Contents

1 Linear Systems

1.1 Fundamental theorems

Consider the linear, autonomous, first-order system of differential equations:

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1.1}$$
$$\mathbf{x}(0) = \mathbf{x}_0, \tag{1.2}$$

where a dot represents $\mathrm{d}/\mathrm{d}t$, $\mathbf{x} \in \mathbb{R}^n$, and $A \in \mathcal{M}_n(\mathbb{R})$, the set of $n \times n$ matrices with coefficients in $\mathbb{R}$. Questions we are typically interested in:

(i) Find the solution

(ii) Describe the behaviour of the solution close to the fixed point $\mathbf{x} = \mathbf{0}$.

When $n = 1$ the equation is easy to solve:

$$\dot{x} = ax \qquad \Rightarrow \qquad x(t) = \mathrm{e}^{at}x_0.$$

In general we might expect to be able to write something like $\mathbf{x} = \mathrm{e}^{tA}\mathbf{x}_0$. But what is e^{tA}?

Definition 1.1. Let $A \in \mathcal{M}_n(\mathbb{R})$, $t \in \mathbb{R}$. Then the *matrix exponential* is

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}. \tag{1.3}$$

For a given T this series is absolutely, uniformly convergent for all $t < T$.

Theorem 1.2. *The initial value problem (1.1)-(1.2) has the unique solution*

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0. \tag{1.4}$$

Lemma 1.3. *If $A = BCB^{-1}$, then $\mathrm{e}^{tA} = B\mathrm{e}^{tC}B^{-1}$.*

Proof.

$$A^2 = BCB^{-1}BCB^{-1} = BC^2B^{-1}.$$

Iterating (more properly induction) gives $A^k = BC^kB^{-1}$ for all k. The result then follows from Definition 1.1 and uniform convergence. □

If A is *semi-simple* (i.e. A can be diagonalised), then there exists B such that

$$A = BCB^{-1}, \quad \text{where} \quad C = \mathrm{diag}(\lambda_1, \ldots, \lambda_n). \tag{1.5}$$

Then

$$\mathrm{e}^{tA} = B\,\mathrm{diag}(\mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_n t})\,B^{-1} \tag{1.6}$$

1.2 Normal forms in two dimensions (Recap from Part A)

If $A \in \mathcal{M}_2(\mathbb{R})$, say $A = (a_{ij})$ with $a_{ij} \in \mathbb{R}$ and $\mathbf{x} = (x_1, x_2)^T$ the system is

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2, \tag{1.7}$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2. \tag{1.8}$$

For $B \in$GL$(2, \mathbb{R})$ (the group of 2×2 invertible matrices with real coefficients)

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad \Rightarrow \qquad B\dot{\mathbf{x}} = BA\mathbf{x} = BAB^{-1}B\mathbf{x} = CB\mathbf{x},$$

where $C = BAB^{-1}$. Thus $\mathbf{y} = B\mathbf{x}$ transforms the system $\dot{\mathbf{x}} = A\mathbf{x}$ into

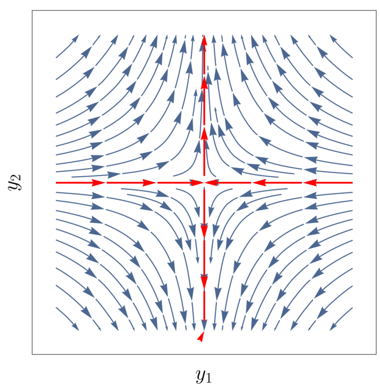
$$\dot{\mathbf{y}} = C\mathbf{y}. \tag{1.9}$$

Depending on the eigenvalues $\lambda_1, \lambda_2 \in \mathrm{Spec}(A)$ (the *spectrum* of A) , we can choose B such that C has one of the following forms:

1. $\lambda_1, \lambda_2 \in \mathbb{R}$.

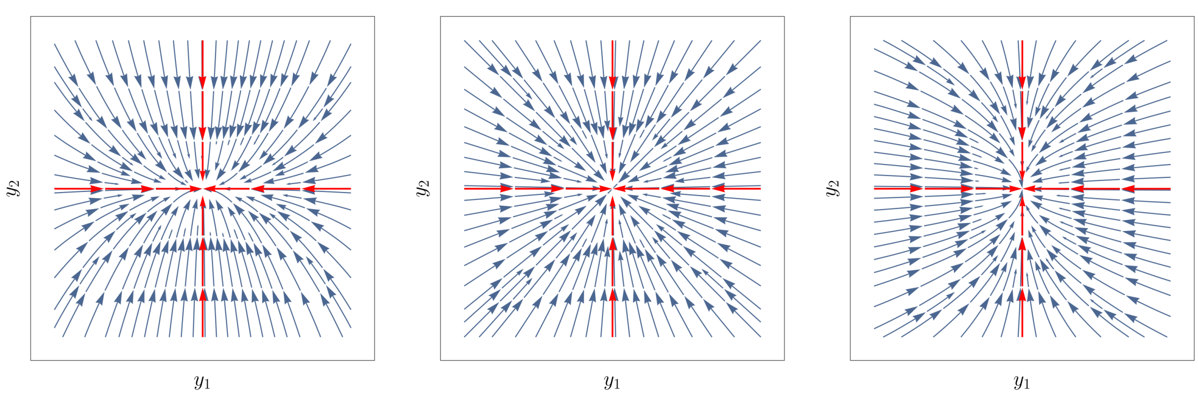
1.1 Saddle: $\lambda_1\lambda_2 < 0$

$$C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} \dot{y}_1 & = & \lambda_1 y_1 \\ \dot{y}_2 & = & \lambda_2 y_2 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} y_1 & = & y_{10}\mathrm{e}^{\lambda_1 t} \\ y_2 & = & y_{20}\mathrm{e}^{\lambda_2 t} \end{array} \qquad \text{WLOG } \lambda_1 < 0 < \lambda_2$$



1.2 Node: $\lambda_1\lambda_2 > 0$ with A semi-simple (i.e. with 2 different eigenvectors).

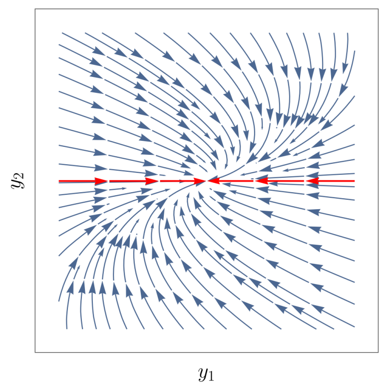
$$C = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} y_1 & = & y_{10}\mathrm{e}^{\lambda_1 t} \\ y_2 & = & y_{20}\mathrm{e}^{\lambda_2 t} \end{array} \quad \Rightarrow \quad y_2 = Cy_1^{\alpha} \quad \text{where } \alpha = \frac{\lambda_2}{\lambda_1} > 0.$$



$\alpha > 1$ $\qquad$ $\alpha = 1$ $\qquad$ $\alpha < 1$

1.3 Degenerate node: $\lambda_1 = \lambda_2$ with A not semi-simple (i.e. the eigenspace is 1-dimensional).

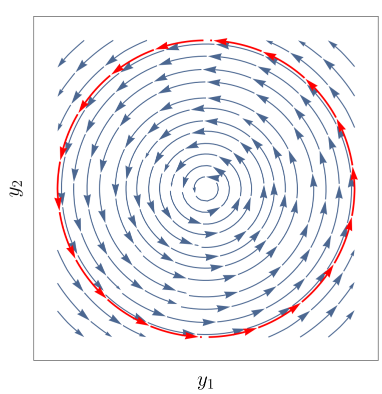
$$C = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} y_1 & = & y_{10}\mathrm{e}^{\lambda t} + y_{20} t \mathrm{e}^{\lambda t} \\ y_2 & = & y_{20}\mathrm{e}^{\lambda t} \end{array} .$$



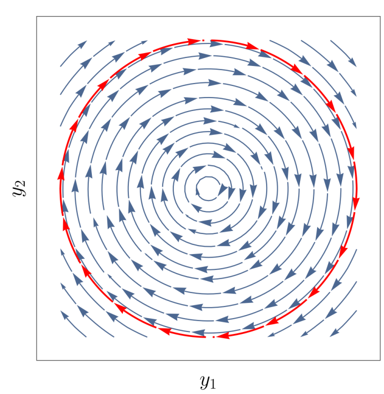
2. $\lambda_1, \lambda_2 \in \mathbb{C}$. Then $\lambda_1 = a + \mathrm{i}b$, $\lambda_2 = a - \mathrm{i}b$ $(a, b$ real).

2.1 Centre: $a = 0$

$$C = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} \dot{y}_1 & = & -by_2 \\ \dot{y}_2 & = & by_1 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} y_1 & = & y_{10}\cos(bt) - y_{20}\sin(bt) \\ y_2 & = & y_{20}\cos(bt) + y_{10}\sin(bt) \end{array}$$



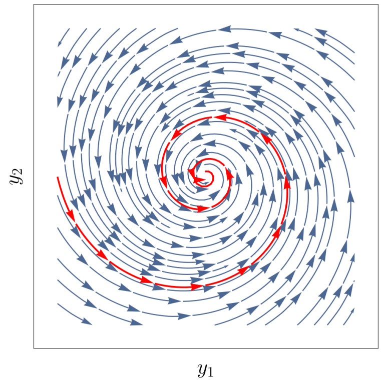
$a = 0, \; b > 0$



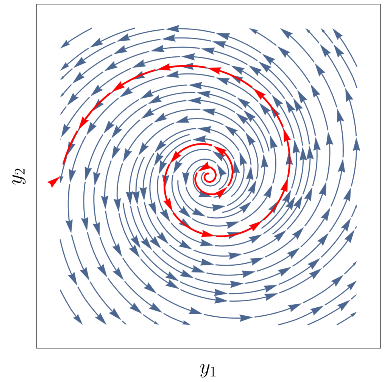
$a = 0, \; b < 0$

2.1 Focus: $a \neq 0$

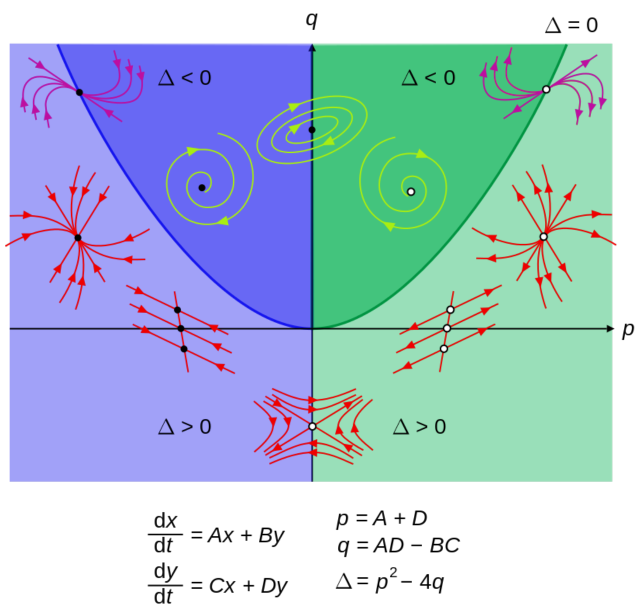
$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



$a < 0, \ b > 0$



$a > 0, \ b > 0$



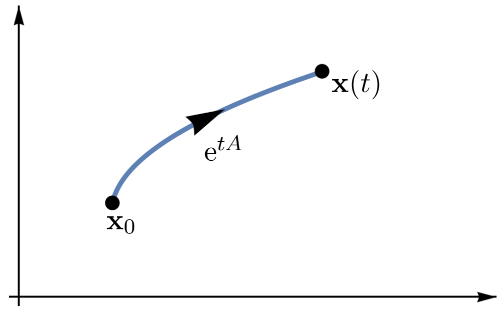
1.3 Linear flows

Consider

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \qquad \mathbf{x} \in \mathbb{R}^n, \quad A \in \mathcal{M}_n(\mathbb{R}), \ n \geq 1.$$

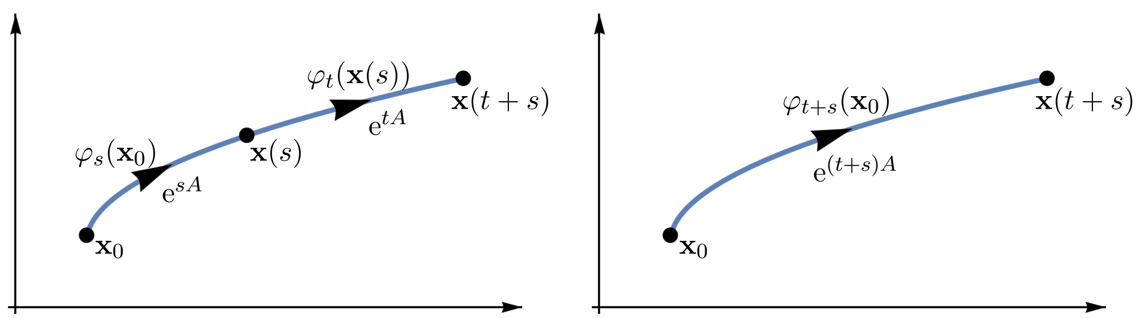
The general solution is $\mathbf{x}(t) = \mathrm{e}^{tA}\mathbf{x}_0$.
Geometrically, e^{tA} is a map, *the linear flow*. Let $\varphi_t = \mathrm{e}^{tA}$. Then

$$\varphi_t : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{with} \qquad \varphi_t(\mathbf{x}_0) = \mathrm{e}^{tA}\mathbf{x}_0 = \mathbf{x}(t). \tag{1.10}$$



Properties:

- $\varphi_0 = \mathbf{1}$ (the identity map)
- $\varphi_{t+s}(\mathbf{x}) = \varphi_t(\varphi_s(\mathbf{x})) = \varphi_s(\varphi_t(\mathbf{x})), \quad \forall \mathbf{x} \in \mathbb{R}^n$



Consider the set of eigenvalues of A:

$$\text{Spec}(A) = \{\lambda_1, \dots, \lambda_n\} \tag{1.11}$$

Definition 1.4. If A is such that $\text{Re}(\lambda) \neq 0$, $\forall \lambda \in \text{Spec}(A)$, then the linear flow e^{tA} is *hyperbolic*. By extension, the system $\dot{\mathbf{x}} = A\mathbf{x}$ is a *hyperbolic system*.

