)$, we must have $u \equiv 0$ in C_{t_0,x_0}. ■

Corollary 23

The Cauchy problem (41) of linear wave equation has at most one classical solution.

Proof. Assume that u_1 and u_2 are two classical solutions of (41). Then $u := u_1 - u_2 \in C^2(\mathbb{R}^n \times [0, \infty))$ satisfies

$$\begin{cases} \Box u = u_{tt} - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, & x \in \mathbb{R}^n. \end{cases}$$

Applying Theorem 22 to u, we conclude $u = 0$ in $\mathbb{R}^n \times [0, \infty)$. ■

2. Existence

The existence of (41) can be established by solving the following two problems:

$$\begin{cases} \Box u := u_{tt} - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^n \end{cases} \tag{43}$$

$$\begin{cases} \Box u := u_{tt} - \triangle u = f(x,t) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R}^n. \end{cases} \tag{44}$$

- If v is the solution of (43) and w is the solution of (44), then $u := v + w$ is the solution of (41).
- We will solve (43) by deriving the explicit solution formula.
- We then solve (44) by reducing it to a problem like (43) using the Duhamel principle.

We now derive the solution formula of (43) when $n = 1, 2, 3$.

Case $n = 1$: Consider the Cauchy problem of 1D homogeneous wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{45}$$

where $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$.

- Observing that $u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u$. We introduce $v = u_t + u_x$. Then $v_t - v_x = 0$ in $\mathbb{R} \times (0, \infty)$. By the method of Characteristics, we have

$$v(x,t) = v_0(x+t),$$

where $v_0(x) := v(x,0)$.

- So $u_t + u_x = v_0(x+t)$. Let $u_0(x) := u(x,0)$. Then, by the method of characteristics again, it follows

$$u(x,t) = u_0(x-t) + \int_0^t v_0(x-t+2s)ds$$
$$= u_0(x-t) + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)d\xi.$$

- The initial conditions give $u_0(x) = g(x)$ and $v_0(x) = h(x) + g'(x)$. Therefore

$$u(x,t) = g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} \left(g'(\xi) + h(\xi)\right) d\xi$$
$$= \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi.$$

We therefore obtain the following result.

Theorem 24

Assume that $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then the d'Alembert formula

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(\xi)d\xi$$

gives the unique classical solution of (45)

We next consider the Cauchy problem (41) in high dimensions.

- The general idea is to reduce the high dimensional problems to one-dimensional problem so that the d'Alembert formula can be used.

- This can be achieved by considering the spherical mean.
- Given $x \in \mathbb{R}^n$ and $r > 0$, we use $B_r(x)$ and $\partial B_r(x)$ to denote the ball of radius r with center x and its boundary respectively. Let ω_n denote the surface area of unit sphere, then

$$|\partial B_r(x)| = \omega_n r^{n-1} \quad \text{and} \quad |B_r(x)| = \frac{1}{n}\omega_n r^n.$$

- Let $u \in C^2(\mathbb{R}^n \times [0, \infty))$ be a solution of (41). For a fixed $x \in \mathbb{R}^n$, define

$$U(r, t; x) := \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y, t) d\sigma(y), \quad r > 0$$

which is called the mean value of u over the sphere $\partial B_r(x)$ at time t.

- Notice that
$$\lim_{r\to 0} U(r,t;x) = u(x,t).$$
If we can find a formula for $U(r,t;x)$ for $r > 0$, then we can obtain $u(x,t)$ by taking $r \to 0$.
- Write $U(r,t;x)$ as
$$U(r,t;x) = \frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi,t)\,d\sigma(\xi).$$
Then
$$\begin{aligned}\partial_r U(r,t;x) &= \frac{1}{\omega_n}\int_{|\xi|=1} \nabla u(x+r\xi,t)\cdot \xi\,d\sigma(\xi)\\ &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} \nabla u(y,t)\cdot \frac{y-x}{r}\,d\sigma(y).\end{aligned}$$

Since $(y-x)/r$ is the outward unit normal to $\partial B_r(x)$ at y, we may use the divergence theorem to derive

$$\partial_r U(r,t;x) = \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \triangle u(y,t)dy.$$

- Using polar coordinates, we have

$$\partial_r U(r,t;x) = \frac{1}{\omega_n r^{n-1}} \int_0^r \int_{\partial B_\tau(x)} \triangle u(y,t)d\sigma(y)d\tau.$$

Consequently

$$\partial_r^2 U(r,t;x)$$
$$= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} \triangle u(y,t)d\sigma(y) - \frac{n-1}{\omega_n r^n} \int_{B_r(x)} \triangle u(y,t)dy.$$

.

- By using $u_{tt} - \triangle u = 0$, we have

$$\partial_r^2 U(r,t;x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u_{tt}(y,t)\,d\sigma(y) - \frac{n-1}{r}\partial_r U(r,t;x)$$
$$= \partial_t^2 U(r,t;x) - \frac{n-1}{r}\partial_r U(r,t;x).$$

- By the above expressions, we have

$$\begin{aligned} \lim_{r\to 0} U(r,t;x) &= u(x,t), \\ \lim_{r\to 0} U_r(r,t;x) &= 0, \\ \lim_{r\to 0} U_{rr}(r,t;x) &= \frac{1}{n}\triangle u(x,t). \end{aligned} \tag{46}$$

- Moreover, if u is a C^2 solution of (43), then, for fixed $x \in \mathbb{R}^n$, $U(r,t;x)$ as a function of (r,t) is in $C^2([0,\infty)\times[0,\infty))$ and satisfies the Euler-Poisson-Darboux equation

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{for } r>0,\ t>0, \\ U = G, \quad U_t = H & \text{for } t = 0, \end{cases} \tag{47}$$

where

$$G(r;x) := \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} g(y)d\sigma(y),$$
$$H(r;x) := \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} h(y)d\sigma(y).$$

We hope to transform (47) into the usual 1D wave equation. This can be done easily when $n = 3$. So we consider this case first.

Case $n = 3$. We consider the Cauchy problem (43) of 3D wave equation. The Euler-Poisson-Darboux equation becomes

$$U_{tt} - U_{rr} - \frac{2}{r} U_r = 0.$$

Thus $\partial_r^2(rU) = \partial_t^2(rU)$. Let $\widetilde{U} = rU$, $\widetilde{G} = rG$ and $\widetilde{H} = rH$. Then

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 & \text{for } r > 0,\ t > 0, \\ \widetilde{U} = \widetilde{G}, \quad \widetilde{U}_t = \widetilde{H} & \text{at } t = 0 \text{ and } r > 0. \end{cases}$$

Moreover, in view of (46), we have

$$\widetilde{U} = 0,\ \widetilde{U}_r = u(x,t),\ \widetilde{U}_{rr} = 0 \quad \text{when } r = 0.$$

Thus, we may extend $\widetilde{U}$ to $\mathbb{R} \times [0, \infty)$ by odd reflection, i.e. we set

$$\overline{U}(r,t) = \begin{cases} \widetilde{U}(r,t;x), & r \geq 0,\ t \geq 0, \\ -\widetilde{U}(-r,t;x), & r < 0,\ t \geq 0. \end{cases}$$

Then $\overline{U} \in C^2(\mathbb{R} \times [0,\infty))$ and

$$\begin{cases} \overline{U}_{tt} - \overline{U}_{rr} = 0, & -\infty < r < \infty,\ t > 0, \\ \overline{U}(r,0) = \overline{G}(r), & \overline{U}_r(r,0) = \overline{H}(r), \quad -\infty < r < \infty, \end{cases}$$

where

$$\overline{G}(r) = \begin{cases} \widetilde{G}(r;x), & r \geq 0, \\ -\widetilde{G}(-r;x), & r < 0, \end{cases} \qquad \overline{H}(r) = \begin{cases} \widetilde{H}(r;x), & r \geq 0, \\ -\widetilde{H}(-r;x), & r < 0. \end{cases}$$

By the d'Alembert formula,

$$\overline{U}(r,t) = \frac{1}{2}\left(\overline{G}(r+t) + \overline{G}(r-t)\right) + \frac{1}{2}\int_{r-t}^{r+t} \overline{H}(s)ds.$$

Thus

$$\widetilde{U}(r,t;x)$$
$$= \begin{cases} \frac{1}{2}\left(\widetilde{G}(r+t) + \widetilde{G}(r-t)\right) + \frac{1}{2}\int_{r-t}^{r+t} \widetilde{H}(s)ds, & r > t > 0, \\ \frac{1}{2}\left(\widetilde{G}(r+t) - \widetilde{G}(t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \widetilde{H}(s)ds, & 0 \le r \le t. \end{cases}$$

Consequently, for $t > 0$ we have

$$u(x,t) = \lim_{r\to 0} \frac{1}{r}\widetilde{U}(r,t;x) = \widetilde{G}'(t) + \widetilde{H}(t).$$

Using the definition of $\widetilde{G}$ and $\widetilde{H}$, and the fact $|\partial B_r(x)| = 4\pi t^2$ in $\mathbb{R}^3$ we obtain

Theorem 25 (Kirchoff formula)

Let $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$. Then

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=t} g(y)d\sigma(y)\right) + \frac{1}{4\pi t}\int_{|y-x|=t} h(y)d\sigma(y)$$
$$= \frac{1}{4\pi t^2}\int_{|y-x|=t} \left(g(y) + \nabla g(y)\cdot(y-x) + th(y)\right) d\sigma(y)$$

gives the unique solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ of the Cauchy problem (43) for 3D wave equation.

Case $n = 2$:

- The procedure for $n = 3$ does not work for 2D wave equations.
- We use the Hadamard's method of descent to derive the solution formula for 2D wave equation from the Kirchoff formula for 3D wave equation.
- Write $x = (x_1, x_2)$ and $\bar{x} = (x, x_3)$ and consider the Cauchy problem of the 3D wave equation

$$\begin{cases} U_{tt} - \triangle U - U_{x_3 x_3} = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\ U(\bar{x}, 0) = g(x), \quad U_t(\bar{x}, 0) = h(x), \quad \bar{x} \in \mathbb{R}^3, \end{cases}$$

where $\triangle$ denotes 2D Laplacian, i.e. $\triangle U = U_{x_1 x_1} + U_{x_2 x_2}$.

- By the Kirchoff formula,

$$U(x, x_3, t) = U(\bar{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\bar{y}-\bar{x}|=t} g(y)d\sigma(\bar{y})\right) + \frac{1}{4\pi t}\int_{|\bar{y}-\bar{x}|=t} h(y)d\sigma(\bar{y})$$

  where $y = (y_1, y_2)$ and $\bar{y} = (y, y_3)$. Since g and h do not depend on y_3, U is independent of x_3 and hence it is a solution of the Cauchy problem (43) of 2D wave equation.
- We simplify U by rewriting the two integrals over the sphere $|\bar{y} - \bar{x}| = t$.

- The sphere $|\bar{y} - \bar{x}| = t$ is a union of the two hemispheres

$$y_3 = \phi_\pm(y) := x_3 \pm \sqrt{t^2 - |y - x|^2},$$

where $|y - x| \leq t$. On both hemispheres, we have

$$d\sigma(\bar{y}) = \sqrt{1 + |\nabla \phi_\pm(y)|^2} dy = \frac{t}{\sqrt{t^2 - |y - x|^2}} dy.$$

Therefore

$$U(x,t) = \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{|y-x|<t} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}} dy \right) + \frac{1}{2\pi} \int_{|y-x|<t} \frac{h(y)}{\sqrt{t^2 - |y - x|^2}} dy.$$

This immediately gives the following result.

