Further Partial Differential Equations (2023)
Problem Sheet 3

1. **Asymptotic analysis of Stefan problems**

   (a) Show that the transcendental relation (2.12) between β and St may be parameterized as

   $$\mathrm{St} = \sqrt{\pi}\xi \mathrm{e}^{\xi^2}\mathrm{erf}(\xi), \qquad \beta = \frac{2\sqrt{\xi}\mathrm{e}^{-\xi^2/2}}{\pi^{1/4}\sqrt{\mathrm{erf}(\xi)}}, \tag{1}$$

   where $0 < \xi < \infty$. By taking the limits $\xi \to 0$ and $\xi \to \infty$, derive the asymptotic expressions (2.13).

Solution

Define $\xi = \beta\sqrt{\mathrm{St}}/2$. Substituting into the transcendental equation (2.12) gives

$$\sqrt{\pi}\xi \mathrm{e}^{\xi^2}\mathrm{erf}(\xi) = \mathrm{St}, \tag{2}$$

$$\beta = \frac{2\xi}{\mathrm{St}} = \frac{2\sqrt{\xi}\mathrm{e}^{-\xi^2/2}}{\pi^{1/4}\sqrt{\mathrm{erf}(\xi)}}. \tag{3}$$

As $\xi \to 0$, $\mathrm{erf}(\xi) \sim 2\xi/\sqrt{\pi}$ so $\mathrm{St} \sim 2\xi^2$ in (2) and $\beta \to \sqrt{2}$ in (3).
As $\xi \to \infty$, $\mathrm{erf} \to 1$, so (2) and (3) give respectively

$$\mathrm{St} \sim \sqrt{\pi}\xi \mathrm{e}^{\xi^2}, \tag{4}$$

$$\beta \sim \frac{2}{\pi^{1/4}}\xi^{1/2}\mathrm{e}^{-\xi^2/2}. \tag{5}$$

Equation (4) gives

$$\log\left(\frac{\mathrm{St}}{\sqrt{\pi}}\right) \sim \xi^2 + \text{higher order logarithmic terms}, \tag{6}$$

and so

$$\beta \sim \frac{2}{\sqrt{\mathrm{St}}}\sqrt{\log\left(\frac{\mathrm{St}}{\sqrt{\pi}}\right)}. \tag{7}$$

2. **Similarity solutions in the two-phase Stefan problem**

Consider the two-phase Stefan problem (2.15) in the limit $t \to 0$. Show that the leading-order behaviour is given by

$$u(x,t) \sim \begin{cases} f(\eta) & 0 < \eta < \beta, \\ g(\eta) & \beta < \eta < \infty, \end{cases} \qquad s(t) \sim \beta\sqrt{t}, \qquad \eta = \frac{x}{\sqrt{t}},$$

where

$$g(\eta) = \theta\left(\frac{\operatorname{erfc}\left(\eta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)}{\operatorname{erfc}\left(\beta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)} - 1\right), \qquad f(\eta) = \left(1 - \frac{\operatorname{erf}\left(\eta\sqrt{\mathrm{St}}/2\right)}{\operatorname{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)}\right),$$

and β satisfies the transcendental equation

$$\frac{\beta\sqrt{\pi}}{2\sqrt{\mathrm{St}}} = \frac{\mathrm{e}^{-\beta^2\mathrm{St}/4}}{\operatorname{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)} - \frac{K\theta\mathrm{e}^{-\beta^2\mathrm{St}/4\kappa}}{\sqrt{\kappa}\operatorname{erfc}\left(\beta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)}.$$

Solution

Substitute in the similarity solution form given in the question. (Note that you can obtain the form of this similarity solution by using a scaling argument.) This transforms the problem to

$$f'' + \frac{\mathrm{St}}{2}\eta f' = 0, \qquad \eta < \beta, \tag{8}$$

$$g'' + \frac{\mathrm{St}}{2\kappa}\eta g' = 0, \qquad \eta > \beta, \tag{9}$$

$$f(0) = 1, \tag{10}$$

$$g \to -\theta \qquad \text{as} \quad \eta \to \infty, \tag{11}$$

$$f(\beta) = g(\beta) = 0, \tag{12}$$

$$Kg'(\beta) - f'(\beta) = \frac{\beta}{2}. \tag{13}$$

The solution follows straightforwardly from this.