NB: Since the real part of all the eigenvalues are all different from zero, hyperbolic flows are controlled by exponential contraction or expansion close to the fixed point.

Definition 1.5. Let $E \subset \mathbb{R}^n$. Then E is an *invariant set* of φ if $\varphi_t(E) \subseteq E \ \ \forall t \in \mathbb{R}$.

Example 1.1. Let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ, then $E = \text{Span}(\mathbf{v})$ is an invariant set.

Proof.

$$E = \text{Span}(\mathbf{v}) = \{c\mathbf{v} : c \in \mathbb{R}\}.$$

But

$$\varphi_t(c\mathbf{v}) = e^{tA}c\mathbf{v} = ce^{tA}\mathbf{v} = ce^{\lambda t}\mathbf{v} = \tilde{c}\mathbf{v} \in E \qquad \forall c$$

□

We can now construct three subspaces depending on the real part of the eigenvalues. First, consider the case where A is semi-simple and $\text{Spec}(A) = \{\lambda_1, \dots, \lambda_n\}$. We write the eigenvalues and eigenvectors of A as $A\mathbf{w}_j = \lambda_j \mathbf{w}_j$ (for $j = 1, \dots, n$) where

$$\lambda_j = a_j + ib_j, \quad a_j,\ b_j \in \mathbb{R}, \qquad \mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j, \quad \mathbf{u}_j,\ \mathbf{v}_j \in \mathbb{R}^n.$$

Definition 1.6. The *stable*, *center*, *unstable* linear subspaces are defined, respectively, as

- $E^s = \operatorname{Span}(\mathbf{u}_j, \mathbf{v}_j |\ a_j < 0)$ (stable linear subspace)
- $E^c = \operatorname{Span}(\mathbf{u}_j, \mathbf{v}_j |\ a_j = 0)$ (centre linear subspace)
- $E^u = \operatorname{Span}(\mathbf{u}_j, \mathbf{v}_j |\ a_j > 0)$ (unstable linear subspace)

Define the dimensions of the stable (s), centre (c) and unstable (u) linear subspaces :

$$n_s = \dim(E^s), \qquad n_c = \dim(E^c), \qquad n_u = \dim(E^u).$$

Then $n = n_s + n_c + n_u$. By construction E^s, E^c, and E^n are invariant sets.

In the case where the unstable and centre subspaces are empty, we have:

Lemma 1.7. *If all the eigenvalues have negative real part, then $\forall \mathbf{x}_0 \in \mathbb{R}^n$, the origin is* stable*. That is, we have*

$$\lim_{t\to\infty} e^{tA}\mathbf{x}_0 = \mathbf{0}, \tag{1.12}$$

and $\forall \mathbf{x}_0 \neq 0$

$$\lim_{t\to-\infty} |e^{tA}\mathbf{x}_0| = \infty. \tag{1.13}$$

NB: For a general system the same result holds for all $\mathbf{x}_0 \in E^s$.

If A is not semi-simple, then we take $\mathbf{w}_j$ to be the generalised eigenvectors (See Perko, p.33). For a degenerate eigenvalue λ with multiplicity m, the *generalised eigenvectors* of A given by m linearly independent solutions of

$$(A - \lambda\mathbf{1})^k\mathbf{w} = \mathbf{0}, \qquad k = 1, \ldots, m. \tag{1.14}$$

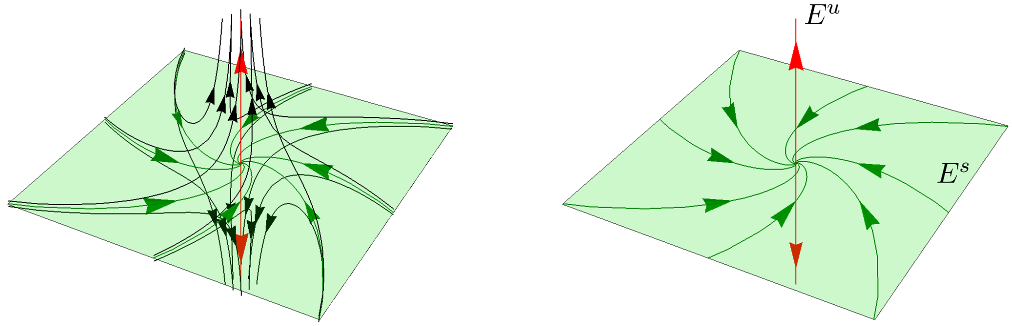
These generalised eigenvectors form a basis of the eigenspace of eigenvalue λ.

Example 1.2.

$$A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathrm{i}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \mathrm{i}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\lambda_1 = -2 + \mathrm{i}, \qquad \lambda_2 = -2 - \mathrm{i}, \qquad \lambda_3 = 3.$$

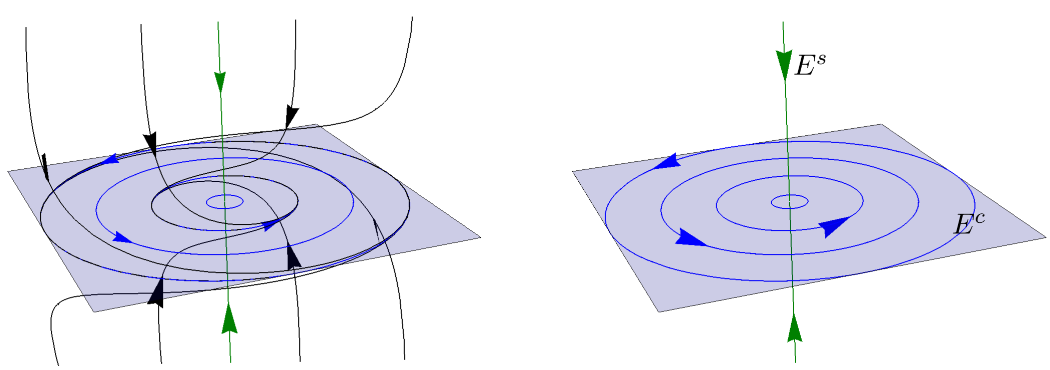


Example 1.3.

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} \mathrm{i} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathrm{i} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -\mathrm{i} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \mathrm{i} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\lambda_1 = \mathrm{i}, \qquad \lambda_2 = -\mathrm{i}, \qquad \lambda_3 = -2.$$

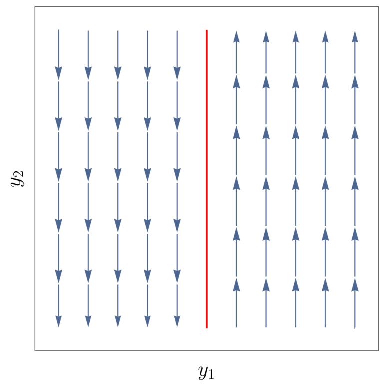


Example 1.4.

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\lambda_1 = 0, \qquad \lambda_2 = 0.$$

NB $\mathbf{w}_2$ is a generalised eigenvector: it satisfies $A^2\mathbf{w}_2 = \mathbf{0}$ but not $A\mathbf{w}_2 = \mathbf{0}$. $E^c = \mathbb{R}^2$. The typical way to find $\mathbf{w}_2$ is to solve $(A - \lambda I)\mathbf{w}_2 = \mathbf{w}_1$. In this case since $n_c = 2$ and the system is two-dimensional there is nothing else $\mathbf{w}_2$ could be.



2 Nonlinear systems

2.1 Existence and uniqueness

Consider the nonlinear, autonomous, first-order system of differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.1}$$

where $\mathbf{f} : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the *vector field.* NB autonomous means $\partial_t \mathbf{f} = \mathbf{0}$, i.e. $\mathbf{f}$ does not depend explicitly on t. In general, this equation cannot be solved explicitly.
Questions we might be interested in:
What are the possible solutions (from a geometric point of view)?
What is the stability of such solutions (how do nearby solutions behave?)?

Theorem 2.1. *Let E be an open subset of $\mathbb{R}^n$ containing $\mathbf{x}_0$ and let $\mathbf{f} \in C^1(E)$. Then there exists $c > 0$ such that*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

has one and only one solution $\mathbf{x}(t)$ on $[-c, c]$.

Remark 2.2. The proof is by Picard's method, after showing that $\mathbf{f} \in C^1(E)$ implies that $\mathbf{f}$ is locally Lipschitz on E.

Remark 2.3. This is a *local* result. It guarantees the existence of a unique solution but only for a short time.

Remark 2.4. If $\mathbf{x}(t)$ is a solution of the equation [not the initial condition] then so is $\mathbf{x}(t+\sigma)$ for any $\sigma \in \mathbb{R}$. This is a consequence of $\partial_t \mathbf{f} = \mathbf{0}$. Thus we also have existence and uniqueness on an interval $t \in [t_0 - c, t_0 + c]$ when the initial condition is replaced by $\mathbf{x}(t_0) = \mathbf{x}_0$.

Remark 2.5. For the rest of this course, unless otherwise specified, we will assume that the maximum interval of existence is $\mathbb{R}$ (we are interested in global behaviour).

Remark 2.6. The general conditions guaranteeing the existence of global solutions are not obvious.

We will find it useful to highlight the parametric dependence of the solution on the initial condition by writing $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$.

2.2 Flows, asymptotic sets, and invariant sets

We assume that the maximum interval of existence is $\mathbb{R}$ (i.e. solutions are defined for all time for all initial conditions).
Let E be an open subset of $\mathbb{R}^n$, and $\mathbf{f} \in C^1(E)$. For $\mathbf{x}_0 \in E$, let $\mathbf{x}(t; \mathbf{x}_0)$ be the solution to (2.1). Then

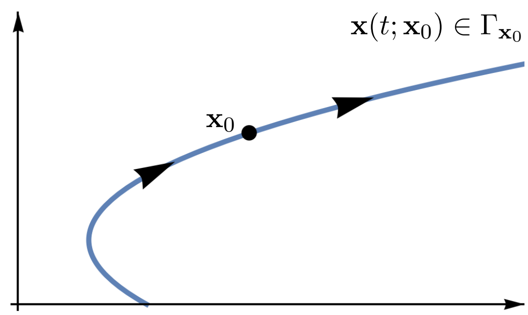
Definition 2.7. An *orbit* or *trajectory* based on $\mathbf{x}_0$ is the curve $\Gamma_{\mathbf{x}_0} \subset E$ defined by

$$\Gamma_{\mathbf{x}_0} = \{\mathbf{x}(t;\mathbf{x}_0) \mid t \in \mathbb{R}\} \tag{2.2}$$

Definition 2.8. The *flow* of the differential equation (2.1) is the map $\varphi_t : E \to E$ such that

$$\varphi_t(\mathbf{x}_0) = \mathbf{x}(t;\mathbf{x}_0). \tag{2.3}$$

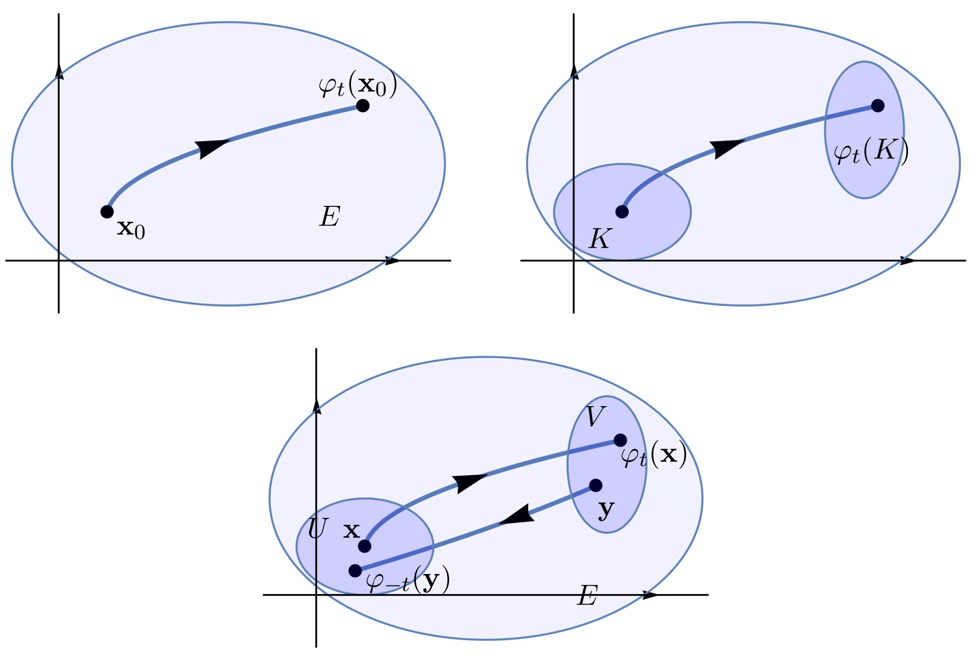
The space $E \subseteq \mathbb{R}^n$ on which the solutions live is called the *phase space*.