Theorem 26 (Poisson formula)

Let $g \in C^3(\mathbb{R}^2)$ *and* $h \in C^2(\mathbb{R}^2)$. *Then*

$$u(x,t) = \partial_t \left(\frac{t}{2\pi} \int_{|y|<1} \frac{g(x+ty)}{\sqrt{1-|y|^2}} dy \right) + \frac{t}{2\pi} \int_{|y|<1} \frac{h(x+ty)}{\sqrt{1-|y|^2}} dy$$
$$= \frac{1}{2\pi} \int_{|y-x|<t} \frac{g(y) + th(y) + \nabla g(y) \cdot (y-x)}{\sqrt{t^2 - |y-x|^2}} dy$$

gives the unique solution in $C^2(\mathbb{R}^2 \times [0,\infty))$ *of the Cauchy problem (43) for 2D wave equation.*

The procedures for $n = 2, 3$ can be extended to derive solution formulae of the Cauchy problems (43) for higher dimensional wave equations.

Since the procedure is lengthy and boring, we state the results without proofs.

Theorem 27

If $g \in C^{[n/2]+2}(\mathbb{R}^n)$ and $h \in C^{[n/2]+1}(\mathbb{R}^n)$, then (43) has a unique solution $u \in C^2([0,\infty) \times \mathbb{R}^n)$, where $[n/2]$ denotes the greatest integer not greater than $n/2$.

Moreover, if $n \geq 3$ is odd, then, with $\gamma_n = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-2)$,

$$
\begin{aligned}
u(x,t) = & \frac{1}{\gamma_n} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(\frac{t^{n-2}}{|\partial B_t(x)|} \int_{\partial B_t(x)} g d\sigma \right) \\
& + \frac{1}{\gamma_n} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(\frac{t^{n-2}}{|\partial B_t(x)|} \int_{\partial B_t(x)} h d\sigma \right)
\end{aligned}
$$

while, if $n \geq 2$ is even, then, with $\gamma_n = 2 \cdot 4 \cdot \ldots \cdot (n-2) \cdot n$,

$$u(x,t) = \frac{1}{\gamma_n} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(\frac{t^n}{|B_t(x)|} \int_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} dy \right)$$
$$+ \frac{1}{\gamma_n} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(\frac{t^n}{|B_t(x)|} \int_{B_t(x)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}} d\sigma \right).$$

Remark.

- Given $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Theorem 22 shows that $u(x_0, t_0)$ is completely determined by the values of f and g in the ball $|x - x_0| \leq t_0$.
- When $n \geq 3$ is odd, by the solution formula this result can be strengthened: $u(t_0, x_0)$ depends only on the values of f and g (and derivatives) on the sphere $|x - x_0| = t_0$. This is called the Huygens' principle.

Duhamel Principle

We now consider the inhomogeneous problem (44), i.e.

$$\begin{cases} u_{tt} - \triangle u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R}, \end{cases} \tag{48}$$

where $f \in C^{[n/2]+1}(\mathbb{R}^n \times [0,\infty))$. We use the Duhamel principle, i.e. for any $s \geq 0$, we first consider the homogeneous problem

$$\begin{cases} w_{tt} - \triangle w = 0 & \text{in } \mathbb{R}^n \times (s,\infty), \\ w = 0, \quad w_t = f(\cdot,s), & \text{when } t = s \end{cases} \tag{49}$$

which has a unique solution, denoted as $w(x,t;s)$; we then define

$$u(x,t) = \int_0^t w(x,t;s)ds. \tag{50}$$

The following result shows that u is the solution of (48).

Theorem 28

Let $f \in C^{[n/2]+1}(\mathbb{R}^n \times [0,\infty))$. Then the u defined by (50) is the unique solution of (48) in $C^2(\mathbb{R}^n \times [0,\infty))$.

Proof. Clearly $u(x,0) = 0$ and

$$u_t(x,t) = w(x,t;t) + \int_0^t w_t(x,t;s)ds = \int_0^t w_t(x,t;s)ds.$$

So $u(x,0) = 0$. Moreover

$$u_{tt}(x,t) = w_t(x,t;t) + \int_0^t w_{tt}(x,t;s)ds = f(x,t) + \int_0^t \triangle w(x,t;s)ds$$
$$= f(x,t) + \triangle \int_0^t w(x,t;s)ds = f(x,t) + \triangle u(x,t).$$

■

We conclude this section by giving the explicit solution formulae of (48) for $n = 1, 2, 3$.

- When $n = 1$, by the d'Alembert formula the solution of (49) is given by

$$w(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy.$$

Therefore the solution of (48) for $n = 1$ is given by

$$\begin{aligned} u(x, t) &= \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y, s) dyds \\ &= \frac{1}{2} \int_0^t \int_{x-\tau}^{x+\tau} f(y, t-\tau) dyd\tau. \end{aligned}$$

- When $n = 3$, by the Kirchoff formula the solution of (49) is

$$w(x,t;s) = \frac{1}{4\pi(t-s)} \int_{|y-x|=t-s} f(y;s)\,d\sigma(y).$$

Therefore, the solution of (48) is

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi} \int_0^t \int_{|y-x|=t-s} \frac{f(y,s)}{t-s}\,d\sigma(y)\,ds \\ &= \frac{1}{4\pi} \int_0^t \int_{|y-x|=\tau} \frac{f(y,t-\tau)}{\tau}\,d\sigma(y)\,d\tau \\ &= \frac{1}{4\pi} \int_{|y-x|\le t} \frac{f(y,t-|y-x|)}{|y-x|}\,dy \end{aligned}$$

which is called the retarded potential because $u(x,t)$ depends on the values of f at the earlier times $t' = t - |y-x|$.

- When $n = 2$, by Poisson formula the solution of (49) is given by

$$w(x,t;s) = \frac{1}{2\pi} \int_{|y-x|<t-s} \frac{f(y,s)}{\sqrt{(t-s)^2 - |y-x|^2}} dy.$$

Therefore the solution of (48) is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \int_0^t \int_{|y-x|<t-s} \frac{f(y,s)}{\sqrt{(t-s)^2 - |y-x|^2}} dyds \\ &= \frac{1}{2\pi} \int_0^t \int_{|y-x|<\tau} \frac{f(y,t-\tau)}{\sqrt{\tau^2 - |y-x|^2}} dyd\tau. \end{aligned}$$

IV. Nonlinear Wave Equations

1. Local Existence of Semiliner Wave Equations

- We will consider the Cauchy problem of semi-linear wave equation

$$\begin{cases} \Box u := u_{tt} - \triangle u = F(u, \partial u), & \text{in } \mathbb{R}^n \times (0, T], \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^n \end{cases} \tag{51}$$

where $\partial u = (\partial_t u, \nabla u)$ and $F \in C^\infty$ satisfies $F(0,0) = 0$.

- Under certain conditions on g and h, we will establish a local existence result, i.e. there is a small $T > 0$ such that (51) has a unique solution in $\mathbb{R}^n \times [0, T]$.
- The proof is based on the Picard iteration which defines a sequence $\{u_m\}$; the solution of (51) is obtained by the limit of this sequence.

- The sequence $\{u_m\}$ is defined by solving the Cauchy problem of linear wave equation

$$\begin{cases} \Box u_m = F(u_{m-1}, \partial u_{m-1}), & \text{in } \mathbb{R}^n \times (0, T], \\ u_m(x,0) = g(x), \quad \partial_t u_m(x,0) = h(x), & x \in \mathbb{R}^n \end{cases} \tag{52}$$

for $m = 0, 1, \cdots$, where we set $u_{-1} = 0$.

- So it is necessary to understand the Cauchy problems of linear wave equations deeper.
- We need some knowledge on Sobolev spaces.

1.1. The Sobolev spaces H^s

For any fixed $s \in \mathbb{R}$, $H^s := H^s(\mathbb{R}^n)$ denotes the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

where $\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) dx$ is the Fourier transform of f.

- H^s is a Hilbert space and $H^0 = L^2$.
- If $s \geq 0$ is an integer, then $\|f\|_{H^s} \approx \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2}$.
- $H^{s_2} \subset H^{s_1}$ for any $-\infty < s_1 \leq s_2 < \infty$.
- H^{-s} is the dual space of H^s for any $s \in \mathbb{R}$.
- If $s > k + n/2$ for some integer $k \geq 0$, then $H^s \hookrightarrow C^k(\mathbb{R}^n)$ compactly and there is a constant C_s such that

$$\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty} \leq C_s \|f\|_{H^s}, \quad \forall f \in H^s.$$

- Given integer $k \geq 0$, $C^k([0, T], H^s)$ consists of functions $f(x,t)$ such that $t \to \|\partial_t^j f(\cdot, t)\|_{H^s}$ is continuous on $[0, T]$ for $j = 0, \cdots, k$. It is a Banach space under the norm

$$\sum_{j=0}^{k} \max_{0 \leq t \leq T} \|\partial_t^j f(\cdot, t)\|_{H^s}.$$

- $L^1([0, T], H^s)$ consists of functions $f(x,t)$ such that

$$\int_0^T \|f(\cdot, t)\|_{H^s} dt < \infty.$$

1.2. Solutions of linear wave equations

Let $\Box = \partial_t^2 - \triangle$ denote the d'Alembertian. We first establish the following energy estimate.

Lemma 29

For any $u \in C^2(\mathbb{R}^n \times [0, T])$ there holds

$$\|\partial u(\cdot, t)\|_{L^2} \le \|\partial u(\cdot, 0)\|_{L^2} + \int_0^t \|\Box u(\cdot, \tau)\|_{L^2} d\tau, \quad 0 \le t \le T.$$

Proof. Fix $T_0 > T$ and consider the energy

$$E(t) := \int_{|x| \le T_0 - t} \left(|u_t(x, t)|^2 + |\nabla u(x, t)|^2 \right) dx.$$

From the proof of Theorem 22 we have

$$\frac{d}{dt} E(t) \le 2 \int_{|x| \le T_0 - t} u_t(x, t) \Box u(x, t) dx.$$

By the Cauchy-Schwartz inequality we can obtain

$$\begin{aligned}\frac{d}{dt}E(t) &\le 2\left(\int_{|x|\le T_0-t}|u_t(x,t)|^2dx\right)^{1/2}\left(\int_{|x|\le T_0-t}|\Box u(x,t)|^2dx\right)^{1/2}\\ &= 2E(t)^{1/2}\|\Box u(\cdot,t)\|_{L^2(B_{T_0-t}(0))}.\end{aligned}$$

Therefore $\frac{d}{dt}E(t)^{1/2} \le \|\Box u(\cdot,t)\|_{L^2(B_{T_0-t}(0)}$. Consequently

$$\begin{aligned}\|\partial u(\cdot,t)\|_{L^2(B_{T_0-t}(0))} = E(t)^{1/2} &\le E(0)^{1/2} + \int_0^t \|\Box u(\cdot,\tau)\|_{L^2(B_{T_0-t}(0))}d\tau\\ &\le \|\partial u(\cdot,0)\|_{L^2} + \int_0^t \|\Box u(\cdot,\tau)\|_{L^2}d\tau.\end{aligned}$$

Letting $T_0 \to \infty$ gives the desired inequality. ■

The energy estimate in Lemma 29 can be extended as follows.

Theorem 30

Let $u \in C^\infty(\mathbb{R}^n \times [0, T])$. Then, for any $s \in \mathbb{R}$, there is a constant C depending on T such that

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot, t)\|_{H^s} \leq C \left(\sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot, 0)\|_{H^s} + \int_0^t \|\Box u(\cdot, \tau)\|_{H^s} d\tau \right)$$

for $0 \leq t \leq T$.