3. **Linear stability of a two-dimensional Stefan problem**
Consider the linear stability of the free boundary problem depicted in Figure 2.2 in the limit $\mathrm{St} \to 0$. Assume that the free boundary is moving at constant speed V under a constant temperature gradient $-\lambda_{1,2}$ in each phase before being perturbed, so the solutions take the form

$$u_1(x, y, t) = -\lambda_1(x - Vt) + \tilde{u}_1(x, y, t), \qquad u_2(x, y, t) = -\lambda_2(x - Vt) + \tilde{u}_2(x, y, t)$$

and the position of the free boundary is given by

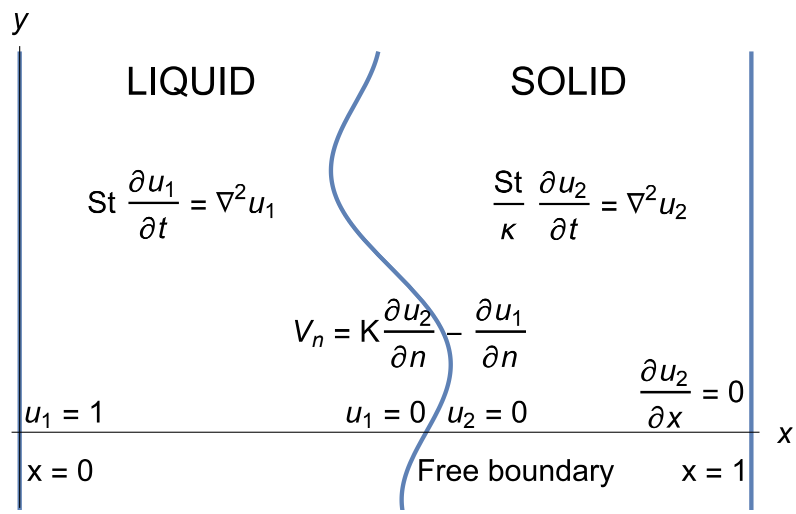
$$x = Vt + \xi(y, t).$$

By linearising the problem with respect to $\tilde{u}_1$, $\tilde{u}_2$ and ξ, show that perturbations with wavenumber $k > 0$ and growth rate σ are possible provided

$$\frac{\sigma}{Vk} = -\frac{\lambda_1 + K\lambda_2}{\lambda_1 - K\lambda_2}.$$

Solution

We consider the following problem with $\mathrm{St} \to 0$:



We set

$$
\begin{aligned}
u_1 &= -\lambda_1(x - Vt) + \tilde{u}_1, \\
u_2 &= -\lambda_2(x - Vt) + \tilde{u}_2, \\
x &= Vt + \xi(y, t).
\end{aligned}
$$

If the free boundary is $x = f(y, t)$ then the unit normal is

$$
\boldsymbol{n} = \frac{\left(1, -\dfrac{\partial f}{\partial y}\right)}{\sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2}},
$$

the normal derivative is

$$
\frac{\partial u}{\partial n} = \frac{1}{\sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2}} \left(\frac{\partial u}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial u}{\partial y} \right),
$$

and the normal velocity is

$$
V_n = \frac{\dfrac{\partial f}{\partial t}}{\sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2}}.
$$

Now in our case, $f = Vt + \xi(y, t)$, so the free boundary conditions are

$$
K\left(-\lambda_2 + \frac{\partial \tilde{u}_2}{\partial x} - \frac{\partial \xi}{\partial y}\frac{\partial \tilde{u}_2}{\partial y}\right) - \left(-\lambda_1 + \frac{\partial \tilde{u}_1}{\partial x} - \frac{\partial \xi}{\partial y}\frac{\partial u_1}{\partial y}\right) = V + \frac{\partial \xi}{\partial t}
$$

on $x = Vt + \xi(y, t)$. Considering this at O(1) gives

$$
-K\lambda_2 + \lambda_1 = V \qquad \text{on} \quad x = Vt
$$

and at next order,

$$K\frac{\partial \tilde{u}_2}{\partial x} - \frac{\partial \tilde{u}_1}{\partial x} = \frac{\partial \xi}{\partial t} \qquad \text{on} \quad x = Vt.$$

Since $u_1 = u_2 - 0$ on the interface, this gives

$$-\lambda_1 \xi + \tilde{u}_1 = -\lambda_2 \xi + \tilde{u}_2 = 0 \qquad \text{on} \quad x = Vt.$$

The leading-order equations for $\mathrm{St} \to 0$ are

$$\begin{aligned} \nabla^2 \tilde{u}_1 &= 0, & x &< Vt, \\ \nabla^2 \tilde{u}_2 &= 0, & x &> Vt, \end{aligned}$$

We no longer need to consider the conditions on $x = 0$ and $x = 1$ since we are now just performing a local analysis. Our only requirement is that the perturbations decay away so we seek solutions of the form

$$\begin{aligned} \tilde{u}_1 &= A\exp(\sigma t + \mathrm{i}ky + k(x - Vt)), \\ \tilde{u}_2 &= B\exp(\sigma t + \mathrm{i}ky - k(x - Vt)), \\ \xi &= C\exp(\sigma t + \mathrm{i}ky). \end{aligned}$$

These satisfy Laplace's equation and decay away from the interface. The interface conditions give

$$-Kk\lambda_1 - Bk\lambda_2 = \sigma$$

and

$$\begin{pmatrix} Kk & k & \sigma \\ 1 & 0 & -\lambda_1 \\ 0 & 1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Non-trivial solutions require the determinant of this matrix to be zero, which gives

$$\frac{\sigma}{kV} = -\frac{1}{V}\left(K\lambda_1 + \lambda_2\right) = -\frac{K\lambda_1 + \lambda_2}{\lambda_1 - K\lambda_2}$$

as required.