Properties of flows:

- $\varphi_0 = \mathbf{1}$ (the identity map)
- $\varphi_{t+s}(\mathbf{x}) = \varphi_t(\varphi_s(\mathbf{x})) = \varphi_s(\varphi_t(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathbb{R}^n$
- Let U be a neighborhood of $\mathbf{x}_0$ and $V = \varphi_t(U)$, then

$$\varphi_{-t}(\varphi_t(\mathbf{x})) = \mathbf{x}, \qquad \forall\, \mathbf{x} \in U \tag{2.4}$$
$$\varphi_t(\varphi_{-t}(\mathbf{y})) = \mathbf{y}, \qquad \forall\, \mathbf{y} \in V \tag{2.5}$$



2.2.1 Invariant sets

Definition 2.9. Consider a vector field $\mathbf{f} \in C^1(E)$, defining a flow $\varphi_t : E \to E$. Then $S \subseteq E$ is an *invariant set* of φ_t if

$$\varphi_t(S) \subseteq S \quad \forall\, t \in \mathbb{R}. \tag{2.6}$$

Example 2.1.

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= y + x^2 \end{aligned} \qquad \varphi_t(\mathbf{x}_0) = \begin{bmatrix} x_0 \mathrm{e}^{-t} \\ y_0 \mathrm{e}^{t} + \dfrac{x_0^2}{3}(\mathrm{e}^{t} - \mathrm{e}^{-2t}) \end{bmatrix}$$

Then the sets

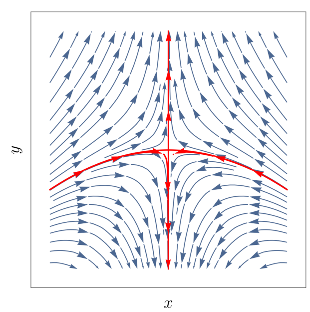
$$S_1 = \{\mathbf{x} \in \mathbb{R}^2 \mid y = -x^2/3\}, \qquad S_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid x = 0\}$$

are both invariant.

Proof. S_2 is invariant because $x_0 = 0 \Rightarrow x = x_0 \mathrm{e}^t = 0$. If $\mathbf{x}_0 \in S_1$ then $y_0 = -x_0^2/3$. Then

$$\varphi_t(\mathbf{x}_0) = \begin{bmatrix} x_0 \mathrm{e}^{-t} \\ -\dfrac{x_0^2}{3}\mathrm{e}^{t} + \dfrac{x_0^2}{3}(\mathrm{e}^{t} - \mathrm{e}^{-2t}) \end{bmatrix} = \begin{bmatrix} x_0 \mathrm{e}^{-t} \\ -\dfrac{x_0^2}{3}\mathrm{e}^{-2t} \end{bmatrix} \in S_1.$$

□



2.2.2 Attracting sets

Definition 2.10. A point $\mathbf{p} \in E$ is an *ω-limit point* of $\varphi_t(\mathbf{x})$ if there exists a sequence of times $t_1 < t_2 < \ldots < t_n$, with $t_i \to \infty$ as $i \to \infty$ such that

$$\lim_{i\to\infty} \varphi_{t_i}(\mathbf{x}) = \mathbf{p}. \tag{2.7}$$

Similarly, a point $\mathbf{p} \in E$ is an *α-limit point* of $\varphi_t(\mathbf{x})$ if there exists a sequence of time $t_1 > t_2 > \ldots > t_n$, with $t_i \to -\infty$ as $i \to \infty$ such that

$$\lim_{i\to\infty} \varphi_{t_i}(\mathbf{x}) = \mathbf{p}. \tag{2.8}$$

Example 2.2.

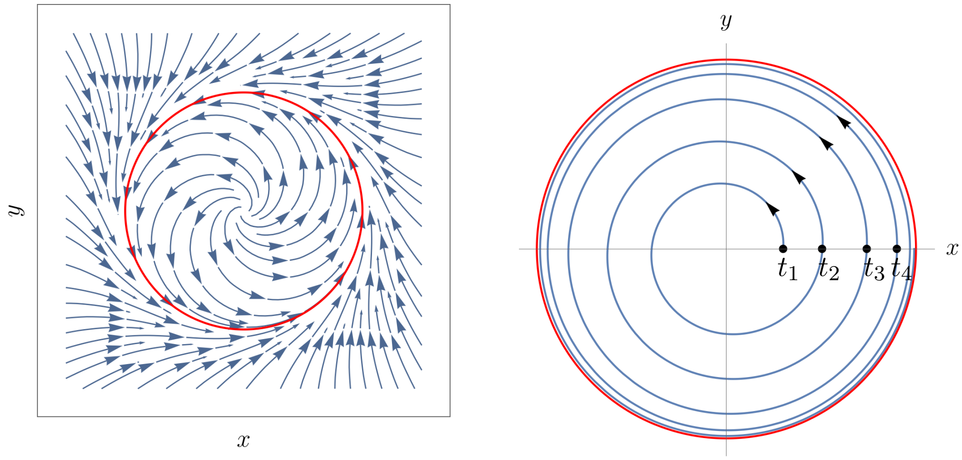
$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \end{aligned}$$

Write $x = r\cos\theta$, $y = r\sin\theta$ to give

$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= 1\end{aligned}$$

The circle $r = 1$ is an ω-limit set, i.e. each point on the circle is an ω-limit point. The point $r = 0$ is an α-limit point.

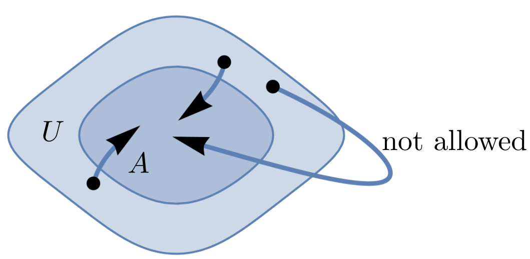
Remark 2.11. Note that no point on the circle $r = 1$ is a limit point of the flow.



Definition 2.12. An closed invariant set $A \subseteq E$ is called an *attracting set* of (2.1) if there is some neighbourhood U of A such that

$$\varphi_t(U) \subseteq U \quad \forall\ t \geq 0 \qquad \text{and} \qquad A = \bigcap_{t>0} \varphi_t(U).$$

Here a neighbourhood of A is any open set containing A.



The *domain of attraction* is the set of all initial conditions that have A as ω-limit set. That is

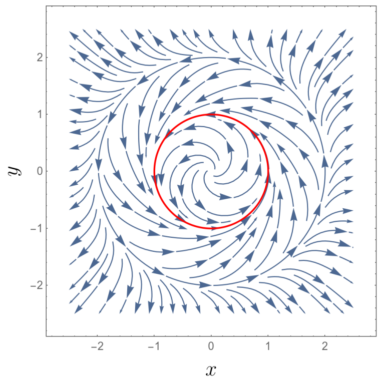
$$D(A) = \bigcup_{t\leq 0} \varphi_t(U) \tag{2.9}$$

Example 2.3.

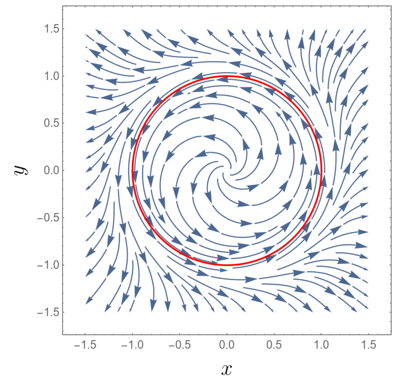
$$\begin{aligned}\dot{x} &= -y + x\left(1 - \frac{5}{4}x^2 - \frac{5}{4}y^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right)\\ \dot{y} &= x + y\left(1 - \frac{5}{4}x^2 - \frac{5}{4}y^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right)\end{aligned}$$

$$\begin{aligned}\dot{r} &= r\left(1 - \frac{5}{4}r^2 + \frac{1}{4}r^4\right) = r(1-r^2)\left(1 - \frac{r^2}{4}\right)\\ \dot{\theta} &= 1\end{aligned}$$

The circle $r = 1$ is an attracting set. The domain of attraction is $\{\mathbf{x} \mid 0 < r < 2\}$.



Example 2.3



Example 2.4

Example 2.4.

$$\dot{x} = -y + x(1 - 2x^2 - 2y^2 + x^4 + 2x^2y^2 + y^4) \qquad \dot{r} = r(1 - 2r^2 + r^4)$$
$$\dot{y} = x + y(1 - 2x^2 - 2y^2 + x^4 + 2x^2y^2 + y^4) \qquad \dot{\theta} = 1$$

The circle $r = 1$ is an invariant set, but not an attracting set.

2.2.3 Attractors

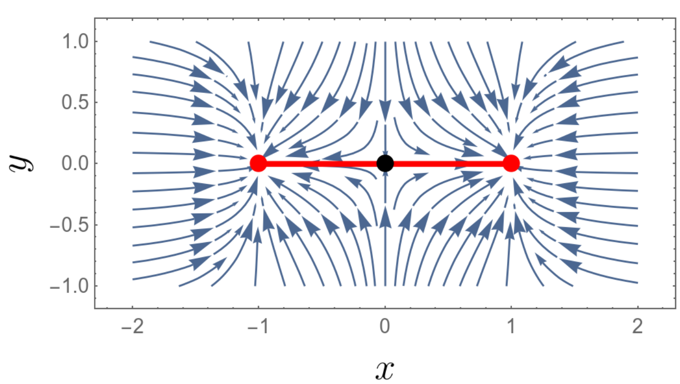
Definition 2.13. An attracting set with a dense orbit is called an *attractor*.

Definition 2.14. A orbit $\Gamma \in A$ is *dense* if for all $\epsilon > 0$ and all points $\mathbf{x} \in A$ there exists some point $\tilde{\mathbf{x}} \in \Gamma$ such that $|\mathbf{x} - \tilde{\mathbf{x}}| < \epsilon$. A dense orbit goes as close as wanted to any point of A.

Example 2.5.

$$\begin{aligned} \dot{x} &= x - x^3, \\ \dot{y} &= -y \end{aligned}$$

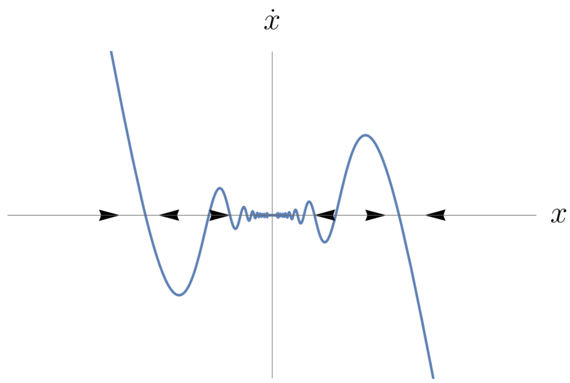
The interval $I_1 = \{(x, 0) \mid -1 \leq x \leq 1\}$ is an attracting set. But it is not an attractor, since it does not have a dense orbit. The interval $I_2 = \{(x, 0) \mid -1 \leq x \leq 0\}$ is an invariant set, has a dense orbit, but it is not an attracting set. The only attractors are the two points $(-1, 0)$ and $(1, 0)$.



Example 2.6.

$$\dot{x} = \begin{cases} -x^4 \sin(\pi/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{2.10}$$

Fixed points at $x = 0$, $x = \pm 1/n$. The set $A = [-1, 1]$ is an attracting set, because the neighbourhood $U = [-1 - \epsilon, 1 + \epsilon]$ for $\epsilon > 0$ is such that $\varphi_t(U) \subseteq U \ \forall t \geq 0$ and $\varphi_t(U) \to A$ as $t \to \infty$. The set A itself is not an attractor (there is no dense orbit). But each fixed point $x = \pm 1/(2n - 1)$ is an attractor. The point $x = 0$ is not an attracting set.



In two dimensions attracting sets are well characterised:

Theorem 2.15 (Poincaré-Bendixson theorem)**.** *Suppose $E \subset \mathbb{R}^2$ is an open subset of the plane and $\mathbf{f} \in C^1(E)$. If $D \subset E$ is compact (i.e. closed and bounded) such that $\mathbf{x}(t) \in D$ for all $t \geq 0$ where $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then the orbit either is a limit cycle, approaches a limit cycle as $t \to \infty$, or approaches an equilibrium point.*

Another way to say this is that either $\omega(\varphi_t(\mathbf{x}_0))$ contains a critical point or $\omega(\varphi_t(\mathbf{x}_0))$ is a periodic orbit. In three or more dimensions attractors can be much more exotic. The Lorenz system has a *strange attractor*. An attractor is called *strange* if it has fractal dimension.

2.3 Stability

As usual we consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in E \subseteq \mathbb{R}^n$, $f \in C^1(E)$.

The simplest solutions are fixed points.

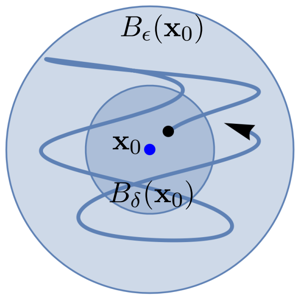
Definition 2.16. A *fixed point* $\mathbf{x}_0$ is a constant solution of the system, that is $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.

We want a definition of the intuitive idea of stability: "solutions close to a given invariant set remain close to that set for all time."