Proof. Consider only $s \in \mathbb{Z}$. We may assume that the right hand side is finite. There are three cases to be considered.

Case 1: $s = 0$. We need to establish

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot, t)\|_{L^2} \lesssim \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot, 0)\|_{L^2} + \int_0^t \|\Box u(\cdot, \tau)\|_{L^2} d\tau. \tag{53}$$

To see this, we first use Lemma 29 to obtain

$$\|\partial u(\cdot, t)\|_{L^2} \lesssim \|\partial u(\cdot, 0)\|_{L^2} + \int_0^t \|\Box u(\cdot, \tau)\|_{L^2} d\tau. \tag{54}$$

By the fundamental theorem of Calculus we can write

$$u(x, t) = u(x, 0) + \int_0^t u_t(x, \tau) d\tau.$$

Thus it follows from the Minkowski inequality that

$$\|u(\cdot,t)\|_{L^2} \leq \|u(\cdot,0)\|_{L^2} + \int_0^t \|u_t(\cdot,\tau)\|_{L^2}\, d\tau.$$

Adding this inequality to (54) gives

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot,t)\|_{L^2} \lesssim \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot,0)\|_{L^2} + \int_0^t \|\Box u(\cdot,\tau)\|_{L^2}\, d\tau$$
$$+ \int_0^t \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\cdot,\tau)\|_{L^2}\, d\tau.$$

An application of the Gronwall inequality then gives (53).

Case 2: $s \in \mathbb{N}$. Let β be any multi-index with $|\beta| \le s$. We apply (53) to $\partial_x^\beta u$ to obtain

$$\sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(\cdot,t)\|_{L^2} \lesssim \sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(\cdot,t)\|_{L^2} + \int_0^t \|\Box \partial_x^\beta u(\cdot,\tau)\|_{L^2} d\tau$$
$$\lesssim \sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(\cdot,0)\|_{L^2} + \int_0^t \|\partial_x^\beta \Box u(\cdot,\tau)\|_{L^2} d\tau.$$

Summing over all β with $|\beta| \le s$ we obtain

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(\cdot,t)\|_{H^s} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha u(\cdot,0)\|_{H^s} + \int_0^t \|\Box u(\cdot,\tau)\|_{H^s} d\tau.$$

Case 3: $s \in -\mathbb{N}$. We consider

$$v(\cdot, t) := (I - \triangle)^s u(\cdot, t).$$

Since $-s \in \mathbb{N}$, we can apply the estimate established in Case 2 to v to derive that

$$\sum_{|\alpha| \le 1} \|\partial^\alpha v(\cdot, t)\|_{H^{-s}} \lesssim \sum_{|\alpha| \le 1} \|\partial^\alpha v(\cdot, 0)\|_{H^{-s}} + \int_0^t \|\Box v(\cdot, \tau)\|_{H^{-s}} d\tau.$$

Since $\Box$ and $(I - \triangle)^s$ commute, we have

$$\Box v(\cdot, \tau) = (I - \triangle)^s \Box u(\cdot, \tau).$$

Therefore

$$\|\Box v(\cdot,\tau)\|_{H^{-s}} = \|\Box u(\cdot,\tau)\|_{H^s}.$$

Consequently

$$\sum_{|\alpha|\le 1} \|\partial^\alpha v(\cdot,t)\|_{H^{-s}} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha v(\cdot,0)\|_{H^{-s}} + \int_0^t \|\Box u(\cdot,\tau)\|_{H^s} d\tau.$$

Since $\|\partial^\alpha v(\cdot,t)\|_{H^{-s}} = \|\partial^\alpha u(\cdot,t)\|_{H^s}$, the proof is complete. ∎

We now prove the following existence and uniqueness result for the Cauchy problem of linear wave equation

$$\begin{cases} \Box u = f(x,t), & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = g(x), \quad \partial_t u(x,0) = h(x), & x \in \mathbb{R}^n \end{cases} \tag{55}$$

Theorem 31

If $g, h \in C^\infty(\mathbb{R}^n)$ *and* $f \in C^\infty(\mathbb{R}^n \times [0,T])$, *then (55) has a unique solution* $u \in C^\infty(\mathbb{R}^n \times [0,T])$. *If in addition there is* $s \in \mathbb{R}$ *such that*

$$g \in H^{s+1}(\mathbb{R}^n), \quad h \in H^s(\mathbb{R}^n) \quad \text{and} \quad f \in L^1([0,T], H^s(\mathbb{R}^n)),$$

then

$$u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$$

and, for $0 \le t \le T$ *there holds the estimate*

$$\sum_{|\alpha| \le 1} \|\partial^\alpha u(\cdot, t)\|_{H^s} \lesssim \|g\|_{H^{s+1}} + \|h\|_{H^s} + \int_0^t \|f(\cdot, \tau)\|_{H^s} d\tau.$$

Proof. The existence and uniqueness follow from the previous chapter. The remaining part is a consequence of Theorem 30. ∎

1.3. Solutions of semilinear wave equations

We next consider the semi-linear wave equation (51), i.e.

$$\begin{aligned} \Box u = F(u, \partial u) \qquad & \text{in } \mathbb{R}^n \times (0, T], \\ u(\cdot, 0) = g, \quad u_t(\cdot, 0) = h, & \end{aligned} \tag{56}$$

where $F \in C^\infty$ satisfies $F(0,0) = 0$.

- For this equation, there holds the finite propagation speed property, i.e. if $u \in C^2(\mathbb{R}^n \times [0, T])$ is a solution with $u(x,0) = u_t(x,0) = 0$ for $|x - x_0| \le t_0$, then $u \equiv 0$ in the backward light cone $\mathcal{C}_{x_0,t_0}$. (see Exercise)

Theorem 32

If $g, h \in C_0^\infty(\mathbb{R}^n)$, then there is a $T > 0$ such that (56) has a unique solution $u \in C_0^\infty(\mathbb{R}^n \times [0, T])$.

Proof. 1. We first prove uniqueness. Let u and $\tilde{u}$ be two solutions. Then $v := u - \tilde{u}$ satisfies

$$v_{tt} - \triangle v = R, \quad v(0, \cdot) = 0, \quad v_t(0, \cdot) = 0,$$

where $R := F(u, \partial u) - F(\tilde{u}, \partial \tilde{u})$. It is clear that

$$|R| \le C(|v| + |\partial v|).$$

In view of Theorem 30, we have

$$\sum_{|\alpha| \le 1} \|\partial^\alpha v(\cdot, t)\|_{L^2} \lesssim \int_0^t \|R(\cdot, \tau)\|_{L^2} d\tau \lesssim \int_0^t \sum_{|\alpha| \le 1} \|\partial^\alpha v(\cdot, \tau)\|_{L^2} d\tau.$$

By Gronwall inequality, $\sum_{|\alpha|\le 1}\|\partial^\alpha v\|_{L^2} = 0$. Thus $0 = v = u - \tilde{u}$.

2. Next we prove existence. We first fix an integer $s \ge n+2$.

- We use the Picard iteration. Let $u_{-1} = 0$ and define u_m, $m \ge 0$, successively by

$$\begin{aligned} &\Box u_m = F(u_{m-1}, \partial u_{m-1}) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ &u_m(\cdot,0) = g, \quad \partial_t u_m(\cdot,0) = h. \end{aligned} \tag{57}$$

  By Theorem 31, all u_m are in $C^\infty(\mathbb{R}^n \times [0,\infty))$.

- For any index γ satisfying $|\gamma| \le s$ we have

$$\Box \partial^\gamma u_m = \partial^\gamma [F(u_{m-1}, \partial u_{m-1})].$$

- Therefore, it follows from Theorem 30 that

$$\sum_{|\beta|\leq 1} \|\partial^\beta \partial^\gamma u_m(\cdot, t)\|_{L^2}$$

$$\leq C_0 \left(\sum_{|\beta|\leq 1} \|\partial^\beta \partial^\gamma u_m(\cdot, 0)\|_{L^2} + \int_0^t \|\partial^\gamma [F(u_{m-1}, \partial u_{m-1})]\|_{L^2} d\tau \right)$$

for all γ with $|\gamma| \leq s$. Summing over all such γ gives

$$\sum_{|\alpha|\leq s+1} \|\partial^\alpha u_m(\cdot, t)\|_{L^2}$$

$$\leq C_0 \left(\sum_{|\alpha|\leq s+1} \|\partial^\alpha u_m(\cdot, 0)\|_{L^2} + \int_0^t \sum_{|\alpha|\leq s} \|\partial^\alpha [F(u_{m-1}, \partial u_{m-1})]\|_{L^2} d\tau \right)$$

- Let

$$A_m(t) := \sum_{|\alpha|\leq s+1} \|\partial^\alpha u_m(\cdot, t)\|_{L^2}.$$

Then

$$A_m(t) \leq C_0\Big(A_m(0) + \int_0^t \sum_{|\alpha|\leq s} \|\partial^\alpha[F(u_{m-1}, \partial u_{m-1})]\|_{L^2} d\tau\Big).$$

By using (57) it is easy to show that

$$A_m(0) \leq A_0, \quad m = 0, 1, \cdots$$

for some number A_0 independent of m; in fact we can take A_0 to be a multiple of $\|g\|_{H^{s+1}} + \|h\|_{H^s}$.

- Consequently

$$A_m(t) \le C_0\Big(A_0 + \int_0^t \sum_{|\alpha|\le s} \|\partial^\alpha[F(u_{m-1}, \partial u_{m-1})]\|_{L^2} d\tau\Big). \tag{58}$$

Step 1. We show that there is $0 < T \le 1$ independent of m such that

$$A_m(t) \le 2C_0A_0, \quad \forall 0 \le t \le T \text{ and } m = 0, 1, \cdots . \tag{59}$$

- We prove (59) by induction on m. Since $F(0,0) = 0$ and $u_{-1} = 0$, we can obtain (59) with $m = 0$ from (58). Next we assume that (59) is true for $m = k$ and show that it is also true for $m = k + 1$. During the argument we will indicate the choice of T.

In view of (58), we have

$$A_{k+1}(t) \le C_0\Big(A_0 + \int_0^t \sum_{|\alpha|\le s} \|\partial^\alpha[F(u_k, \partial u_k)]\|_{L^2} d\tau\Big). \quad (60)$$

Observing that $\partial^\alpha[F(u_k, \partial u_k)]$ is the sum of the terms

$$a(u_k, \partial u_k)\partial^{\beta_1} u_k \cdots \partial^{\beta_l} u_k \partial^{\gamma_1} \partial u_k \cdots \partial^{\gamma_m} \partial u_k$$

where $|\beta_1| + \cdots + |\beta_l| + |\gamma_1| + \cdots + |\gamma_m| = |\alpha|$. Therefore $|\beta_j| \le |\alpha|/2$ and $|\gamma_j| \le |\alpha|/2$ except one of the multi-indices.

So $\partial^\alpha[F(u_k, \partial u_k)]$ is the sum of finitely many terms, each is a product of derivatives of u_k in which at most one factor where u_k is differentiated more than $|\alpha|/2 + 1 \le s/2 + 1$ times.