4. **OPTIONAL (will not be marked) A solid–liquid interface with a density change**

Consider the one-dimensional Stefan problem for melting of a solid considered in lectures. The full system behaviour may be described by equations expressing conservation of mass, momentum and total energy, which are given respectively by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v\right) = 0, \tag{14}$$

$$\frac{\partial}{\partial t}\left(\rho v\right) + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0, \tag{15}$$

$$\frac{\partial}{\partial t}\left(\rho h + \frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left(pv - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right) = 0, \tag{16}$$

where ρ is the density, v the velocity, p the pressure, T the temperature and

$$h = \begin{cases} c(T - T_{\mathrm{m}}) + L & T > T_{\mathrm{m}} \\ c(T - T_{\mathrm{m}}) & T < T_{\mathrm{m}}. \end{cases}$$

is the *enthalpy* of the system, which is the total energy per unit mass, including heat. Here, c is the specific heat and L the latent heat.

Suppose that liquid occupies a region $0 \le x \le s(t)$ and solid occupies a region $x > s(t)$.

(a) Show that when the density of the fluid and the solid are the same then $v = 0$ and the temperature in the liquid and the solid is described by the one-dimensional heat equation

$$\frac{\partial}{\partial t}\left(\rho c T\right) - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = 0. \tag{17}$$

(b) Now suppose that the densities in the solid and the liquid phases are different. Integrate (14) over a domain $x_1 < x < x_2$ that contains the interface (so $x_1 < s(t)$ and $x_2 > s(t)$). Divide the integral into $x_1 \le x \le s(t)$ and $s(t) \le x \le x_2$ and take the limit as $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ to show that the following jump condition is satisfied by the density:

$$\left[\rho\right]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[\rho v\right]_-^+ . \tag{18}$$

(c) By performing an identical process for (15) and (16) obtain the jump conditions

$$\left[\rho v\right]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[\rho v^2 + p\right]_-^+ , \tag{19}$$

$$\left[\rho h + \frac{1}{2}\rho v^2\right]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[pv - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right]_-^+ . \tag{20}$$

(d) Explain how these reduce to the Stefan condition presented in lectures when the solid and liquid densities are equal.

Solution

(a) Substitution of constant ρ into (14) gives v as an arbitrary function of time. Since the liquid occupies the region $0 \leq x \leq s(t)$, the boundary $x = 0$ is fixed and so $v = 0$ here and hence $v = 0$ everywhere. Substitution into (15) gives constant pressure gradient p. Substitution into (16) gives the required heat equation.

(b) Equation (14) only applies provided the variables are continuous, and so does not hold across jumps. We thus consider the integrated conservative version,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho\,\mathrm{d}x = [\rho v]_{x_1}^{x_2},$$

where $x_1 < s(t) < x_2$. We divide the integral into parts to the left and right of the jump,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{s(t)} \rho\,\mathrm{d}x + \int_{s(t)}^{x_2} \rho\,\mathrm{d}x = [\rho v]_{x_1}^{x_2} \int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t}\mathrm{d}x + \rho|_{x_1}\frac{\mathrm{d}s}{\mathrm{d}t} + \int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t}\,\mathrm{d}x - \rho|_{x_2}\frac{\mathrm{d}s}{\mathrm{d}t} \quad = [\rho v]_{x_1}^{x_2}$$

using Leibniz' rule. Then, taking the limit $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ and recognizing that

$$\lim_{x_1 \to s(t)^-} \int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t}\,\mathrm{d}x = 0, \qquad \lim_{x_2 \to s(t)^+} \int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t}\,\mathrm{d}x = 0,$$

we obtain the required result,

$$[\rho]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = [\rho v]_-^+ .$$

(c) This may be found easily by following the same steps as above.

(d) When the solid and liquid densities are equal, (19) gives $[p]_-^+ = 0$, so the pressure is continuous across the interface, and

$$\rho L \frac{\mathrm{d}s}{\mathrm{d}t} = -\left[k\frac{\partial T}{\partial x}\right]_-^+ \tag{21}$$

if we assume that the temperature is continuous across the interface. This is precisely the Stefan condition from the lectures.