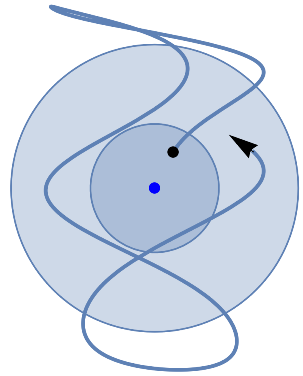
Let $B_\delta(\mathbf{x})$ denote the open ball of radius δ around $\mathbf{x}$, i.e.

$$B_\delta(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{x}_0| < \delta\}.$$

Definition 2.17. The fixed point $\mathbf{x}_0$ is *(Lyapunov) stable* if $\forall \ \epsilon > 0$, $\exists \ \delta > 0$ such that $\forall \ \mathbf{x} \in B_\delta(\mathbf{x}_0)$ and $t \geq 0$, we have $\varphi_t(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0)$. If $\mathbf{x}_0$ is not stable it is *(Lyapunov) unstable*.



Stable: trajectories starting within $B_\delta(\mathbf{x}_0)$ stay within $B_\epsilon(\mathbf{x}_0)$



This trajectory leaves $B_\epsilon(\mathbf{x}_0)$. Either the point $\mathbf{x}_0$ is unstable, or we need to choose a smaller δ.

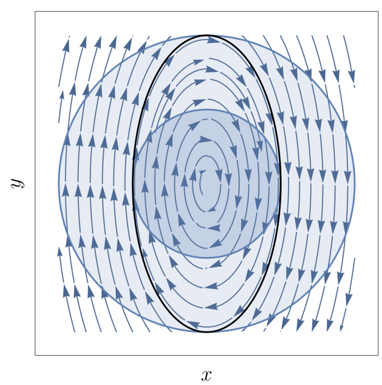
Example 2.7.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -4x \end{aligned} \qquad \Rightarrow \qquad 4x^2 + y^2 = c$$

Then

$$x_0^2 + y_0^2 < \delta^2 \quad \Rightarrow \quad c < 4\delta^2 \quad \Rightarrow \quad x^2 + y^2 < 4\delta^2.$$

Thus points starting in in $B_\delta(\mathbf{0})$ stay inside $B_{2\delta}(\mathbf{0})$ so choose $\delta = \epsilon/2$.



We also have the following stronger notion of stability:

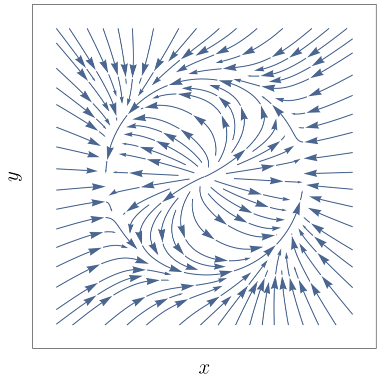
Definition 2.18. The fixed point $\mathbf{x}_0$ is *asymptotically stable* if (i) it is Lyapunov stable and (ii) $\exists\ \delta > 0$ such that $\varphi_t(\mathbf{x}) \to \mathbf{x}_0$ as $t \to \infty$ for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$.

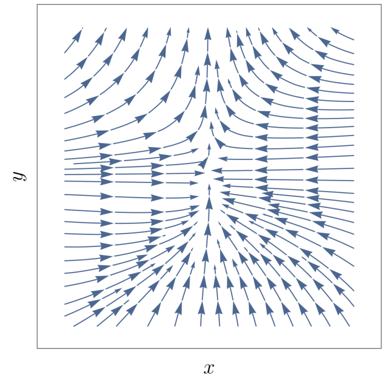
At first sight it might seem that we do not need (i) above, but to see that we do consider the following example.

Example 2.8.

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= \sin^2 \frac{\theta}{2} \end{aligned}$$

All trajectories starting from $\mathbf{x}_0 \neq \mathbf{0}$ tend to $(1,0)$. But $(1,0)$ is not stable (because trajectories starting at $\theta = \epsilon$ do a full loop and settle down at $\theta = 2\pi$—they do not stay close to (1,0)).





locally near $x = 1$, $y = 0$

For linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ the origin $\mathbf{x}_0 = \mathbf{0}$ is always a fixed point. If A is a semi-simple matrix, we have

- $\mathbf{x} = \mathbf{0}$ is asymptotically stable if $\mathrm{Re}(\lambda) < 0, \quad \forall\, \lambda \in \mathrm{Spec}(A)$.
- $\mathbf{x} = \mathbf{0}$ is stable if $\mathrm{Re}(\lambda) \leq 0, \quad \forall\, \lambda \in \mathrm{Spec}(A)$.

Remark 2.19. The first property remains true for non semi-simple A, but not the second one (can you find a counter-example?).

2.4 Lyapunov functions

Example 2.9. Consider the system

$$\begin{aligned} \dot{x} &= -x - y, \\ \dot{y} &= x - y. \end{aligned}$$

Define

$$V(\mathbf{x}) = \frac{1}{2}(x^2 + y^2).$$

Then, by the chain rule,

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = x(-x - y) + y(x - y) = -x^2 - y^2 = -2V \leq 0$$

with equality only if $x = y = 0$. Therefore

$$V(\mathbf{x}(t)) = V(\mathbf{x}(0))\mathrm{e}^{-2t} \to 0 \text{ as } t \to \infty$$

for all $\mathbf{x}_0$. Therefore

$$\mathbf{x} \to \mathbf{0} \quad \text{as } t \to \infty$$

i.e. the origin is a globally asymptotically stable equilibrium point.

More generally consider the usual system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

with $\mathbf{x} \in E \subseteq \mathbb{R}^n$, $\mathbf{f} \in C^1(E)$. Assume that this system has a fixed point $\mathbf{x}_0$ (so that $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$). Now consider a function $V(\mathbf{x})$ defined for $\mathbf{x} \in E$. The derivative of V along the solution trajectory $\varphi_t(\mathbf{x})$ at $\mathbf{x}$ is

$$\dot{V}(\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t} V(\varphi_t(\mathbf{x}))\bigg|_{t=0} = \nabla V(\mathbf{x}) \cdot \dot{\mathbf{x}} = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}).$$

Theorem 2.20. *Let W be an open subset of E and $\mathbf{x}_0 \in W$. Suppose there exists a function $V : W \to \mathbb{R}$, $V \in C^1(W)$ satisfying $V(\mathbf{x}_0) = 0$ and $V(\mathbf{x}) > 0\ \forall\ \mathbf{x} \in W \setminus \{\mathbf{x}_0\}$. Then,*

(i) if $\dot{V}(\mathbf{x}) \le 0\ \forall\ \mathbf{x} \in W \setminus \{\mathbf{x}_0\}$ then $\mathbf{x}_0$ is stable;

(ii) if $\dot{V}(\mathbf{x}) < 0\ \forall\ \mathbf{x} \in W \setminus \{\mathbf{x}_0\}$, then $\mathbf{x}_0$ is asymptotically stable;

(iii) if $\dot{V}(\mathbf{x}) > 0\ \forall\ \mathbf{x} \in W \setminus \{\mathbf{x}_0\}$, then $\mathbf{x}_0$ is unstable.

The function V is known as a Lyapunov function.

Proof. (Perko, p.131) (i) Given $\epsilon > 0$ sufficiently small that $\overline{B_\epsilon(\mathbf{x}_0)} \subset W$, let m_ϵ be the minimum of the continuous function V on the compact set

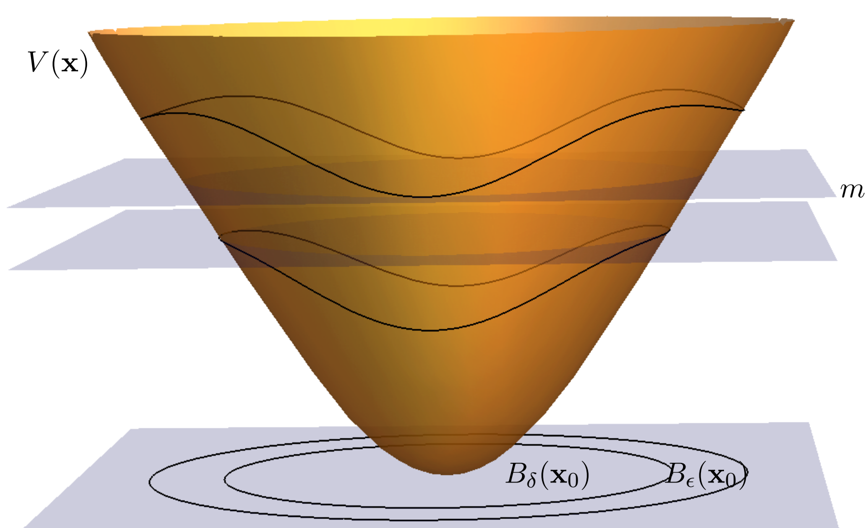
$$S_\epsilon = \partial B_\epsilon(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{x}_0| = \epsilon\}.$$

Since $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_0$ we have $m_\epsilon > 0$. Since V is continuous and $V(\mathbf{x}_0) = 0$ it follows that there exists $\delta > 0$ such that $|\mathbf{x} - \mathbf{x}_0| < \delta$ implies $V(\mathbf{x}) < m_\epsilon$.

Now $\dot{V}(\mathbf{x}) \le 0$ means that V is decreasing along trajectories of (2.1). Thus for all $\boldsymbol{\xi}_0 \in B_\delta(\mathbf{x}_0)$ and $t \ge 0$ we have

$$V(\varphi_t(\boldsymbol{\xi}_0)) \le V(\boldsymbol{\xi}_0) < m_\epsilon.$$

This implies that $\phi_t(\boldsymbol{\xi}_0) \in B_\epsilon(\mathbf{x}_0)\ \forall t$. Indeed, suppose for a contradiction that $\exists t_1 \in \mathbb{R}$ and $\boldsymbol{\xi}_0 \in B_\delta(\mathbf{x}_0)$ such that $\varphi_{t_1}(\boldsymbol{\xi}_0) \in S_\epsilon$. Then, since m_ϵ is the minimum of V on S_ϵ we must have $V(\varphi_{t_1}(\boldsymbol{\xi}_0)) \ge m_\epsilon$, contradicting the inequality above. Thus we have found the required $\delta > 0$ such that $|\varphi_t(\boldsymbol{\xi}_0) - \mathbf{x}_0| < \epsilon$ for all $\boldsymbol{\xi}_0 \in B_\delta(\mathbf{x}_0)$ and $t \ge 0$.



(ii) Now suppose that $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in W \setminus \{\mathbf{x}_0\}$, so that V is strictly decreasing along trajectories. Let $\{t_k\}$ be any sequence with $t_k \to \infty$ as $k \to \infty$. We first show that if $\varphi_{t_k}(\boldsymbol{\xi}_0) \to \mathbf{y}_0$ as $k \to \infty$ then $\mathbf{y}_0 = \mathbf{x}_0$. Since V is continuous $V(\phi_{t_k}(\boldsymbol{\xi}_0)) \to V(\mathbf{y}_0)$ as $t_k \to \infty$. Since V is strictly decreasing along trajectories, for any t,

$$V(\phi_t(\boldsymbol{\xi}_0)) > V(\mathbf{y}_0).$$

But it $\mathbf{y}_0 \neq \mathbf{x}_0$ then for $s > 0$ we have $V(\varphi_s(\mathbf{y}_0)) < V(\mathbf{y}_0)$ and by continuity it follows that for fixed s and all $\mathbf{y}$ sufficiently close to $\mathbf{y}_0$ we have $V(\varphi_s(\mathbf{y})) < V(\mathbf{y}_0)$. Since $\varphi_{t_k}(\boldsymbol{\xi}_0) \to \mathbf{y}_0$ we can choose $\mathbf{y} = \varphi_{t_k}(\boldsymbol{\xi}_0)$ for t_k sufficiently large to give

$$V(\varphi_s(\mathbf{y})) = V(\varphi_{s+t_k}(\boldsymbol{\xi}_0)) < V(\mathbf{y}_0),$$

contradicting the inequality above.

Now consider the sequence $\varphi_{t_k}(\boldsymbol{\xi}_0)$. Since $\varphi_{t_k}(\boldsymbol{\xi}_0) \in \overline{B_\epsilon(\mathbf{x}_0)}$ $\forall k$ and $\overline{B_\epsilon(\mathbf{x}_0)}$ is compact there is a subsequence $\varphi_{t_{k_n}}(\boldsymbol{\xi}_0)$ which converges to some point in $\overline{B_\epsilon(\mathbf{x}_0)}$. We have just shown that any convergent subsequence must converge to $\mathbf{x}_0$. It follows that the sequence $\varphi_{t_k}(\boldsymbol{\xi}_0)$ itself must converge to $\mathbf{x}_0$, and therefore $\varphi_t(\boldsymbol{\xi}_0) \to \mathbf{x}_0$ as $t \to \infty$.

(iii) Let M be the maximum of $V(\mathbf{x})$ on the compact set $\overline{B_\epsilon(\mathbf{x}_0)}$. Since V is now strictly increasing along trajectories, given any $\delta > 0$ and $\boldsymbol{\xi}_0 \in B_\delta(\mathbf{x}_0)$,

$$V(\varphi_t(\boldsymbol{\xi}_0)) > V(\boldsymbol{\xi}_0) > 0$$

for all t. Thus

$$\inf_{t \geq 0} \dot{V}(\varphi_t(\boldsymbol{\xi}_0)) = m > 0.$$

Then

$$V(\varphi_t(\boldsymbol{\xi}_0)) - V(\boldsymbol{\xi}_0) \geq mt.$$

Therefore for $t > M/m$

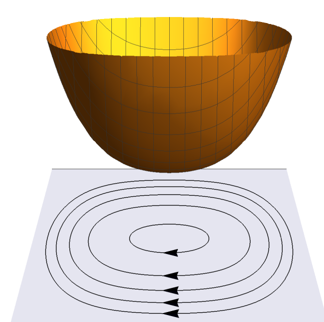
$$V(\varphi_t(\boldsymbol{\xi}_0)) > mt > M,$$

i.e. $\varphi_t(\boldsymbol{\xi}_0)$ lies outside $\overline{B_\epsilon(\mathbf{x}_0)}$.