For $\partial^\gamma u_k$ with $|\gamma| \leq s/2+1$, by Sobolev embedding we have for $r > n/2+1+s/2$ that

$$\sum_{|\gamma|\leq s/2+1} |\partial^\gamma u_k(x,t)| \leq C \sum_{|\gamma|\leq r} \|\partial^\gamma u_k(\cdot,t)\|_{L^2}.$$

Since $s \geq n+2$, we have $s+1 > n/2+1+s/2$ and thus by induction hypothesis

$$\begin{aligned}\sum_{|\gamma|\leq s/2+1} |\partial^\gamma u_k(x,t)| &\leq C \sum_{|\gamma|\leq s+1} \|\partial^\gamma u_k(\cdot,t)\|_{L^2} \\ &\leq CA_k(t) \leq 2CC_0A_0. \end{aligned} \tag{61}$$

Therefore

$$|\partial^\alpha[F(u_k,\partial u_k)]| \le C_{A_0} \sum_{|\beta|\le s+1} |\partial^\beta u_k|, \quad \forall |\alpha| \le s.$$

Consequently, by the induction hypothesis, we have

$$\sum_{|\alpha|\le s} \|\partial^\alpha[F(u_k,\partial u_k)]\|_{L^2} \le C_{A_0} A_k(t) \le C_{A_0}. \tag{62}$$

In view of (60), we obtain

$$A_{k+1}(t) \le C_0\left(A_0 + C_{A_0} t\right) \le C_0\left(A_0 + C_{A_0} T\right), \quad 0 \le t \le T.$$

So, by taking $0 < T \le 1$ so small that $C_{A_0} T \le A_0$, we obtain $A_{k+1}(t) \le 2C_0 A_0$ for $0 \le t \le T$. This completes the proof of (59).

Step 2. Next we show that $\{u_m\}$ is convergent under the norm

$$\|u\| := \max_{0\le t\le T} \sum_{|\alpha|\le s+1} \|\partial^\alpha u(\cdot,t)\|_{L^2}.$$

To this end, consider

$$E_m(t) := \sum_{|\alpha|\le s+1} \|\partial^\alpha (u_{m+1}-u_m)(\cdot,t)\|_{L^2}.$$

By the definition of $\{u_m\}$, we have

$$\Box(u_{m+1}-u_m) = R_m \quad \text{in } \mathbb{R}^n\times(0,T],$$
$$(u_{m+1}-u_m)|_{t=0} = 0, \quad \partial_t(u_{m+1}-u_m)|_{t=0} = 0,$$

where

$$R_m := F(u_m, \partial u_m) - F(u_{m-1}, \partial u_{m-1}).$$

By the same argument for deriving (58), we obtain

$$E_m(t) \leq C_0 \int_0^t \sum_{|\alpha| \leq s} \|\partial^\alpha R_m(\cdot, \tau)\|_{L^2} d\tau.$$

By (59) and the similar argument for deriving (62) we have

$$\sum_{|\alpha| \leq s} \|\partial^\alpha R_m(\cdot, t)\|_{L^2} \leq C E_{m-1}(t).$$

Thus

$$E_m(t) \leq C \int_0^t E_{m-1}(\tau) d\tau, \quad m = 1, 2, \cdots .$$

Consequently

$$E_m(t) \le \frac{(Ct)^m}{m!} \sup_{0\le t\le T} E_0(t), \quad m = 0, 1, \cdots .$$

So $\sum_m E_m(t) \le C_0$. Therefore $\{u_m\}$ converges to some function u under the norm $|||\cdot|||$. By Sobolev embedding, we can conclude $u_m \to u$ in $C^{s+[(1-n)/2]}(\mathbb{R}^n \times [0, T])$ and hence in $C^2(\mathbb{R}^n \times [0, T])$ since $s \ge n+2$. By taking $m \to \infty$ in (57) we obtain that u is a solution of (56).

Step 3. The T obtained in Step 1 depends on s. If we can show (59), i.e.

$$\sum_{|\alpha|\le s+1} \|\partial^\alpha u_m(\cdot, t)\|_{L^2} \le A_s, \quad 0 \le t \le T$$

for all $m = 0, 1, \cdots$ with $T > 0$ independent of s, then we can conclude that $u \in C^\infty(\mathbb{R}^n \times [0, T])$.

- We now fix $s_0 \geq n + 3$ and let $T > 0$ be such that

$$\max_{0 \leq t \leq T} \sum_{|\alpha| \leq s_0 + 1} \|\partial^\alpha u_m(\cdot, t)\|_{L^2} \leq C_0 < \infty, \quad m = 0, 1, \cdots$$

  and show that for all $s \geq s_0$ there holds

$$\max_{0 \leq t \leq T} \sum_{|\alpha| \leq s+1} \|\partial^\alpha u_m(t, \cdot)\|_{L^2} \leq C_s < \infty, \quad \forall m. \tag{63}$$

- We show (63) by induction on s. Assume that (63) is true for some $s \geq s_0$, we show it is also true with s replaced by $s + 1$.

By the induction hypothesis and Sobolev embedding,

$$\max_{(x,t)\in\mathbb{R}^n\times[0,T]} \sum_{|\alpha|\le s+1-[(n+2)/2]} |\partial^\alpha u_m(x,t)| \le A_s < \infty, \quad \forall m.$$

Since $s \ge n+3$, we have $[(s+4)/2] \le s+1-[(n+2)/2]$. So

$$\max_{(x,t)\in\mathbb{R}^n\times[0,T]} \sum_{|\alpha|\le(s+4)/2} |\partial^\alpha u_m(x,t)| \le A_s, \quad \forall m.$$

This is exactly (61) with s replaced by $s+2$. Same argument there can be used to derive that

$$\max_{0\le t\le T} \sum_{|\alpha|\le s+2} \|\partial^\alpha u_m(\cdot,t)\|_{L^2} \le C_{s+1} < \infty, \quad \forall m.$$

We complete the induction argument and obtain a C^∞ solution. ∎

- The interval of existence for semi-linear wave equation could be very small.
- The following theorem gives a criterion on extending solutions which is important in establishing global existence results.

Theorem 33 (Continuation principle)

Assume that u be the solution of the Cauchy problem (56) with $g, h \in C_0^\infty(\mathbb{R}^n)$*. Let*

$$T_* := \sup \{ T > 0 : u \text{ satisfies (56) on } [0, T] \} .$$

If $T_* < \infty$*, then*

$$\sum_{|\alpha| \leq (n+6)/2} |\partial^\alpha u(t, x)| \notin L^\infty(\mathbb{R}^n \times [0, T_*]). \tag{64}$$

Proof. Assume that (64) does not hold, then

$$\sup_{[0,T_*)\times\mathbb{R}^n} \sum_{|\alpha|\le(n+6)/2} |\partial^\alpha u(t,x)| \le C < \infty.$$

Applying the argument in deriving (59) we have

$$\sup_{\mathbb{R}^n\times[0,T_*)} \sum_{|\alpha|\le s_0+1} \|\partial^\alpha u(\cdot,t)\|_{L^2} \le C_0 < \infty$$

where $s_0 = n+3$. By the argument in Step 3 of the proof of Theorem 32 we obtain for all $s \ge s_0$ that

$$\sup_{[0,T_*)\times\mathbb{R}^n} \sum_{|\alpha|\le s+1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \le C_s < \infty.$$

So u can be extend to $u \in C^\infty([0, T_*] \times \mathbb{R}^n)$.

Since $g, h \in C_0^\infty(\mathbb{R}^n)$, by the finite speed of propagation we can find a number R (possibly depending on T_*) such that $u(x,t) = 0$ for all $|x| \geq R$ and $0 \leq t < T_*$. Consequently

$$u(x, T_*) = \partial_t u(x, T_*) = 0 \quad \text{when } |x| \geq R.$$

Thus, $u(x, T_*)$ and $\partial_t u(x, T_*)$ are in $C_0^\infty(\mathbb{R}^n)$, and can be used as initial data at $t = T_*$ to extend u beyond T_* by theorem 32. This contradicts the definition of T_*. ■

2. Invariant Vector Fields in Minkowski Space

First are some conventions. We will set

$$\mathbb{R}^{1+n} := \{(t, x) : t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n\},$$

where t denotes the time and $x := (x^1, \cdots, x^n)$ the space variable. We sometimes write $t = x^0$ and use

$$\partial_0 = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_j := \frac{\partial}{\partial x^j} \text{ for } j = 1, \cdots, n.$$

For any multi-index $\alpha = (\alpha_0, \cdots, \alpha_n)$ and any function $u(t, x)$ we write

$$|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \partial^\alpha u := \partial_0^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u.$$

Given any function $u(t, x)$, we use

$$|\partial_x u|^2 := \sum_{j=1}^{n} |\partial_j u|^2 \quad \text{and} \quad |\partial u|^2 := |\partial_0 u|^2 + |\partial_x u|^2.$$

We will use Einstein summation convention: *any term in which an index appears twice stands for the sum of all such terms as the index assumes all of a preassigned range of values.*

- A Greek letter is used for index taking values $0, \cdots, n$.
- A Latin letter is used for index taking values $1, \cdots, n$.

For instance

$$b^\mu \partial_\mu u = \sum_{\mu=0}^{n} b^\mu \partial_\mu u \quad \text{and} \quad b^j \partial_j u = \sum_{j=1}^{n} b^j \partial_j u.$$

2.1 Vector fields and tensor fields

- We use $x = (x^0, x^1, \cdots, x^n)$ to denote the natural coordinates in $\mathbb{R}^{1+n}$, where $x^0 = t$ denotes time variable.
- A vector field X in $\mathbb{R}^{1+n}$ is a first order differential operator of the form

$$X = \sum_{i=0}^{n} X^\mu \frac{\partial}{\partial x^\mu} = X^\mu \partial_\mu,$$

where X^μ are smooth functions. We will identify X with (X^μ).

- The collection of all vector fields on $\mathbb{R}^{1+n}$ is called the tangent space of $\mathbb{R}^{1+n}$ and is denoted by $T\mathbb{R}^{1+n}$.

- For any two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$, one can define the Lie bracket

$$[X, Y] := XY - YX.$$

Then

$$\begin{aligned}
[X, Y] &= (X^\mu \partial_\mu)(Y^\nu \partial_\nu) - (Y^\nu \partial_\nu)(X^\mu \partial_\mu) \\
&= X^\mu Y^\nu \partial_\mu \partial_\nu + X^\mu (\partial_\mu Y^\nu) \partial_\nu - Y^\nu X^\mu \partial_\nu \partial_\mu - Y^\nu (\partial_\nu X^\mu) \partial_\mu \\
&= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \partial_\nu = (X(Y^\mu) - Y(X^\mu)) \partial_\mu .
\end{aligned}$$

So $[X, Y]$ is also a vector field.

- A linear mapping $\eta : T\mathbb{R}^{1+n} \to \mathbb{R}$ is called a 1-form if

$$\eta(fX) = f\eta(X), \quad \forall f \in C^\infty(\mathbb{R}^{1+n}), X \in T\mathbb{R}^{1+n}.$$

For each $\mu = 0, 1, \cdots, n$, we can define the 1-form dx^μ by

$$dx^\mu(X) = X^\mu, \quad \forall X = X^\mu \partial_\mu \in T\mathbb{R}^{1+n}.$$

Then for any 1-form η we have

$$\eta(X) = X^\mu \eta(\partial_\mu) = \eta_\mu dx^\mu(X), \quad \text{where } \eta_\mu := \eta(\partial_\mu).$$

Thus any 1-form in $\mathbb{R}^{1+n}$ can be written as $\eta = \eta_\mu dx^\mu$ with smooth functions η_μ. We will identify η with (η_μ).

- A bilinear mapping $T : T\mathbb{R}^{1+n} \times T\mathbb{R}^{1+n} \to \mathbb{R}$ is called a (covariant) 2-tensor field if for any $f \in C^\infty(\mathbb{R}^{1+n})$ and $X, Y \in T\mathbb{R}^{1+n}$ there holds

$$T(fX, Y) = T(X, fY) = fT(X, Y).$$

It is called symmetric if $T(X,Y) = T(Y,X)$ for all vector fields X and Y.