$\square$

Example 2.10. The damped nonlinear spring

$$m\ddot{x} + k(x + x^3) + \alpha\dot{x} = 0, \qquad \alpha > 0, \qquad \begin{aligned} \dot{x} &= y, \\ m\dot{y} &= -k(x + x^3) - \alpha y \end{aligned}$$



(a) If $\alpha = 0$ [no damping] then the energy

$$E = \frac{m\dot{x}^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right)$$

is conserved. Take

$$V = \frac{my^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right).$$

Then
(i) $V(\mathbf{0}) = 0$; (ii) $V(\mathbf{x}) > 0 \ \forall \ \mathbf{x} \neq \mathbf{0}$; (iii) $\dot{V}(\mathbf{x}) = 0 \ \forall \ \mathbf{x}$.
Thus V is a Lyapunov function and $(0,0)$ is stable (but not asymptotically stable).

Note that any system in the form

$$\ddot{x} = -\frac{\partial W}{\partial x}$$

for some potential $W(x)$ can be treated in a similar way. Such systems are said to be conservative.
(b) When $\alpha \neq 0$ we might try E again. However, we find

$$\dot{E} = -\alpha y^2.$$

Thus E is enough to prove stability but not asymptotic stability, because $\dot{E} \not< 0 \ \forall y$. Try perturbing E as follows:

$$V = E + axy + bx^2.$$

Then

$$\begin{aligned}\dot{V} &= \dot{E} + \frac{\mathrm{d}}{\mathrm{d}t}\left(axy + bx^2\right) = -\alpha y^2 + a\dot{x}y + ax\dot{y} + 2bx\dot{x} \\ &= -\alpha y^2 + ay^2 - a\frac{k}{m}(x^2 + x^4) - \frac{a\alpha}{m}xy + 2bxy.\end{aligned}$$

Choose

$$b = \frac{a\alpha}{2m}$$

to eliminate the xy term, giving

$$\dot{V} = -(\alpha - a)y^2 - a\frac{k}{m}(x^2 + x^4).$$

If $a > 0$ is small enough then both coefficients are negative so that $\dot{V} < 0$ for all $(x,y) \neq (0,0)$. Also if a is small enough then we still have $V > 0$ for all $(x,y) \neq (0,0)$. Thus $(0,0)$ is asymptotically stable.

3 Local analysis

As usual, consider the nonlinear, autonomous, first-order system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{3.1}$$

with vector field $\mathbf{f} : E \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{f} \in C^1(E)$. Suppose that $\mathbf{x}_0$ is a fixed point of the system, that is $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$. How can we determine the stability of this fixed point algorithmically?

The basic idea is to look at nearby solutions by expanding $\mathbf{x}$ close to $\mathbf{x}_0$, that is, we write

$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}, \tag{3.2}$$

where $\boldsymbol{\xi}$ is small. Inserting (3.2) into (3.1) and Taylor expanding, remembering that $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ because $\mathbf{x}_0$ is a fixed point, we have for each component i,

$$\dot{\xi}_i = f_i(\mathbf{x}_0 + \boldsymbol{\xi}) = f_i(\mathbf{x}_0) + \sum_j \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)\xi_j + \cdots = \sum_j \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)\xi_j + \cdots .$$

We can write this succinctly as

$$\dot{\boldsymbol{\xi}} = D\mathbf{f}(\mathbf{x}_0)\boldsymbol{\xi} + \mathcal{O}(|\boldsymbol{\xi}|^2) \tag{3.3}$$

where $D\mathbf{f}(\mathbf{x}_0)$ is the *Jacobian matrix* associated with a vector field $\mathbf{f}$:

$$[D\mathbf{f}(\mathbf{x}_0)]_{ij} = \left[\frac{\partial f_i}{\partial x_j}\right]\Bigg|_{\mathbf{x}=\mathbf{x}_0} . \tag{3.4}$$

Remark 3.1. Since $\mathbf{x}_0$ is constant $D\mathbf{f}(\mathbf{x}_0)$ is a constant matrix.

The *variational equations* or *linearised equations* are given by the linear system obtained by dropping the nonlinear terms in (3.3):

$$\dot{\boldsymbol{\xi}} = D\mathbf{f}(\mathbf{x}_0)\boldsymbol{\xi} \tag{3.5}$$

Equation (3.5) is a linear equation with a constant matrix. We know we can solve it and that the stability of $\boldsymbol{\xi} = \mathbf{0}$ is determined by $\mathrm{Spec}(D\mathbf{f}(\mathbf{x}_0))$. The central problem of *local analysis* is to relate the stability of $\boldsymbol{\xi} = \mathbf{0}$ for (3.5) to the stability of $\boldsymbol{\xi} = \mathbf{0}$ for (3.1).

3.1 Stable manifold theorem

Before we talk about stable and unstable manifolds, we had better be clear about what we mean by manifold.

Definition 3.2. Let X be a metric space and let A and B be subsets of X. A *homeomorphism* of A onto B is a continuous one-to-one map $h : A \to B$ of A onto B such that $h^{-1} : B \to A$ is continuous. The sets A and B are called *homeomorphic* or *topologically equivalent* if there is a homeomorphism of A onto B.

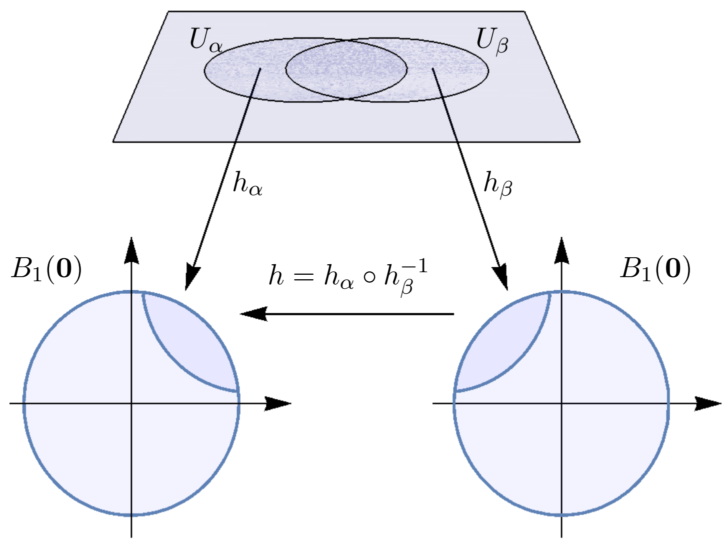
Definition 3.3. An *n-dimensional differentiable manifold*, $\mathcal{M}$, is a connected metric space with an open covering $\{U_\alpha\}$ (i.e. $\mathcal{M} = \cup_\alpha U_\alpha$) such that

(1) for all α, U_α is homeomorphic $B_1(\mathbf{0})$, the open unit ball in $\mathbb{R}^n$, i.e. for all α there exists a homeomorphism $\mathbf{h}_\alpha : U \to B_1(\mathbf{0})$ of U onto $B_1(\mathbf{0})$;

(2) if $U_\alpha \cap U_\beta \neq \emptyset$ and $\mathbf{h}_\alpha : U_\alpha \to B_1(\mathbf{0})$, $\mathbf{h}_\beta : U_\beta \to B_1(\mathbf{0})$ are homeomorphisms, then $\mathbf{h}_\alpha(U_\alpha \cap U_\beta)$ and $\mathbf{h}_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$\mathbf{h} = \mathbf{h}_\alpha \circ \mathbf{h}_\beta^{-1} : \mathbf{h}_\beta(U_\alpha \cap U_\beta) \to \mathbf{h}_\alpha(U_\alpha \cap U_\beta)$$

is differentiable and for all $\mathbf{x} \in \mathbf{h}_\beta(U_\alpha \cap U_\beta)$, the Jacobian determinant $\det D\mathbf{h}(\mathbf{x}) \neq 0$. The pair $(U_\alpha, \mathbf{h}_\alpha)$ is called a *chart* for the manifold $\mathcal{M}$, and the set of all charts is called an *atlas* for $\mathcal{M}$.



3.1.1 Basic idea in $\mathbb{R}^2$

Consider the system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{3.6}$$

and assume without loss of generality that $x = y = 0$ is a fixed point.
The linearised system is

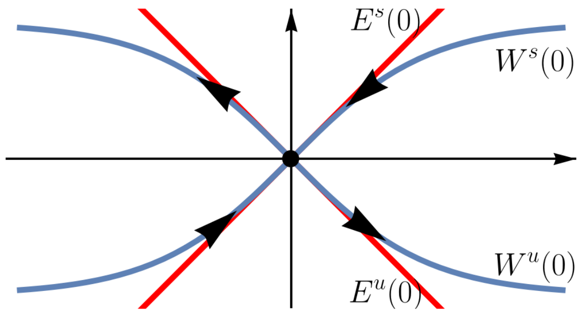
$$\begin{aligned} \dot{\xi} &= \partial_x f(0,0)\xi + \partial_y f(0,0)\eta \\ \dot{\eta} &= \partial_x g(0,0)\xi + \partial_y g(0,0)\eta \end{aligned} \tag{3.7}$$

Suppose that the eigenvalues of the Jacobian matrix

$$D\mathbf{f}(0) = \begin{bmatrix} \partial_x f(0,0) & \partial_y f(0,0) \\ \partial_x g(0,0) & \partial_y g(0,0) \end{bmatrix} \tag{3.8}$$

are real (but non-vanishing) with opposite sign. Then the linear system has a one-dimensional stable linear subspace, and a one-dimensional unstable linear subspace.

The nonlinear system has two trajectories which converge to the fixed point for large positive time, and these curves are tangent to the stable linear subspace at the origin. They form the stable manifold. It also has two trajectories which converge to the fixed point for large negative time, and these curves are tangent to the unstable linear subspace at the origin. These form the unstable manifold.



Explicitly, for this system, we define the stable and unstable manifolds as

$$W^s(0) = \{(x, y) \in \mathbb{R}^2 \mid \varphi_t(x, y) \to 0 \text{ as } t \to \infty\} \tag{3.9}$$

$$W^u(0) = \{(x, y) \in \mathbb{R}^2 \mid \varphi_t(x, y) \to 0 \text{ as } t \to -\infty\} \tag{3.10}$$

More generally we have

Theorem 3.4 (Stable manifold)**.** *Let E be an open subset of $\mathbb{R}^n$ containing $\mathbf{x}_0$ and let $\varphi_t : E \to E$ be the flow of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Suppose that the spectrum of $D\mathbf{f}(x_0)$ is composed of k eigenvalues with positive real parts and $(n-k)$ eigenvalues with negative real parts.*
Then,

- *there exists, in a neighbourhood of $\mathbf{x}_0$, a $(n-k)$-dimensional manifold $W^s_{loc}(\mathbf{x}_0)$ tangent to E^s such that $\forall t \geq 0$, $\varphi_t(W^s_{loc}) \subseteq W^s_{loc}$ and $\forall \mathbf{x} \in W^s_{loc}$, $\varphi_t(\mathbf{x}) \to \mathbf{x}_0$ as $t \to \infty$.*
- *there exists, in a neighbourhood of $\mathbf{x}_0$, a k-dimensional manifold $W^u_{loc}(\mathbf{x}_0)$ tangent to E^u such that $\forall t \leq 0$, $\varphi_t(W^u_{loc}) \subseteq W^u_{loc}$ and $\forall \mathbf{x} \in W^u_{loc}$, $\varphi_t(\mathbf{x}) \to \mathbf{x}_0$ as $t \to -\infty$.*

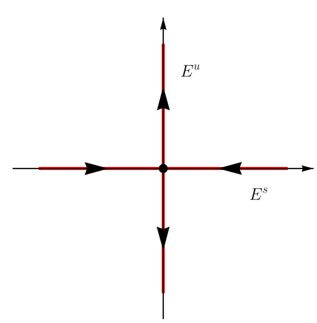
Moreover, W^s_{loc} and W^u_{loc} are as smooth as $\mathbf{f}$.

The existence of **local** stable and and unstable manifolds allows us to define ***global*** *stable and unstable manifolds* as follows:

$$W^s(\mathbf{x}_0) = \bigcup_{t \leq 0} \varphi_t\left(W^s_{\text{loc}}(\mathbf{x}_0)\right) \qquad\qquad W^u(\mathbf{x}_0) = \bigcup_{t \geq 0} \varphi_t\left(W^u_{\text{loc}}(\mathbf{x}_0)\right)$$

Example 3.1.

$$\begin{aligned} \dot{x} &= -x - y^2 \\ \dot{y} &= y + x^2 \end{aligned} \qquad (0,0) \text{ fixed point,} \qquad D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



W^s_{loc} is analytic (since $\mathbf{f}$ is analytic) and tangent to E^s so is of the form

$$y = \sum_{i=0}^{\infty} a_i x^i \qquad \text{with } a_0 = a_1 = 0.$$

Locally $y = a_2 x^2 + O(x^3)$. Can we find a_2? Differentiating gives

$$\begin{aligned}
& \dot{y} = 2a_2 x\dot{x} + O(\dot{x}x^2) \\
\Rightarrow \quad & y + x^2 = 2a_2 x(-x - y^2) + O(x^3) \\
\Rightarrow \quad & a_2 x^2 + x^2 = 2a_2 x(-x) + O(x^3) \\
\Rightarrow \quad & a_2 + 1 = -2a_2 \qquad \Rightarrow \qquad a_2 = -1/3
\end{aligned}$$

Thus

$$y = -\frac{x^2}{3} + O(x^3).$$

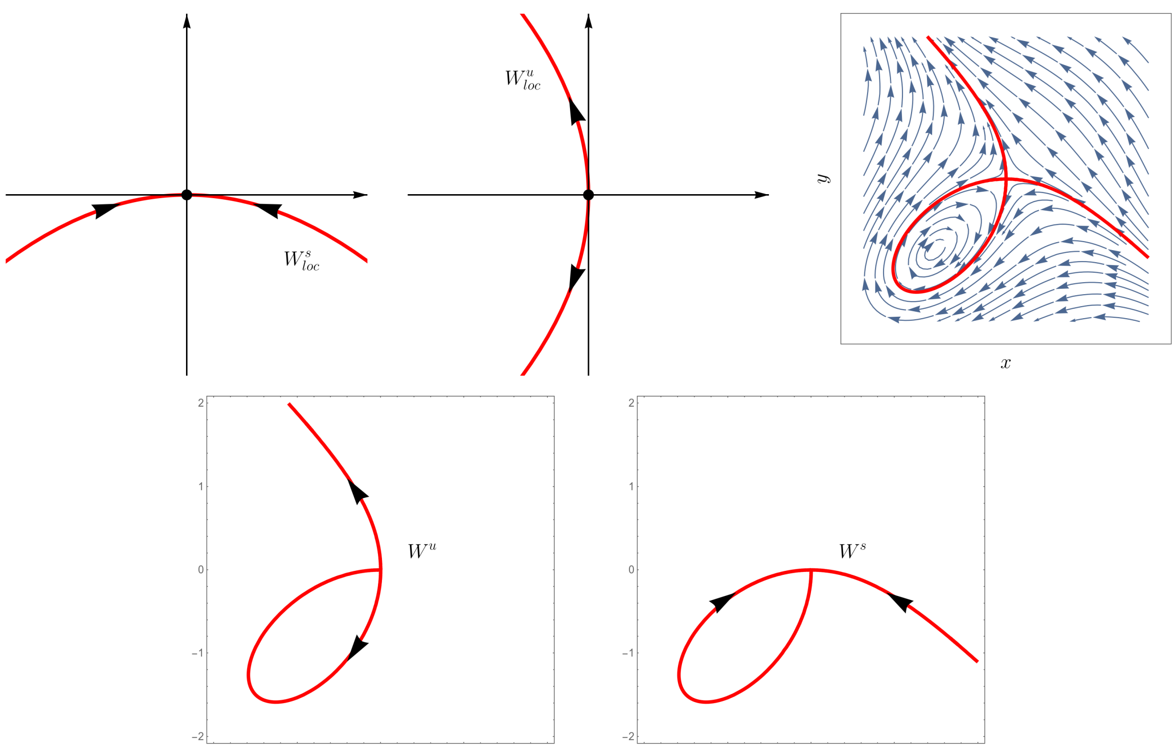
A similar calculation for W^u_{loc} gives

$$x = -\frac{y^2}{3} + O(y^3).$$

What about the global manifold? Here we are in luck because we observe that $J = 3xy + y^3 + x^3$ is conserved:

$$\begin{aligned}
\dot{J} &= 3\dot{x}y + 3x\dot{y} + 3y^2\dot{y} + 3x^2\dot{x} \\
&= -3y(x + y^2) + 3x(y + x^2) + 3y^2(y + x^2) - 3x^2(x + y^2) \\
&= -3yx - 3y^3 + 3xy + 3x^3 + 3y^3 + 3x^2y^2 - 3x^3 - 3x^2y^2 \\
&= 0.
\end{aligned}$$

Thus the stable and unstable manifolds must both have $J = 0$. Maclaurin trisectrix (1742).



Remark 3.5. W^s and W^u are not solution curves (they are unions of curves).

Remark 3.6. If $\mathbf{f}$ is analytic, it follows that W^s and W^u are also analytic.

Remark 3.7. If $W^s \cap W^u \neq \emptyset$, then $W^s \cap W^u$ is a *homoclinic manifold*. The property of the homoclinic manifold is that any initial condition on the manifold ends up asymptotically for negative and positive time on the same fixed point.

3.1.2 Hyperbolicity and stability

Definition 3.8. If $\mathrm{Re}(\lambda) \neq 0$ for all $\lambda \in \mathrm{Spec}(D\mathbf{f}(\mathbf{x}_0))$, then $\mathbf{x}_0$ is an *hyperbolic fixed point.*

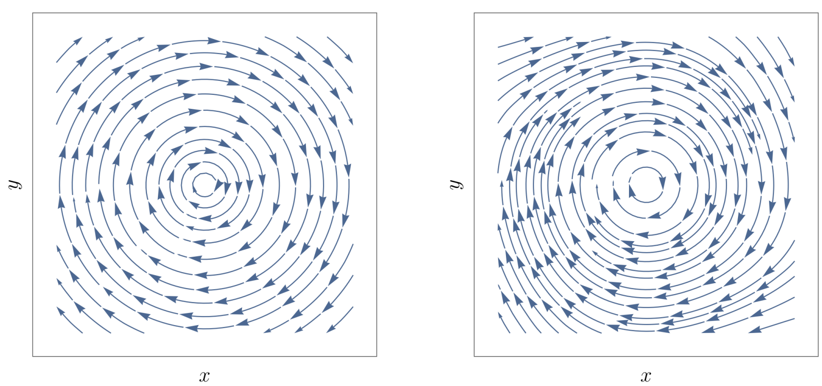
The stability of hyperbolic fixed points is fully determined by the linearisation of the vector field around the fixed point:

Theorem 3.9. *If $Re(\lambda) < 0$ for all $\lambda \in Spec(D\mathbf{f}(\mathbf{x}_0))$, then $\mathbf{x}_0$ is asymptotically stable. If there exists $\lambda \in Spec(D\mathbf{f}(\mathbf{x}_0))$ such that $Re(\lambda) > 0$, then $\mathbf{x}_0$ is unstable.*

To illustrate the necessity of hyperbolicity consider the following example

Example 3.2. The nonlinearly damped harmonic oscillator

$$\ddot{x}+\epsilon x^2\dot{x}+x = 0 \qquad \Rightarrow \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \epsilon x^2 y \end{aligned} \qquad \Rightarrow \qquad D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



$\epsilon = 0$ $\qquad$ $\epsilon > 0$

The linearised system is stable, but gives no information on the stability of the nonlinear system. We try the Lyapunov function (the energy of the linear system)

$$E = \frac{y^2}{2} + \frac{x^2}{2}$$

which gives

$$\dot{E} = -\epsilon x^2 y^2$$

which as a Lyapunov function is enough for stability but not asymptotic stability. If we try

$$L = \frac{y^2}{2} + \frac{x^2}{2} + \alpha x^3 y - \beta x y^3$$

we find

$$\dot{L} \quad = \quad -\alpha x^4 - \beta y^4 - (\epsilon - 3\alpha - 3\beta)x^2y^2 - \alpha\epsilon x^5 y + 3\beta\epsilon x^3 y^3.$$

The first three terms are of the right sign if α and β are small enough and $\epsilon > 0$. We need to group the remaining terms with terms that dominate them. We see

$$\dot{L} \quad = \quad -\alpha x^4(1+\epsilon xy) - \beta y^4 - (\epsilon - 3\alpha - 3\beta)x^2y^2\left(1 - \frac{3\beta\epsilon xy}{\epsilon - 3\alpha - 3\beta}\right) < 0$$

for small enough $(x, y) \neq (0, 0)$. Thus the origin is asymptotically stable for $\epsilon > 0$.

3.2 The centre manifold

Recall the construction of the stable and unstable manifolds: they are defined locally as the unique manifolds tangent to the stable and unstable linear subspaces of the linearised equations. Then the global stable and unstable manifolds are defined as the evolution in (negative and positive respectively) time of these local manifolds.

$$W^{s,u} \text{ defined as the set of points such that } \mathbf{x} \to \mathbf{x}_0 \text{ as } t \to \pm\infty.$$

What happens if one of the eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ has zero real part? In this case, the linearised equations have a non-empty centre subspace.

Theorem 3.10 (Centre manifold)**.** *Let $\varphi_t : E \subseteq \mathbb{R}^n \to E$ be the flow of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with fixed point $\mathbf{x}_0$ where $f \in C^r(E)$. Suppose that the spectrum of $D\mathbf{f}(\mathbf{x}_0)$ has k eigenvalues with zero real part and $(n-k)$ eigenvalues with non-zero real part. Then there exists, in a neighbourhood of $\mathbf{x}_0$ a k-dimensional manifold $W^c_{loc}(\mathbf{x}_0)$ that is*

(i) tangent to E^c at $\mathbf{x}_0$;

(ii) of class C^r;

(iii) invariant under the flow.

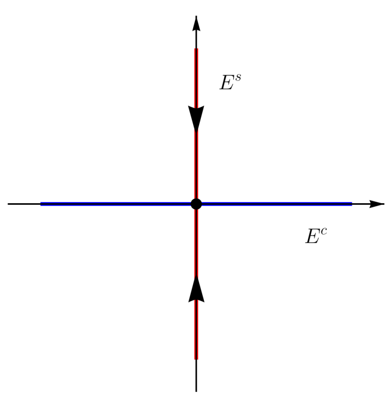
The problem is that the centre manifold may not be unique, as shown by the following example.

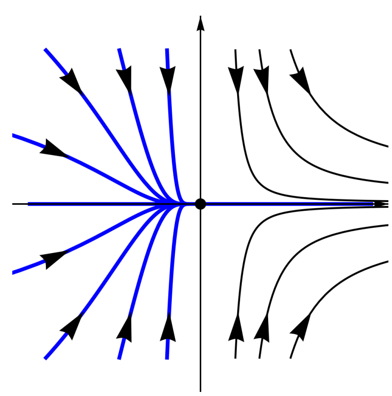
Example 3.3 (Perko p.116)**.**

$$\begin{aligned}\dot{x} &= x^2 \\ \dot{y} &= -y\end{aligned} \quad \Rightarrow \quad \begin{aligned}x &= \frac{1}{1/x_0 - t} \\ y &= y_0 \mathrm{e}^{-t}\end{aligned} \quad \Rightarrow \quad y = y_0 \mathrm{e}^{-1/x_0}\mathrm{e}^{1/x}$$

Linearised system is

$$\begin{aligned}\dot{x} &= 0 \\ \dot{y} &= -y\end{aligned}$$





Every single curve $y = Ce^{1/x}$, with $x \leq 0$, is tangent to E^c at $(0,0)$. Any solution curve to the left of the origin, patched with the positive x-axis at the origin, would give a one-dimensional centre manifold of class C^∞. However, there is only one curve for which the centre manifold would be *analytic* (the same smoothness as $\mathbf{f}$). This is the curve corresponding to $C = 0$, which gives the x-axis. Often we are interested in the smoothest centre manifold.

We can combine the two manifold theorems.

Theorem 3.11. *Given an open subset $E \subseteq R$, a vector field $\mathbf{f} \in C^r(E)$, $r \geq 1$, with a fixed point $\mathbf{x}_0$, and a set of eigenvalues $\Lambda = Spec\,(D\mathbf{f}(\mathbf{x}_0))$, we have*

- *k_s eigenvalues $\lambda \in \Lambda$ with $Re(\lambda) < 0$, with linear subspace E^s,*
- *k_u eigenvalues $\lambda \in \Lambda$ with $Re(\lambda) > 0$, with linear subspace E^u,*
- *k_c eigenvalues $\lambda \in \Lambda$ with $Re(\lambda) = 0$, with linear subspace E^c,*

with $k_s + k_u + k_c = n$. Then there exist

- *a unique k_s-dimensional manifold W^s of class C^r tangent to E^s at $\mathbf{x}_0$,*
- *a unique k_u-dimensional manifold W^u of class C^r tangent to E^u at $\mathbf{x}_0$,*
- *a k_c-dimensional manifold W^c of class C^r tangent to E^c at $\mathbf{x}_0$.*

Furthermore W^s, W^u and W^c are invariant under the flow of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Note that W^s and W^u are unique, but W^c need not be.

3.3 Reduction to the centre manifold

Consider again a fixed point $\mathbf{x}_0$. If the unstable manifold is non-empty, the fixed point is unstable. Suppose the unstable manifold is empty and the system has both a non-empty stable and centre manifold. What is the stability of a fixed point in this case?
Basic idea: The stability is governed by the dynamics on the centre manifold.

Without loss of generality we assume that the original system has been brought, by a linear transformation, to the canonical form:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x},\mathbf{y}), \qquad && \mathbf{x} \in \mathbb{R}^{\dim W^c} \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x},\mathbf{y}) \qquad && \mathbf{y} \in \mathbb{R}^{\dim W^s} \end{aligned} \tag{3.11}$$

where $(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{0}, \mathbf{0})$ is a fixed point (i.e. $\mathbf{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}(\mathbf{0},\mathbf{0}) = \mathbf{0}$) and

$$\begin{aligned} \mathrm{Re}(\lambda) &= 0 \ \ \forall\, \lambda \in \ \mathrm{Spec}(A), \\ \mathrm{Re}(\lambda) &< 0 \ \ \forall\, \lambda \in \ \mathrm{Spec}(B). \end{aligned} \tag{3.12}$$

Note that we also assume that $\mathbf{f}$ and $\mathbf{g}$ are nonlinear at the origin (the Jacobian $\partial(\mathbf{f},\mathbf{g})/\partial(\mathbf{x},\mathbf{y})$ vanishes at $(\mathbf{0},\mathbf{0})$).