- Let
$$(\mathbf{m}_{\mu\nu}) = \text{diag}(-1, 1, \cdots, 1)$$
be the $(1+n) \times (1+n)$ diagonal matrix. We define $\mathbf{m} : T\mathbb{R}^{1+n} \times T\mathbb{R}^{1+n} \to \mathbb{R}$ by
$$\mathbf{m}(X,Y) := \mathbf{m}_{\mu\nu} X^\mu Y^\nu$$
for all $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$ in $T\mathbb{R}^{1+n}$. It is easy to check $\mathbf{m}$ is a symmetric 2-tensor field on $\mathbb{R}^{1+n}$. We call $\mathbf{m}$ the Minkowski metric on $\mathbb{R}^{1+n}$. Clearly
$$\mathbf{m}(X,X) = -\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + (X^n)^2.$$

- A vector field X in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called space-like, time-like, or null if

$$\mathbf{m}(X, X) > 0, \quad \mathbf{m}(X, X) < 0, \quad \text{or} \quad \mathbf{m}(X, X) = 0$$

respectively. Consider the three vector fields $X_1 = 2\partial_0 - \partial_1$, $X_2 = \partial_0 - \partial_1$ and $X_3 = \partial_0 - 2\partial_1$. Then X_1 is time-like, X_2 is null, and X_3 is space-like.

- In $(\mathbb{R}^{1+n}, \mathbf{m})$ we define the d'Alembertian

$$\Box = \mathbf{m}^{\mu\nu}\partial_\mu\partial_\nu, \quad \text{where } (\mathbf{m}^{\mu\nu}) := (\mathbf{m}_{\mu\nu})^{-1}.$$

In terms of the coordinates $(t, x^1, \cdots, x^n)$, $\Box = -\partial_t^2 + \triangle$, where $\triangle = \partial_1^2 + \cdots + \partial_n^2$.

2.2 Energy-momentum tensor

- In order to derive the general energy estimates related to $\Box u = 0$, we introduce the so called energy-momentum tensor.
- To see how to write down this tensor, we consider a vector field $X = X^\mu \partial_\mu$ with constant X^μ. Then for any smooth function u we have

$$
\begin{aligned}
(Xu)\Box u &= X^\rho \partial_\rho u \, \mathbf{m}^{\mu\nu} \partial_\mu \partial_\nu u \\
&= \partial_\mu \left(X^\rho \mathbf{m}^{\mu\nu} \partial_\nu u \partial_\rho u \right) - X^\rho \mathbf{m}^{\mu\nu} \partial_\mu \partial_\rho u \partial_\nu u.
\end{aligned}
$$

Using the symmetry of $(\mathbf{m}^{\mu\nu})$ we can obtain

$$
X^\rho \mathbf{m}^{\mu\nu} \partial_\mu \partial_\rho u \partial_\nu u = \partial_\rho \left(\frac{1}{2} X^\rho \mathbf{m}^{\mu\nu} \partial_\mu u \partial_\nu u \right).
$$

Therefore $(Xu)\Box u = \partial_\nu \left(Q[u]^\nu_\mu X^\mu\right)$, where

$$Q[u]^\nu_\mu = \mathbf{m}^{\nu\rho}\partial_\rho u \partial_\mu u - \frac{1}{2}\delta^\nu_\mu \left(\mathbf{m}^{\rho\sigma}\partial_\rho u \partial_\sigma u\right)$$

in which δ^ν_μ denotes the Kronecker symbol, i.e. $\delta^\nu_\mu = 1$ when $\mu = \nu$ and 0 otherwise.

- This motivates to introduce the symmetric 2-tensor

$$Q[u]_{\mu\nu} := \mathbf{m}_{\mu\rho} Q[u]^\rho_\nu = \partial_\mu u \partial_\nu u - \frac{1}{2}\mathbf{m}_{\mu\nu}\left(\mathbf{m}^{\rho\sigma}\partial_\rho u \partial_\sigma u\right)$$

which is called the energy-momentum tensor associated to $\Box u = 0$. Then for any vector fields X and Y we have

$$Q[u](X, Y) = (Xu)(Yu) - \frac{1}{2}\mathbf{m}(X, Y)\mathbf{m}(\partial u, \partial u)$$

- For a 1-form η in $(\mathbb{R}^{1+n}, \mathbf{m})$, its divergence is a function defined by
$$\mathrm{div}\eta := \mathbf{m}^{\mu\nu}\partial_\mu \eta_\nu.$$
For a symmetric 2-tensor field T in $(\mathbb{R}^{1+n}, \mathbf{m})$, its divergence is a 1-form defined by
$$(\mathrm{div}\, T)_\rho := \mathbf{m}^{\mu\nu}\partial_\mu T_{\nu\rho}.$$
- The divergence of the energy-momentum tensor is
$$\begin{aligned}(\mathrm{div} Q[u])_\rho &= \mathbf{m}^{\mu\nu}\partial_\mu Q[u]_{\nu\rho} \\ &= \mathbf{m}^{\mu\nu}\partial_\mu \left(\partial_\nu u \partial_\rho u - \frac{1}{2}\mathbf{m}_{\nu\rho}\left(\mathbf{m}^{\sigma\eta}\partial_\sigma u \partial_\eta u\right)\right) \\ &= \mathbf{m}^{\mu\nu}\partial_\mu\partial_\nu u \partial_\rho u = (\Box u)\partial_\rho u.\end{aligned}$$

- Let X be a vector field. Using $Q[u]$ we can introduce the 1-form

$$P_\mu := Q[u]_{\mu\nu} X^\nu.$$

Then its divergence is

$$\begin{aligned}
\mathrm{div} P = \mathbf{m}^{\mu\nu}\partial_\mu P_\nu &= \mathbf{m}^{\mu\nu}\partial_\mu \left(\mathbf{Q}[u]_{\nu\rho} X^\rho\right) \\
&= \mathbf{m}^{\mu\nu}\partial_\mu Q[u]_{\nu\rho} X^\rho + \mathbf{m}^{\mu\nu} Q[u]_{\nu\rho}\partial_\mu X^\rho \\
&= (\mathrm{div} Q[u])_\rho X^\rho + \mathbf{m}^{\mu\nu} Q[u]_{\nu\rho}\partial_\mu X^\rho \\
&= \Box u\, \partial_\rho u\, X^\rho + \mathbf{m}^{\mu\nu} Q[u]_{\nu\rho}\mathbf{m}^{\rho\eta}\partial_\mu X_\eta \\
&= (\Box u) X u + \frac{1}{2} Q[u]^{\mu\rho}\left(\partial_\mu X_\rho + \partial_\rho X_\mu\right).
\end{aligned}$$

where $Q[u]^{\mu\nu} := \mathbf{m}^{\mu\rho}\mathbf{m}^{\sigma\nu} Q[u]_{\rho\sigma}$ and $X_\eta := \mathbf{m}_{\rho\eta} X^\rho$.

- For a vector field X, we define

$$^{(X)}\pi_{\mu\nu} := \partial_\mu X_\nu + \partial_\nu X_\mu$$

which is called the deformation tensor of X with respect to $\mathbf{m}$. Then we have

$$\mathrm{div} P = \partial_\mu(\mathbf{m}^{\mu\nu} P_\nu) = (\Box u)Xu + \frac{1}{2} Q[u]^{\mu\nu}\,{}^{(X)}\pi_{\mu\nu}. \tag{65}$$

- Assume that u vanishes for large $|x|$ at each t. Then for any $t_0 < t_1$, we integrate $\mathrm{div} P$ over $[t_0, t_1] \times \mathbb{R}^n$ and note that ∂_t is the upward unit normal to each slice $\{t\} \times \mathbb{R}^n$, we obtain

$$\iint_{[t_0,t_1]\times\mathbb{R}^n} \mathrm{div} P dxdt = \int_{\{t=t_1\}} Q[u](X, \partial_t) dx - \int_{\{t=t_0\}} Q[u](X, \partial_t) dx.$$

This together with (65) then implies

Theorem 34

Let $u \in C^2(\mathbb{R}^{1+n})$ that vanishes for large $|x|$ at each t. Then for any vector field X and $t_0 < t_1$ there holds

$$\int_{\{t=t_1\}} Q[u](X,\partial_t)dx = \int_{\{t=t_0\}} Q[u](X,\partial_t)dx + \iint_{[t_0,t_1]\times\mathbb{R}^n} (\Box u)Xudxdt + \frac{1}{2}\iint_{[t_0,t_1]\times\mathbb{R}^n} Q[u]^{\mu\nu}\,{}^{(X)}\pi_{\mu\nu}dxdt. \tag{66}$$

- By choosing X suitably, many useful energy estimates can be derived from Theorem 34.

- For instance, we may take $X = \partial_t$ in Theorem 34. Notice that ${}^{(\partial_t)}\pi = 0$ and

$$Q[u](\partial_t, \partial_t) = \frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right),$$

we obtain for $E(t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^n}(|\partial_t u|^2 + |\nabla u|^2)dx$ the identity

$$E(t) = E(t_0) + \int_{t_0}^{t}\int_{\mathbb{R}^n} \Box u\,\partial_t u dx dt', \quad \forall t \geq t_0.$$

This implies that

$$\frac{d}{dt}E(t) = \int_{\{t\}\times\mathbb{R}^n} \Box u \partial_t u dx \leq \sqrt{2}\|\Box u(\cdot, t)\|_{L^2(\mathbb{R}^n)} E(t)^{1/2}.$$

Therefore

$$\frac{d}{dt}E(t)^{1/2} \le \frac{1}{\sqrt{2}}\|\Box u(\cdot,t)\|_{L^2(\mathbb{R}^n)}.$$

Consequently we obtain the energy estimate

$$E(t)^{1/2} \le E(t_0)^{1/2} + \frac{1}{\sqrt{2}}\int_{t_0}^{t} \|\Box u(\cdot,t')\|_{L^2(\mathbb{R}^n)}dt', \quad \forall t \ge t_0.$$

3.3. Killing vector fields

The identity (66) can be significantly simplified if ${}^{(X)}\pi = 0$. A vector field $X = X^\mu \partial_\mu$ in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called a *Killing vector field* if ${}^{(X)}\pi = 0$, i.e.

$$\partial_\mu X_\nu + \partial_\nu X_\mu = 0 \qquad \text{in } \mathbb{R}^{1+n}.$$

Corollary 35

Let $u \in C^2(\mathbb{R}^{1+n})$ that vanishes for large $|x|$ at each t. Then for any Killing vector field X and $t_0 < t_1$ there holds

$$\int_{\{t=t_1\}} Q[u](X, \partial_t)dx = \int_{\{t=t_0\}} Q[u](X, \partial_t)dx + \iint_{[t_0,t_1]\times\mathbb{R}^n} (\Box u)Xudxdt.$$

- We can determine all Killing vector fields in $(\mathbb{R}^{1+n}, \mathbf{m})$. Write $\pi_{\mu\nu} = {}^{(X)}\pi_{\mu\nu}$, Then

$$\partial_\rho \pi_{\mu\nu} = \partial_\rho\partial_\mu X_\nu + \partial_\rho\partial_\nu X_\mu,$$
$$\partial_\mu \pi_{\nu\rho} = \partial_\mu\partial_\nu X_\rho + \partial_\mu\partial_\rho X_\nu,$$
$$\partial_\nu \pi_{\rho\mu} = \partial_\nu\partial_\rho X_\mu + \partial_\nu\partial_\mu X_\rho.$$