The variables $\mathbf{x}$ define the centre linear subspace. The main idea is to obtain a description of the centre manifold in terms of the variables $\mathbf{x}$. We posit that the centre manifold may be described by

$$\mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{3.13}$$

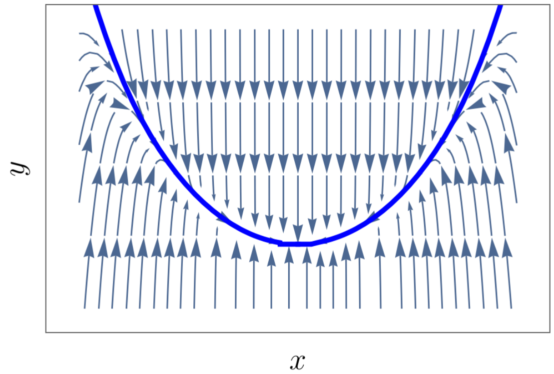


Figure 3.1: Trajectories are attracted exponentially to $W^c \Rightarrow$ dynamics *on* W^c determines the stability.

and we look for a suitable function $\mathbf{h}(\mathbf{x})$. Differentiating gives

$$\dot{\mathbf{y}} = D\mathbf{h}(\mathbf{x})\dot{\mathbf{x}}. \tag{3.14}$$

Hence

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})), \\ \dot{\mathbf{y}} &= D\mathbf{h}(\mathbf{x})\dot{\mathbf{x}} = B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \end{aligned} \tag{3.15}$$

Substituting for $\dot{\mathbf{x}}$ from the first equation into the second equation gives an equation for $\mathbf{h}(\mathbf{x})$:

$$D\mathbf{h}(\mathbf{x})\left(A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))\right) = B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})). \tag{3.16}$$

Close to the origin we can solve it by expanding $\mathbf{h}$ in a Taylor series:

$$\mathbf{h} = \sum_{\mathbf{m},\ ||\mathbf{m}||_1=2}^{||\mathbf{m}||_1=d} \mathbf{h}_\mathbf{m}\mathbf{x}^\mathbf{m} + \mathcal{O}(|\mathbf{x}|^{d+1}), \tag{3.17}$$

and solving for the coefficients $\mathbf{h}_\mathbf{m}$.

Remark 3.12. The Taylor series starts at $||\mathbf{m}||_1 = 2$ because we know the centre manifold is tangent to the centre linear subspace, which is given by $\mathbf{y} = \mathbf{0}$. Thus the derivatives of $\mathbf{h}$ must all be zero at $\mathbf{x} = \mathbf{0}$.

Remark 3.13. In (3.17) we are using the multinomial formalism for a vector $\mathbf{x} = (x_1, \ldots, x_n)$ and postive integer vector $\mathbf{m} = (m_1, \ldots, m_n)$:

$$\mathbf{x}^\mathbf{m} = \prod_{i=1}^{n} x_i^{m_i} \tag{3.18}$$

Once $\mathbf{h}$ is known, the first of equations (3.15) gives the dynamics on the centre manifold:

Theorem 3.14. *The dynamics of (3.11) on its centre manifold W^c at the origin is, for $(\mathbf{x}, \mathbf{y})$ close enough to the origin, given by the dynamics of*

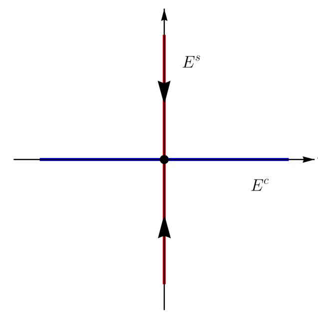
$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})).$$

Example 3.4.

$$\begin{aligned}\dot{x} &= x^2 y - x^5 \\ \dot{y} &= -y + x^2\end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x^2 y - x^5 \\ x^2 \end{bmatrix}.$$

Eigenvalues and linear subspaces:

$$\lambda = -1, \quad X^s = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle, \qquad \lambda = 0, \quad X^c = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$$



On the centre linear subspace $y = 0$. Putting $y = 0$ gives $\dot{x} = -x^5$. Does this mean that the origin is stable?

Denote the centre manifold by

$$y = h(x) = h_2 x^2 + h_3 x^3 + \cdots \qquad \Rightarrow \qquad Dh(x) = 2h_2 x + 3h_3 x^2 + \cdots .$$

Then (3.16) is

$$(2h_2 x + 3h_3 x^2 + \cdots)\left(x^2(h_2 x^2 + h_3 x^3 + \cdots) - x^5\right) = -(h_2 x^2 + h_3 x^3 + \cdots) + x^2.$$

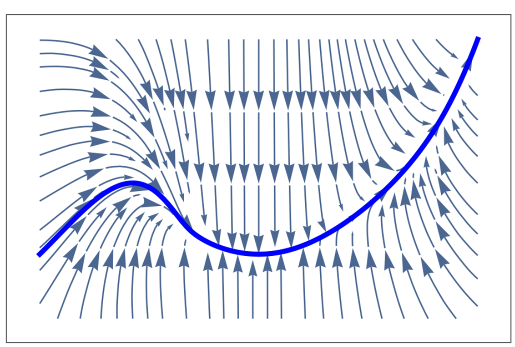
Equating coefficients of powers of x gives

$$h_2 = 1, \qquad h_3 = 0, \qquad \cdots \qquad \Rightarrow \qquad h = x^2 + O(x^4).$$

On the centre manifold the dynamics is given by

$$\dot{x} = x^2 h(x) - x^5 = x^4 - x^5 + O(x^6).$$

Thus the origin is unstable (the dominant term x^4 is positive for positive x).



y

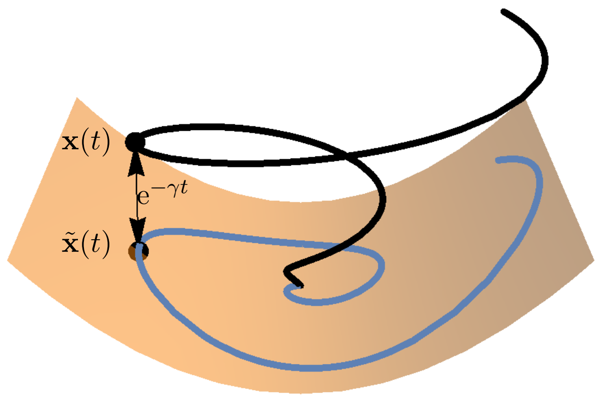
x

Close enough to the origin, the dynamics in the full space is well approximated by the dynamics on the centre manifold:

Theorem 3.15 (Shadowing)**.** *Let $(\mathbf{x}_0, \mathbf{y}_0)$ be close enough to the origin, and $(\mathbf{x}(t), \mathbf{y}(t))$ be the solution of (3.15) starting at $(\mathbf{x}_0, \mathbf{y}_0)$. Then there exists a solution $\tilde{\mathbf{x}}(t)$ on the centre manifold such that*

$$\begin{cases} \mathbf{x}(t) = \tilde{\mathbf{x}}(t) + \mathcal{O}(e^{-\gamma t}), \\ \mathbf{y}(t) = \mathbf{h}(\tilde{\mathbf{x}}(t)) + \mathcal{O}(e^{-\gamma t}), \end{cases} \tag{3.19}$$

for some constant $\gamma > 0$.



3.3.1 The step-by-step method

We start with a system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}), \qquad \mathbf{z} \in \mathbb{R}^p. \tag{3.20}$$

Assume that it has a fixed point at $\mathbf{z}_0$ (i.e. $\mathbf{F}(\mathbf{z}_0) = \mathbf{0}$), and that $M = D\mathbf{F}(\mathbf{z}_0)$ has $n \geq 1$ eigenvalues with zero real parts and $m \geq 1$ eigenvalues with negative real parts ($n + m = p$), and no eigenvalue with positive real part (otherwise the fixed point is unstable).

Step 1: Reduction to a canonical form: Introduce the new variables

$$\mathbf{z} = \mathbf{z}_0 + C\tilde{\mathbf{z}}, \tag{3.21}$$

where C is chosen such that

$$C^{-1}MC = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]. \tag{3.22}$$

where the matrices A and B of respective dimension n and m are such that

$$\begin{aligned} \mathrm{Re}(\lambda) = 0 \quad \forall\, \lambda \in \ \mathrm{Spec}(A), \\ \mathrm{Re}(\lambda) < 0 \quad \forall\, \lambda \in \ \mathrm{Spec}(B). \end{aligned} \tag{3.23}$$

After the change of variable, the new system in the variable $\tilde{\mathbf{z}} = (\mathbf{x}, \mathbf{y})$ is

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \qquad && \mathbf{x} \in \mathbb{R}^n \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}) && \mathbf{y} \in \mathbb{R}^m \end{aligned} \tag{3.24}$$

and $(\mathbf{0}, \mathbf{0})$ is a fixed point.

Step 2: Reduction to the centre manifold: We want to solve

$$D\mathbf{h}(\mathbf{x})\,(A\mathbf{x}+\mathbf{f}(\mathbf{x},\mathbf{h}(\mathbf{x}))) = B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x},\mathbf{h}(\mathbf{x})). \tag{3.25}$$

Close to the origin we expand $\mathbf{h}$ in Taylor series:

$$\mathbf{h} = \sum_{\mathbf{m},\ ||\mathbf{m}||_1=2}^{||\mathbf{m}||_1=d} \mathbf{h}_\mathbf{m}\mathbf{x}^\mathbf{m} + \mathcal{O}(|\mathbf{x}|^{d+1}), \tag{3.26}$$

We first choose $d=2$. Inserting this expansion into (3.25) and expanding $\mathbf{g}$ in power series, we obtain a linear set of equations for $\mathbf{h}_\mathbf{m}$. If there is a non-trivial solution to this set of linear equations, we have the first nonlinear approximation of the centre-manifold. Otherwise, we increase the value of d until we find a non-trivial solution.

Step 3: Dynamics on the centre manifold: We insert the first non-zero approximation

$$\mathbf{h} = \sum_{\mathbf{m},\ ||\mathbf{m}||_1=2}^{||\mathbf{m}||_1=d} \mathbf{h}_\mathbf{m}\mathbf{x}^\mathbf{m}, \tag{3.27}$$

into

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x},\mathbf{h}(\mathbf{x})). \tag{3.28}$$

and we obtain the polynomial system:

$$\dot{\mathbf{x}} = A\mathbf{x} + \sum_{\mathbf{m},\ ||\mathbf{m}||_1=2}^{||\mathbf{m}||_1=d} \mathbf{f}_\mathbf{m}\mathbf{x}^\mathbf{m} + O(|\mathbf{x}|^{d+1}). \tag{3.29}$$

This is still a nonlinear system but of reduced dimension $n < p$. The hope is that it is sufficiently simple to be analysed by elementary means (direct integration, Lyapunov functions,...).

Example 3.5.

$$\begin{aligned}
\dot{z}_1 &= \frac{1}{2}(-z_1 - z_2 + z_3 + z_1^2 - z_2^2 - z_3^2 - 4z_1z_3),\\
\dot{z}_2 &= z_1 + z_3 + z_1z_2 - z_2z_3,\\
\dot{z}_3 &= \frac{1}{2}(z_1 - z_2 - z_3 + z_1^2 + z_2^2 - z_3^2 + 4z_1z_3)
\end{aligned}
\qquad \Rightarrow \qquad M = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\det(M-\lambda)=0 \quad \Rightarrow \quad -\lambda^3-\lambda^2-\lambda-1 = -(1+\lambda)(1+\lambda^2)=0 \qquad \Rightarrow \qquad \lambda = -1, \pm\mathrm{i}.$$

$$\lambda_1=-1,\ \mathbf{w}_1 = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}, \qquad \lambda_2 = \mathrm{i},\ \mathbf{w}_2 = \begin{bmatrix} 1/2 \\ -\mathrm{i} \\ 1/2 \end{bmatrix}, \qquad \lambda_3=-\mathrm{i},\ \mathbf{w}_3 = \begin{bmatrix} 1/2 \\ \mathrm{i} \\ 1/2 \end{bmatrix},$$

So

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} = P^{-1}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

Then

$$\begin{aligned}
\dot{x}_1 &= \dot{z}_1 + \dot{z}_3 = -z_2 + z_1^2 - z_3^2 = -x_2 + x_1 y, \\
\dot{x}_2 &= \dot{z}_2 = z_1 + z_3 + z_1 z_2 - z_2 z_3 = x_1 + x_2 y, \\
\dot{y} &= \dot{z}_1 - \dot{z}_3 = -z_1 + z_3 - z_2^2 - 4 z_1 z_3 = -y - x_2^2 - x_1^2 + y^2.
\end{aligned}$$

So system in canonical form is

$$\begin{aligned}
\dot{x}_1 &= -x_2 + x_1 y, \\
\dot{x}_2 &= x_1 + x_2 y, \\
\dot{y} &= -y - x_1^2 - x_2^2 + y^2
\end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} + \begin{bmatrix} x_1 y \\ x_2 y \\ -x_1^2 - x_2^2 + y^2 \end{bmatrix}.$$

Writing $h = h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2 + \cdots$ gives

$$Dh = [2h_{20} x_1 + h_{11} x_2,\ h_{11} x_1 + 2h_{02} x_2] + \cdots .$$

$$\begin{aligned}
D\mathbf{h}(\mathbf{x})\,(A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))) &= [2h_{20} x_1 + h_{11} x_2,\ h_{11} x_1 + 2h_{02} x_2] \begin{bmatrix} -x_2 + x_1 (h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2) \\ x_1 + x_2 (h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2) \end{bmatrix} + \cdots \\
&= -2h_{20} x_1 x_2 - h_{11} x_2^2 + h_{11} x_1^2 + 2h_{02} x_1 x_2 + \cdots
\end{aligned}$$

$$\begin{aligned}
B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) &= -(h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2) - x_1^2 - x_2^2 + (h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2)^2 + \cdots \\
&= -(h_{20} + 1) x_1^2 - h_{11} x_1 x_2 - (h_{02} + 1) x_2^2 + \cdots
\end{aligned}$$

Equating powers of x_1^2, $x_1 x_2$ and x_2^2 gives

$$h_{11} = -h_{20} - 1, \qquad 2h_{02} - 2h_{20} = -h_{11}, \qquad -h_{11} = -h_{20} - 1.$$

Solving gives

$$h_{20} = -1, \quad h_{11} = 0, \quad h_{02} = -1 \qquad \Rightarrow \qquad h(x_1, x_2) = -x_1^2 - x_2^2 + O(|\mathbf{x}|^3)$$

The dynamics on the centre manifold is then given by

$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_1^3 - x_1 x_2^2 + O(|\mathbf{x}|^4), \\
\dot{x}_2 &= x_1 - x_2 x_1^2 - x_2^3 + O(|\mathbf{x}|^4)
\end{aligned}$$

Writing $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ gives

$$\dot{r} = -r^3 + O(r^4)$$

so that the origin is stable.