- Therefore

$$\partial_\mu \pi_{\nu\rho} + \partial_\nu \pi_{\rho\mu} - \partial_\rho \pi_{\mu\nu} = 2\partial_\mu \partial_\nu X_\rho.$$

If X is a Killing vector field, then ${}^{(X)}\pi = 0$ and hence

$$\partial_\mu \partial_\nu X_\rho = 0 \quad \text{for all } \mu, \nu, \rho.$$

Thus each X_ρ is an affine function, i.e. there are constants $a_{\rho\nu}$ and b_ρ such that

$$X_\rho = a_{\rho\nu} x^\nu + b_\rho.$$

Using ${}^{(X)}\pi = 0$ again we have

$$0 = \partial_\mu X_\nu + \partial_\nu X_\mu = a_{\nu\mu} + a_{\mu\nu}.$$

- Therefore $a_{\mu\nu} = -a_{\nu\mu}$ and thus

$$
\begin{aligned}
X = X^{\mu}\partial_{\mu} = \mathbf{m}^{\mu\nu}X_{\nu}\partial_{\mu} &= \mathbf{m}^{\mu\nu}\left(a_{\nu\rho}x^{\rho} + b_{\nu}\right)\partial_{\mu} \\
&= \sum_{\nu=0}^{n}\left(\sum_{\rho<\nu} + \sum_{\rho>\nu}\right) a_{\nu\rho}x^{\rho}\mathbf{m}^{\mu\nu}\partial_{\mu} + \mathbf{m}^{\mu\nu}b_{\nu}\partial_{\mu} \\
&= \sum_{\nu=0}^{n}\sum_{\rho<\nu} a_{\nu\rho}x^{\rho}\mathbf{m}^{\mu\nu}\partial_{\mu} + \sum_{\rho=0}^{n}\sum_{\nu<\rho} a_{\nu\rho}x^{\rho}\mathbf{m}^{\mu\nu}\partial_{\mu} + \mathbf{m}^{\mu\nu}b_{\nu}\partial_{\mu} \\
&= \sum_{\nu=0}^{n}\sum_{\rho<\nu}\left(a_{\nu\rho}x^{\rho}\mathbf{m}^{\mu\nu}\partial_{\mu} + a_{\rho\nu}x^{\nu}\mathbf{m}^{\mu\rho}\partial_{\mu}\right) + \mathbf{m}^{\mu\nu}b_{\nu}\partial_{\mu} \\
&= \sum_{\nu=0}^{n}\sum_{\rho<\nu} a_{\nu\rho}\left(x^{\rho}\mathbf{m}^{\mu\nu}\partial_{\mu} - x^{\nu}\mathbf{m}^{\mu\rho}\partial_{\mu}\right) + \mathbf{m}^{\mu\nu}b_{\nu}\partial_{\mu}.
\end{aligned}
$$

Thus we obtain the following result on Killing vector fields.

Proposition 36

Any Killing vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ can be written as a linear combination of the vector fields ∂_μ, $0 \le \mu \le n$ and

$$\Omega_{\mu\nu} = (\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu)\, \partial_\rho, \quad 0 \le \mu < \nu \le n.$$

- Since $(\mathbf{m}^{\mu\nu}) = \text{diag}(-1, 1, \cdots, 1)$, the vector fields $\{\Omega_{\mu\nu}\}$ consist of the following elements

$$\Omega_{0i} = x^i \partial_t + t\partial_i, \quad 1 \le i \le n,$$
$$\Omega_{ij} = x^j \partial_i - x^i \partial_j, \quad 1 \le i < j \le n.$$

2.3 Conformal Killing vector fields

- When ${}^{(X)}\pi_{\mu\nu} = f\mathbf{m}_{\mu\nu}$ for some function f, the identity (66) can still be modified into a useful identity. To see this, we use (65) to obtain

$$\begin{aligned}\mathrm{div} P &= \partial_\mu(\mathbf{m}^{\mu\nu} P_\nu) = (\Box u)Xu + \frac{1}{2} f\mathbf{m}^{\mu\nu} Q[u]_{\mu\nu} \\ &= (\Box u)Xu + \frac{1-n}{4} f\mathbf{m}^{\mu\nu}\partial_\mu u \partial_\nu u.\end{aligned}$$

We can write

$$\begin{aligned}f\mathbf{m}^{\mu\nu}\partial_\mu u\partial_\nu u &= \mathbf{m}^{\mu\nu}\partial_\mu(fu\partial_\nu u) - \mathbf{m}^{\mu\nu} u\partial_\mu f\partial_\nu u - fu\Box u \\ &= \mathbf{m}^{\mu\nu}\partial_\mu(fu\partial_\nu u) - \mathbf{m}^{\mu\nu}\partial_\nu\left(\frac{1}{2}u^2\partial_\mu f\right) + \frac{1}{2}u^2\Box f - fu\Box u \\ &= \mathbf{m}^{\mu\nu}\partial_\mu\left(fu\partial_\nu u - \frac{1}{2}u^2\partial_\nu f\right) + \frac{1}{2}u^2\Box f - fu\Box u\end{aligned}$$

Consequently

$$\partial_\mu(\mathbf{m}^{\mu\nu} P_\nu) = (\Box u)Xu + \frac{1-n}{4}\mathbf{m}^{\mu\nu}\partial_\mu\left(fu\partial_\nu u - \frac{1}{2}u^2\partial_\nu f\right) + \frac{1-n}{8}u^2\Box f - \frac{1-n}{4}fu\Box u$$

Therefore, by introducing

$$\widetilde{P}_\mu := P_\mu + \frac{n-1}{4}fu\partial_\mu u - \frac{n-1}{8}u^2\partial_\mu f,$$

we obtain

$$\mathrm{div}\widetilde{P} = \partial_\mu(\mathbf{m}^{\mu\nu}\widetilde{P}_\nu) = \Box u\left(Xu + \frac{n-1}{4}fu\right) - \frac{n-1}{8}u^2\Box f.$$

By integrating over $[t_0, t_1] \times \mathbb{R}^n$ as before, we obtain

Theorem 37

If X is a vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ with ${}^{(X)}\pi = f\mathbf{m}$, then for any smooth function u vanishing for large $|x|$ there holds

$$\int_{t=t_1} \widetilde{Q}(X, \partial_t)dx = \int_{t=t_0} \widetilde{Q}(X, \partial_t)dx - \frac{n-1}{8} \iint_{[t_0,t_1]\times\mathbb{R}^n} u^2 \Box f dxdt$$
$$+ \iint_{[t_0,t_1]\times\mathbb{R}^n} \left(Xu + \frac{n-1}{4} fu \right) \Box u dxdt,$$

where $t_0 \le t_1$ and

$$\widetilde{Q}(X, \partial_t) := Q[u](X, \partial_t) + \frac{n-1}{4} \left(fu\partial_t u - \frac{1}{2} u^2 \partial_t f \right).$$

- A vector field $X = X^\mu \partial_\mu$ in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called conformal Killing if there is a function f such that ${}^{(X)}\pi = f\mathbf{m}$, i.e. $\partial_\mu X_\nu + \partial_\nu X_\mu = f\mathbf{m}_{\mu\nu}$.
- Any Killing vector field is conformal Killing. However, there are vector fields which are conformal Killing but not Killing.
  (i) Consider the vector field

$$L_0 = \sum_{\mu=0}^{n} x^\mu \partial_\mu = x^\mu \partial_\mu.$$

we have $(L_0)^\mu = x^\mu$ and so $(L_0)_\mu = \mathbf{m}_{\mu\nu} x^\nu$. Consequently

$$\begin{aligned} {}^{(L_0)}\pi_{\mu\nu} &= \partial_\mu (L_0)_\nu + \partial_\nu (L_0)_\mu = \partial_\mu (\mathbf{m}_{\nu\eta} x^\eta) + \partial_\nu (\mathbf{m}_{\mu\eta} x^\eta) \\ &= \mathbf{m}_{\nu\eta} \delta_\mu^\eta + \mathbf{m}_{\mu\eta} \delta_\nu^\eta = 2\mathbf{m}_{\mu\nu}. \end{aligned}$$

Therefore L_0 is conformal Killing and ${}^{(L_0)}\pi = 2\mathbf{m}$.

(ii) For each fixed $\mu = 0, 1, \cdots, n$ consider the vector field

$$K_\mu := 2\mathbf{m}_{\mu\nu} x^\nu x^\rho \partial_\rho - \mathbf{m}_{\eta\nu} x^\eta x^\nu \partial_\mu.$$

We have $(K_\mu)^\rho = 2\mathbf{m}_{\mu\nu} x^\nu x^\rho - \mathbf{m}_{\eta\nu} x^\eta x^\nu \delta_\mu^\rho$. Therefore

$$(K_\mu)_\rho = \mathbf{m}_{\rho\eta} (K_\mu)^\eta = 2\mathbf{m}_{\rho\eta} \mathbf{m}_{\mu\nu} x^\nu x^\eta - \mathbf{m}_{\rho\mu} \mathbf{m}_{\nu\eta} x^\nu x^\eta.$$

By direct calculation we obtain

$$^{(K_\mu)}\pi_{\rho\eta} = \partial_\rho (K_\mu)_\eta + \partial_\eta (K_\mu)_\rho = 4\mathbf{m}_{\mu\nu} x^\nu \mathbf{m}_{\rho\eta}.$$

Thus each K_μ is conformal Killing and $^{(K_\mu)}\pi = 4\mathbf{m}_{\mu\nu} x^\nu \mathbf{m}$. The vector field K_0 is due to Morawetz (1961).

All these conformal Killing vector fields can be found by looking at $X = X^\mu \partial_\mu$ with X^μ being quadratic.

- We can determine all conformal Killing vector fields in $(\mathbb{R}^{1+n}, \mathbf{m})$ when $n \geq 2$.

Proposition 38

Any conformal Killing vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ can be written as a linear combination of the vector fields

$$\partial_\mu, \qquad 0 \leq \mu \leq n,$$

$$\Omega_{\mu\nu} = (\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu)\partial_\rho, \quad 0 \leq \mu < \nu \leq n,$$

$$L_0 = \sum_{\mu=0}^{n} x^\mu \partial_\mu,$$

$$K_\mu = \mathbf{m}_{\mu\nu} x^\nu x^\rho \partial_\rho - \mathbf{m}_{\rho\nu} x^\rho x^\nu \partial_\mu, \quad \mu = 0, 1, \cdots, n.$$

Proof. Let X be conformal Killing, i.e. there is f such that

$$^{(X)}\pi_{\mu\nu} := \partial_\mu X_\nu + \partial_\nu X_\mu = f\mathbf{m}_{\mu\nu}. \tag{67}$$

We first show that f is an affine function. Recall that

$$2\partial_\mu\partial_\nu X_\rho = \partial_\mu\pi_{\nu\rho} + \partial_\nu\pi_{\rho\mu} - \partial_\rho\pi_{\mu\nu}.$$

Therefore

$$2\partial_\mu\partial_\nu X_\rho = \mathbf{m}_{\nu\rho}\partial_\mu f + \mathbf{m}_{\rho\mu}\partial_\nu f - \mathbf{m}_{\mu\nu}\partial_\rho f.$$

This gives

$$2\Box X_\rho = 2\mathbf{m}^{\mu\nu}\partial_\mu\partial_\nu X_\rho = (1-n)\partial_\rho f. \tag{68}$$

In view of (67), we have

$$(n+1)f = 2\mathbf{m}^{\mu\nu}\partial_\mu X_\nu$$

This together with (68) gives

$$(n+1)\Box f = 2\mathbf{m}^{\mu\nu}\partial_\mu \Box X_\nu = (1-n)\mathbf{m}^{\mu\nu}\partial_\mu\partial_\nu f = (1-n)\Box f.$$

So $\Box f = 0$. By using again (68) and (67) we have

$$(1-n)\partial_\mu\partial_\nu f = \frac{1-n}{2}(\partial_\mu\partial_\nu f + \partial_\nu\partial_\mu f) = \partial_\mu \Box X_\nu + \partial_\nu \Box X_\mu$$
$$= \Box(\partial_\mu X_\nu + \partial_\nu X_\mu) = \mathbf{m}_{\mu\nu}\Box f = 0.$$

Since $n \geq 2$, we have $\partial_\mu\partial_\nu f = 0$. Thus f is an affine function, i.e. there are constants a_μ and b such that $f = a_\mu x^\mu + b$.

Consequently

$$ {}^{(X)}\pi = (a_\mu x^\mu + b)\mathbf{m}. $$

Recall that ${}^{(L_0)}\pi = 2\mathbf{m}$ and ${}^{(K_\mu)}\pi = 4\mathbf{m}_{\mu\nu}x^\nu \mathbf{m}$. Therefore, by introducing the vector field

$$ \widetilde{X} := X - \frac{1}{2} b L_0 - \frac{1}{4}\mathbf{m}^{\mu\nu} a_\nu K_\mu, $$

we obtain

$$ {}^{(\widetilde{X})}\pi = {}^{(X)}\pi - \frac{1}{2} b \, {}^{(L_0)}\pi - \frac{1}{4}\mathbf{m}^{\mu\nu} a_\nu \, {}^{(K_\mu)}\pi = 0. $$

Thus $\widetilde{X}$ is Killing. We may apply Proposition 36 to conclude that $\widetilde{X}$ is a linear combination of ∂_μ and $\Omega_{\mu\nu}$. The proof is complete. ∎

3. Klainermann-Sobolev Inequality

We turn to global existence of Cauchy problems for nonlinear wave equations

$$\Box u = F(u, \partial u).$$

This requires good decay estimates on $|u(t,x)|$ for large t. Recall the classical Sobolev inequality

$$|f(x)| \le C \sum_{|\alpha| \le (n+2)/2} \|\partial^\alpha f\|_{L^2}, \quad \forall x \in \mathbb{R}^n$$

which is very useful. However, it is not enough for the purpose. To derive good decay estimates for large t, one should replace ∂f by Xf with suitable vector fields X that exploits the structure of Minkowski space. This leads to Klainerman inequality of Sobolev type.

The formulation of Klainerman inequality involves only the constant vector fields

$$\partial_\mu, \quad 0 \leq \mu \leq n$$

and the homogeneous vector fields

$$\begin{aligned} L_0 &= x^\rho \partial_\rho, \\ \Omega_{\mu\nu} &= \left(\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu\right) \partial_\rho, \quad 0 \leq \mu < \nu \leq n. \end{aligned}$$

There are $m+1$ such vector fields, where $m = \frac{(n+1)(n+2)}{2}$. We will use Γ to denote any such vector field, i.e. $\Gamma = (\Gamma_0, \cdots, \Gamma_m)$ and for any multi-index $\alpha = (\alpha_0, \cdots, \alpha_m)$ we adopt the convention $\Gamma^\alpha = \Gamma_0^{\alpha_0} \cdots \Gamma_m^{\alpha_m}$.

It is now ready to state the Klainerman inequality of Sobolev type, which will be used in the proof of global existence.

Theorem 39 (Klainerman)

Let $u \in C^\infty([0,\infty) \times \mathbb{R}^n)$ vanish when $|x|$ is large. Then

$$(1+t+|x|)^{n-1}(1+|t-|x||)|u(t,x)|^2 \le C \sum_{|\alpha| \le \frac{n+2}{2}} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2$$

for $t > 0$ and $x \in \mathbb{R}^n$, where C depends only on n.

We skip the proof of Theorem 39 since the argument is rather lengthy. Before using this result, deeper understanding on the vector fields Γ is necessary.

Lemma 40 (Commutator relations)

Among the vector fields ∂_μ, $\Omega_{\mu\nu}$ *and* L_0 *we have the commutator relations:*

$$
\begin{aligned}
[\partial_\mu, \partial_\nu] &= 0, \\
[\partial_\mu, L_0] &= \partial_\mu, \\
[\partial_\rho, \Omega_{\mu\nu}] &= \left(\mathbf{m}^{\sigma\mu}\delta^\nu_\rho - \mathbf{m}^{\sigma\nu}\delta^\mu_\rho\right)\partial_\sigma, \\
[\Omega_{\mu\nu}, \Omega_{\rho\sigma}] &= \mathbf{m}^{\sigma\mu}\Omega_{\rho\nu} - \mathbf{m}^{\rho\mu}\Omega_{\sigma\nu} + \mathbf{m}^{\rho\nu}\Omega_{\sigma\mu} - \mathbf{m}^{\sigma\nu}\Omega_{\rho\mu}, \\
[\Omega_{\mu\nu}, L_0] &= 0.
\end{aligned}
$$

Therefore, the commutator between ∂_μ *and any other vector field is a linear combination of* $\{\partial_\nu\}$*, and the commutator of any two homogeneous vector fields is a linear combination of homogeneous vector fields.*

Proof. These identity can be checked by direct calculation. As an example, we derive the formula for $[\Omega_{\mu\nu}, \Omega_{\rho\sigma}]$. Recall that

$$\Omega_{\mu\nu} = \left(\mathbf{m}^{\eta\mu} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\mu}\right) \partial_{\eta}.$$

Therefore

$$\begin{aligned}
[\Omega_{\mu\nu},\Omega_{\rho\sigma}] &= \Omega_{\mu\nu} \left(\mathbf{m}^{\eta\rho} x^{\sigma} - \mathbf{m}^{\eta\sigma} x^{\rho}\right) \partial_{\eta} - \Omega_{\rho\sigma} \left(\mathbf{m}^{\eta\mu} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\mu}\right) \partial_{\eta} \\
&= \left(\mathbf{m}^{\gamma\mu} x^{\nu} - \mathbf{m}^{\gamma\nu} x^{\mu}\right) \left(\mathbf{m}^{\eta\rho} \delta^{\sigma}_{\gamma} - \mathbf{m}^{\eta\sigma} \delta^{\rho}_{\gamma}\right) \partial_{\eta} \\
&\quad - \left(\mathbf{m}^{\gamma\rho} x^{\sigma} - \mathbf{m}^{\gamma\sigma} x^{\rho}\right) \left(\mathbf{m}^{\eta\mu} \delta^{\nu}_{\gamma} - \mathbf{m}^{\eta\nu} \delta^{\mu}_{\gamma}\right) \partial_{\eta} \\
&= \mathbf{m}^{\sigma\mu} \left(\mathbf{m}^{\eta\rho} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\rho}\right) \partial_{\eta} - \mathbf{m}^{\rho\mu} \left(\mathbf{m}^{\eta\sigma} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\sigma}\right) \partial_{\eta} \\
&\quad + \mathbf{m}^{\rho\nu} \left(\mathbf{m}^{\eta\sigma} x^{\mu} - \mathbf{m}^{\eta\mu} x^{\sigma}\right) \partial_{\eta} - \mathbf{m}^{\sigma\nu} \left(\mathbf{m}^{\eta\rho} x^{\mu} - \mathbf{m}^{\eta\mu} x^{\rho}\right) \partial_{\eta} \\
&= \mathbf{m}^{\sigma\mu} \Omega_{\rho\nu} - \mathbf{m}^{\rho\mu} \Omega_{\sigma\nu} + \mathbf{m}^{\rho\nu} \Omega_{\sigma\mu} - \mathbf{m}^{\sigma\nu} \Omega_{\rho\mu}.
\end{aligned}$$

This shows the result. ∎

Lemma 41

For any $0 \le \mu, \nu \le n$ *there hold*

$$[\Box, \partial_\mu] = 0, \quad [\Box, \Omega_{\mu\nu}] = 0, \quad [\Box, L_0] = 2\Box$$

Consequently, for any multiple-index α *there exist constants* $c_{\alpha\beta}$ *such that*

$$\Box\Gamma^\alpha = \sum_{|\beta| \le |\alpha|} c_{\alpha\beta}\Gamma^\beta\Box. \tag{69}$$

Proof. Direct calculation. ∎

4. Global Existence in Higher Dimension

We consider in $\mathbb{R}^{1+n}$ the global existence of the Cauchy problem

$$\begin{aligned} \Box u &= F(\partial u) \\ u|_{t=0} &= \varepsilon f, \qquad \partial_t u|_{t=0} = \varepsilon g, \end{aligned} \tag{70}$$

where $n \geq 4$, $\varepsilon \geq 0$ is a number, and $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is a given C^∞ function which vanishes to the second order at the origin:

$$F(0) = 0, \quad \mathbf{D}F(0) = 0. \tag{71}$$

The main result is as follows.

Theorem 42

Let $n \geq 4$ and let $f, g \in C_c^\infty(\mathbb{R}^n)$. If F is a C^∞ function satisfying (71), then there exists $\varepsilon_0 > 0$ such that (70) has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^n)$ for any $0 < \varepsilon \leq \varepsilon_0$.

Proof. Let

$$T_* := \sup\{T > 0 : (70) \text{ has a solution } u \in C^\infty([0,T] \times \mathbb{R}^n)\}.$$

Then $T_* > 0$ by Theorem 33. We only need to show that $T_* = \infty$. Assume that $T_* < \infty$, then Theorem 33 implies

$$\sum_{|\alpha| \leq (n+6)/2} |\partial^\alpha u(t,x)| \notin L^\infty([0, T_*) \times \mathbb{R}^n).$$

We will derive a contradiction by showing that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there holds

$$\sup_{(t,x)\in[0,T_*)\times\mathbb{R}^n} \sum_{|\alpha|\leq(n+6)/2} |\partial^\alpha u(t,x)| < \infty. \tag{72}$$

Step 1. We derive (72) by showing that there exist $A > 0$ and $\varepsilon_0 > 0$ such that

$$A(t) := \sum_{|\alpha|\leq n+4} \|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2} \leq A\varepsilon, \quad 0 \leq t < T_* \tag{73}$$

for $0 < \varepsilon \leq \varepsilon_0$, where the sum involves all invariant vector fields ∂_μ, L_0 and $\Omega_{\mu\nu}$.

In fact, by Klainerman inequality in Theorem 39 we have for any multi-index β that

$$|\partial\Gamma^\beta u(t,x)| \leq C(1+t)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq(n+2)/2} \|\Gamma^\alpha\partial\Gamma^\beta u(t,\cdot)\|_{L^2}.$$

Since $[\Gamma,\partial]$ is either 0 or $\pm\partial$, see Lemma 40, using (73) we obtain for $|\beta| \leq (n+6)/2$ that

$$\begin{aligned}|\partial\Gamma^\beta u(t,x)| &\leq C(1+t)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq n+4} \|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2} \\ &= C(1+t)^{-\frac{n-1}{2}} A(t) \\ &\leq CA\varepsilon(1+t)^{-\frac{n-1}{2}}. \end{aligned} \tag{74}$$

To estimate $|\Gamma^\beta u(t,x)|$, we need further property of u. Since $f, g \in C_0^\infty(\mathbb{R}^n)$, we can choose $R > 0$ such that

$$f(x) = g(x) = 0 \quad \text{for } |x| \geq R.$$

By the finite speed of propagation,

$$u(t,x) = 0, \quad \text{if } 0 \leq t < T_* \text{ and } |x| \geq R + t.$$

To show (72), it suffices to show that

$$\sup_{0 \leq t < T_*, |x| \leq R+t} |\Gamma^\alpha u(t,x)| < \infty, \quad \forall |\alpha| \leq (n+6)/2.$$

For any (t, x) satisfying $0 \le t < T_*$ and $|x| < R + t$, write $x = |x|\omega$ with $|\omega| = 1$. Then

$$\Gamma^\alpha u(t,x) = \Gamma^\alpha u(t,|x|\omega) - \Gamma^\alpha u(t,(R+t)\omega)$$
$$= \int_0^1 \partial_j \Gamma^\alpha u(t,(s|x| + (1-s)(R+t))\omega)ds\,(|x| - R - t)\omega^j.$$

In view of (74), we obtain for all $|\alpha| \le (n+6)/2$ that

$$|\Gamma^\alpha u(t,x)| \le CA\varepsilon(1+t)^{-\frac{n-1}{2}}(R+t-|x|)$$
$$\le CA\varepsilon(1+t)^{-\frac{n-3}{2}}.$$

Step 2. We prove (73).

- Since $u \in C^\infty([0, T_*) \times \mathbb{R}^n)$ and $u(t,x) = 0$ for $|x| \geq R + t$, we have $A(t) \in C([0, T_*))$.
- Using initial data we can find a large number A such that

$$A(0) \leq \frac{1}{4} A\varepsilon. \tag{75}$$

By the continuity of $A(t)$, there is $0 < T < T_*$ such that $A(t) \leq A\varepsilon$ for $0 \leq t \leq T$.

- Let

$$T_0 = \sup\{T \in [0, T_*) : A(t) \leq A\varepsilon, \forall 0 \leq t \leq T\}.$$

Then $T_0 > 0$. It suffices to show $T_0 = T_*$.

We show $T_0 = T_*$ be a contradiction argument. If $T_0 < T_*$, then $A(t) \le A\varepsilon$ for $0 \le t \le T_0$. We will prove that for small $\varepsilon > 0$ there holds

$$A(t) \le \frac{1}{2} A\varepsilon \quad \text{for } 0 \le t \le T_0.$$

By the continuity of $A(t)$, there is $\delta > 0$ such that

$$A(t) \le A\varepsilon \quad \text{for } 0 \le t \le T_0 + \delta$$

which contradicts the definition of T_0.

Step 3. It remains only to prove that there is $\varepsilon_0 > 0$ such that

$$A(t) \le A\varepsilon \text{ for } 0 \le t \le T_0 \Longrightarrow A(t) \le \frac{1}{2} A\varepsilon \text{ for } 0 \le t \le T_0$$

for $0 < \varepsilon \le \varepsilon_0$.

By Klainerman inequality and $A(t) \leq A\varepsilon$ for $0 \leq t \leq T_0$, we have for $|\beta| \leq (n+6)/2$ that

$$|\partial \Gamma^\beta u(t,x)| \leq CA\varepsilon(1+t)^{-\frac{n-1}{2}}, \quad \forall (t,x) \in [0, T_0] \times \mathbb{R}^n. \quad (76)$$

To estimate $\|\partial \Gamma^\alpha u(t,\cdot)\|_{L^2}$ for $|\alpha| \leq n+4$, we use the energy estimate to obtain

$$\|\partial \Gamma^\alpha u(t,\cdot)\|_{L^2} \leq \|\partial \Gamma^\alpha u(0,\cdot)\|_{L^2} + C\int_0^t \|\Box \Gamma^\alpha u(\tau,\cdot)\|_{L^2} d\tau. \quad (77)$$

We write

$$\Box \Gamma^\alpha u = [\Box, \Gamma^\alpha]u + \Gamma^\alpha(F(\partial u))$$

and estimate $\|\Gamma^\alpha(F(\partial u))(\tau,\cdot)\|_{L^2}$ and $\|[\Box, \Gamma^\alpha]u(\tau,\cdot)\|_{L^2}$.

Since $F(0) = \mathbf{D}F(0) = 0$, we can write

$$F(\partial u) = \sum_{j,k=1}^{n} F_{jk}(\partial u)\partial_j u \partial_k u,$$

where F_{jk} are smooth functions. Using this it is easy to see that $\Gamma^\alpha(F(\partial u))$ is a linear combination of following terms

$$F_{\alpha_1\cdots\alpha_m}(\partial u) \cdot \Gamma^{\alpha_1}\partial u \cdot \Gamma^{\alpha_2}\partial u \cdot \cdots \cdot \Gamma^{\alpha_m}\partial u$$

where $m \geq 2$, $F_{\alpha_1\cdots\alpha_m}$ are smooth functions and $|\alpha_1| + \cdots + |\alpha_m| = |\alpha|$ with at most one α_i satisfying $|\alpha_i| > |\alpha|/2$ and at least one α_i satisfying $|\alpha_i| \leq |\alpha|/2$.

- In view of (76), by taking ε_0 such that $A\varepsilon_0 \leq 1$, we obtain $\|F_{\alpha_1\cdots\alpha_m}(\partial u)\|_{L^\infty} \leq C$ for $0 < \varepsilon \leq \varepsilon_0$ with a constant C independent of A and ε.

- Since $|\alpha|/2 \leq (n+4)/2$, using (76) all terms $\Gamma^{\alpha_j}\partial u$, except the one with largest $|\alpha_j|$, can be estimated as

$$\|\Gamma^{\alpha_j}\partial u(t,x)\|_{L^\infty([0,T_0]\times\mathbb{R}^n)} \leq CA\varepsilon(1+t)^{-\frac{n-1}{2}}$$

Therefore

$$\begin{aligned}\|\Gamma^{\alpha}(F(\partial u))(t,\cdot)\|_{L^2} &\leq CA\varepsilon(1+t)^{-\frac{n-1}{2}} \sum_{|\beta|\leq|\alpha|} \|\Gamma^{\beta}\partial u(t,\cdot)\|_{L^2} \\ &\leq CA\varepsilon(1+t)^{-\frac{n-1}{2}} A(t). \qquad (78)\end{aligned}$$

Recall that $[\Box,\Gamma]$ is either 0 or $2\Box$. Thus

$$|[\Box,\Gamma^{\alpha}]u| \lesssim \sum_{|\beta|\leq|\alpha|} |\Gamma^{\beta}\Box u| \lesssim \sum_{|\beta|\leq|\alpha|} |\Gamma^{\beta}(F(\partial u))|.$$

Therefore

$$\|[\Box, \Gamma^\alpha]u(t,\cdot)\|_{L^2} \le C \sum_{|\beta|\le|\alpha|} \|\Gamma^\beta(F(\partial u))(t,\cdot)\|_{L^2}$$
$$\le CA\varepsilon(1+t)^{-\frac{n-1}{2}}A(t). \qquad (79)$$

Consequently, it follows from (77), (78) and (79) that

$$\|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2} \le \|\partial\Gamma^\alpha u(0,\cdot)\|_{L^2} + CA\varepsilon \int_0^t \frac{A(\tau)}{(1+\tau)^{\frac{n-1}{2}}}d\tau$$

Summing over all α with $|\alpha| \le n+4$ we obtain

$$A(t) \le A(0) + CA\varepsilon \int_0^t \frac{A(\tau)}{(1+\tau)^{\frac{n-1}{2}}}d\tau \le \frac{1}{4}A\varepsilon + CA\varepsilon \int_0^t \frac{A(\tau)}{(1+\tau)^{\frac{n-1}{2}}}d\tau.$$

By Gronwall inequality,

$$A(t) \leq \frac{1}{4} A\varepsilon \exp\left(CA\varepsilon \int_0^t \frac{d\tau}{(1+\tau)^{(n-1)/2}} \right), \quad 0 \leq t \leq T_0.$$

For $n \geq 4$, $\int_0^\infty \frac{d\tau}{(1+\tau)^{(n-1)/2}} = \frac{2}{n+2} < \infty$. (This is the reason we need $n \geq 4$ for global existence). We now choose $\varepsilon_0 > 0$ so that

$$\exp\left(\frac{2}{n+2} CA\varepsilon_0 \right) \leq 2.$$

Thus $A(t) \leq A\varepsilon/2$ for $0 \leq t \leq T_0$ and $0 < \varepsilon \leq \varepsilon_0$. The proof is complete. ∎

Remark. The proof does not provide global existence result when $n \leq 3$ in general. However, the argument can guarantee existence on some interval $[0, T_\varepsilon]$, where T_ε can be estimated as

$$T_\varepsilon \geq \begin{cases} e^{c/\varepsilon}, & n = 3, \\ c/\varepsilon^2, & n = 2, \\ c/\varepsilon, & n = 1. \end{cases} \tag{80}$$

In fact, let $A(t)$ be defined as before, the key point is to show that, for any $T < T_\varepsilon$,

$$A(t) \leq A\varepsilon \text{ for } 0 \leq t \leq T \Longrightarrow A(t) \leq \frac{1}{2} A\varepsilon \text{ for } 0 \leq t \leq T$$

The same argument as above gives

$$A(t) \le \frac{1}{4} A\varepsilon \exp\left(CA\varepsilon \int_0^t \frac{d\tau}{(1+\tau)^{(n-1)/2}} \right), \quad 0 \le t \le T.$$

Thus we can improve the estimate to $A(t) \le \frac{1}{2} A\varepsilon$ for $0 \le t \le T$ if T_ε satisfies

$$\exp\left(CA\varepsilon \int_0^{T_\varepsilon} \frac{d\tau}{(1+\tau)^{(n-1)/2}} \right) \le 2$$

When $n \le 3$, the maximal T_ε with this property satisfies (80).

Remark. For $n = 2$ or $n = 3$, the above argument can guarantee global existence when F satisfies stronger condition

$$F(0) = 0, \quad \mathbf{D}F(0) = 0, \quad \cdots, \quad \mathbf{D}^k F(0) = 0, \tag{81}$$

where $k = 5 - n$. Indeed, this condition guarantees that $F(\partial u)$ is a linear combination of the terms

$$F_{j_1 \cdots j_{k+1}}(\partial u)\partial_{j_1} u \cdots \partial_{j_{k+1}} u.$$

Thus $\Gamma^\alpha(F(\partial u))$ is a linear combination of the terms

$$f_{i_1 \cdots i_r}(\partial u)\Gamma^{\alpha_{i_1}}\partial u \cdot \ldots \cdot \Gamma^{\alpha_{i_r}}\partial u,$$

where $r \geq k + 1$, $|\alpha_1| + \cdots + |\alpha_r| = |\alpha|$ and $f_{i_1 \cdots i_r}$ are smooth functions; there are at most one α_i satisfying $\alpha_i > |\alpha|/2$ and at least k of α_i satisfying $|\alpha_i| \leq |\alpha|/2$.

We thus can obtain

$$\|\Gamma^\alpha(F(\partial u))(t,\cdot)\|_{L^2} \le CA\varepsilon(1+t)^{-\frac{(n-1)k}{2}}A(t),$$
$$\|[\Box,\Gamma^\alpha]u(t,\cdot)\|_{L^2} \le CA\varepsilon(1+t)^{-\frac{(n-1)k}{2}}A(t).$$

Therefore

$$A(t) \le \frac{1}{4}A\varepsilon \exp\left(CA\varepsilon\int_0^t \frac{d\tau}{(1+\tau)^{((n-1)k)/2}}\right).$$

Since $k = 5-n$, $\int_0^\infty \frac{d\tau}{(1+\tau)^{((n-1)k)/2}}$ converges for $n=2$ or $n=3$.

The condition (81) is indeed too restrictive. In next lecture we relax it to include quadratic terms when $n=3$ using the so-called null condition introduced by Klainerman.