Remark 3.16. We could do this without switching to canonical variables, so long as we made sure the Taylor expansion of the centre manifold agreed with the centre subspace at leading order, i.e. we could set

$$z_3 = h(z_1, z_2) = z_1 + h_{20} z_1^2 + h_{11} z_1 z_2 + h_{02} z_2^2 + \cdots ,$$

or

$$z_1 = h(z_2, z_3) = z_3 + a_{20} z_2^2 + a_{11} z_2 z_3 + a_{02} z_3^2 + \cdots .$$

However, we would need to expand h to $O(\{z_1, z_2\}^3)$ to capture the dynamics. In canonical variables we only needed to expand to $O(\{x_1, x_2\}^2)$ because the right-hand side of the equations for x_1 and x_2 involve y multiplied by a linear term in x_1 or x_2.

4 Bifurcations

4.1 Local bifurcations for vector fields

Consider the nonlinear, autonomous, first-order system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) \qquad \text{where} \quad \mathbf{x} \in E \subseteq \mathbb{R}^n, \quad \boldsymbol{\mu} \in \mathbb{R}^p \tag{4.1}$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)$ is a vector of parameters.
Questions we might be interested in include:
How does the dynamics change when the parameters are varied?
What are the special values where qualitative changes occur?
Can the possible changes be classified?
Can they be obtained algorithmically?

What do we mean by "special values"? We are interested in parameter values for which the system is not structurally stable.

Definition 4.1 (Topological equivalence)**.** Two vector fields $\mathbf{f}$ and $\mathbf{g}$ and associated flows $\varphi_t(\mathbf{x})$ and $\psi_t(\mathbf{x})$ are topologically equivalent if $\exists$ a homeomorphism (1-1, continuous, with continuous inverse) $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$, and a map $\tau(t, \mathbf{x}) \to \mathbb{R}$, strictly increasing on t, such that,

$$\tau(t+s, \mathbf{x}) = \tau(s, \mathbf{x}) + \tau(t, \varphi_s(\mathbf{x})), \qquad \text{and} \qquad \psi_{\tau(t,\mathbf{x})}(\mathbf{h}(\mathbf{x})) = \mathbf{h}(\varphi_t(\mathbf{x})).$$

Definition 4.2 (Structural stability)**.** The vector field $\mathbf{f}$ is structurally stable if for all continuously differentiable vector fields $\mathbf{v}$ there exists $\epsilon_v > 0$ such that $\mathbf{f}$ is topologically equivalent to $\mathbf{f} + \epsilon\mathbf{v}$ for all $0 < \epsilon < \epsilon_v$.

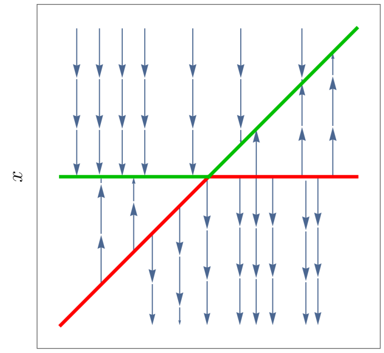
If we change parameters for a given $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu})$ then we will have structural stability in general except for certain special values of $\boldsymbol{\mu}$ (i.e. certain surfaces in $\boldsymbol{\mu}$-space).

Definition 4.3. We define a *bifurcation point* $\boldsymbol{\mu}_c$ as a point in parameter space where f is not structurally stable. A *bifurcation* (change in structure of the solution) will occur when the parameters are varied to pass through these points. The *bifurcation set* is the locus in $\boldsymbol{\mu}$-space of bifurcation values. If we plot, for example, the amplitude of the fixed points and periodic orbits as the parameters are varied this is called a *bifurcation diagram.*

Example 4.1 (Transcritical bifurcation)**.** Take $p = n = 1$ and consider the system

$$\dot{x} = \mu x - x^2 \tag{4.2}$$

Equilibrium points are $x = 0$ and $x = \mu$.



The "Jacobian" is $Df = \mu - 2x$ with eigenvalue $\lambda = \mu - 2x$. Thus $x = 0$ is stable for $\mu < 0$ but unstable for $\mu > 0$. Similarly $x = \mu$ is unstable for $\mu < 0$ but stable for $\mu > 0$. There is a switch in stability as the two solution branches cross.

Remark 4.4. Loss of structural stability at the bifurcation point $\mu = 0$ here is indicated by the change in stability of the fixed points. Bifurcation points for fixed points can be identified as places where the number or stability of fixed points changes.

We consider the problem of determining bifurcations at fixed points. Take $\mu \in \mathbb{R}$ $(p = 1)$. Then the fixed points are given by the solution of

$$\mathbf{f}(\mathbf{x}, \mu) = \mathbf{0}.$$

For values of μ for which a solution can be found, the solution defines *a branch of equilibria* $\mathbf{x} = \mathbf{x}(\mu)$. Along this branch, define the matrix

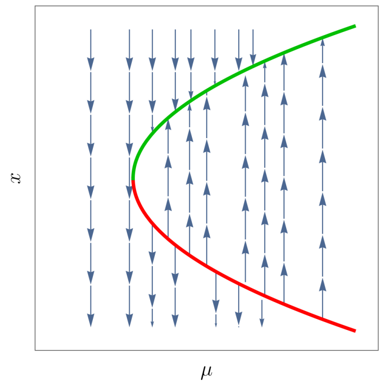
$$D(\mu) = D_{\mathbf{x}}\mathbf{f}|_{(\mathbf{x}(\mu),\mu)}\,.$$

Suppose that there is a value μ_0 for which $D(\mu)$ has only eigenvalues with non-zero real parts (i.e. the fixed point is hyperbolic for that value). Then, $D(\mu)$ is invertible and the local branch of equilibria can be continued locally. We increase (or decrease) μ up to a critical *bifurcation value* μ_c where $D(\mu)$ is not invertible. At this point, the branch of equilibria is non-smooth.

Example 4.2 (Saddle-node or fold bifurcation)**.** Consider the system

$$\dot{x} = \mu - x^2$$

Here the equilibrium points are $x = \pm\mu^{1/2}$ which exist only for $\mu > 0$. The Jacobian $Df = -2x$. At the critical point $Df = \mp 2\mu^{1/2}$, and this vanishes when $\mu = 0$, which is therefore the bifurcation point. For a one dimensional system there is just one eigenvalue λ and it is equal to Df. Thus $x = +\mu^{1/2}$ is stable and $x = -\mu^{1/2}$ is unstable.



Example 4.3 (Supercritical pitchfork bifurcation)**.** Consider the system

$$\dot{x} = \mu x - x^3$$

The equilibrium points are $x = 0$ and $x = \pm\mu^{1/2}$. For $\mu > 0$ there are three equilibrium points, while for $\mu < 0$ there is only one. The Jacobian $Df = \mu - 3x^2$. For the equilibrium value $x = 0$, $Df = \mu$. This is zero when $\mu = 0$ which is the bifurcation point. For $x = 0$, $\lambda < 0$ when $\mu < 0$ and $\lambda > 0$ when $\mu > 0$. Thus the equilibrium point $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$.

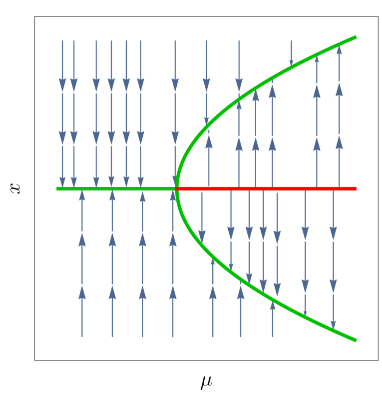
For $x = \pm\mu^{1/2}$, $\lambda = Df = -2\mu$. Since these branches only exist for $\mu > 0$ they have $\lambda < 0$ and are therefore stable.

Example 4.4 (Subcritical pitchfork bifurcation)**.** Consider the system

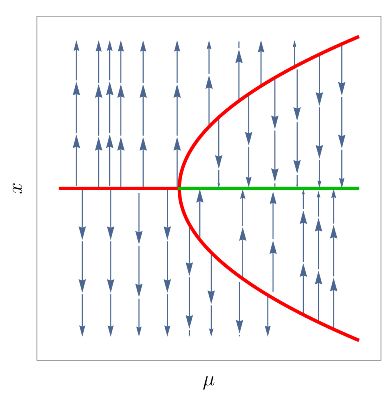
$$\dot{x} = -\mu x + x^3$$

The equilibrium points are $x = 0$ and $x = \pm\mu^{1/2}$. For $\mu > 0$ there are three equilibrium points, while for $\mu < 0$ there is only one. The Jacobian $Df = -\mu + 3x^2$. For the equilibrium value $x = 0$, $Df = -\mu$. This is zero when $\mu = 0$ which is the bifurcation point. For $x = 0$, $\lambda > 0$ when $\mu < 0$ and $\lambda < 0$ when $\mu > 0$. Thus the equilibrium point $x = 0$ is unstable for $\mu < 0$ and stable for $\mu > 0$.

For $x = \pm\mu^{1/2}$, $\lambda = Df = 2\mu$. Since these branches only exist for $\mu > 0$ they have $\lambda > 0$ and are therefore unstable.



supercritical pitchfork



subcritical pitchfork

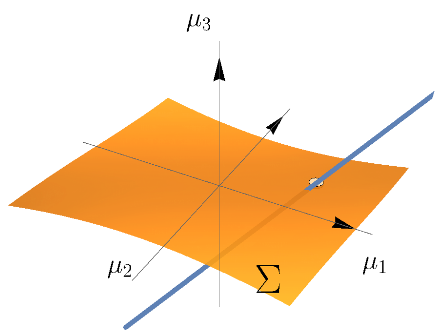
Let us first introduce the notion of the co-dimension of a bifurcation. As an example consider the case $m = 1$, $p = 3$:

$$\dot{x} = f(x, \boldsymbol{\mu}) \qquad \text{where} \quad x \in \mathbb{R}, \quad \boldsymbol{\mu} \in \mathbb{R}^3 \tag{4.3}$$

Let Σ be the bifurcation set, i.e.

$$\Sigma = \{\boldsymbol{\mu} \in \mathbb{R}^3 \mid D_x f(x) = 0 \text{ for some } x \text{ with } f(x, \boldsymbol{\mu}) = 0\}.$$

Since Σ is defined by one constraint on $\boldsymbol{\mu}$ generically it will be a two-dimensional manifold in three dimensional $\boldsymbol{\mu}$-space. A generic line in $\boldsymbol{\mu}$-space will therefore intersect Σ in a point.



Thus, following this line, we have a system

$$\dot{y} = g(y, \lambda), \qquad \lambda \in \mathbb{R} \tag{4.4}$$

which has the same bifurcation behaviour as (4.3) but a single parameter. We say the bifurcation has *co-dimension 1* and the equation (4.4) is its normal form.

Example 4.5 (Two parameter family)**.** Consider the system

$$\dot{x} = \mu_1 + \mu_2 x - x^2. \tag{4.5}$$

If we set $\mu_2 = 0$ we see there is a saddle-node bifurcation at $\mu_1 = 0$. If we set $\mu_1 = 0$ then we see there is a transcritical bifurcation at $\mu_2 = 0$. In general the fixed points are

$$x = \frac{\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1}}{2} \qquad \text{provided } \mu_2^2 + 4\mu_1 > 0.$$

There is a single (non-hyperbolic) fixedpoint when $\mu_2^2 + 4\mu_1 = 0$, and no fixed points if $\mu_2^2 + 4\mu_1 < 0$. Thus the bifurcation set is the parabola $\mu_2^2 + 4\mu_1 = 0$. Along a curve passing *through* any point on the parabola we will see a saddle-node bifurcation. To see a transcritical bifurcation we need to follow a curve in parameter space which is tangential to the parabola.

Example 4.6 (General co-dimension 1 problem in 1 dimension)**.** Consider the system

$$\dot{x} = f(x, \mu).$$

Suppose that $x = 0$ is an equilibrium point and that there is a bifurcation at $\mu = 0$. This implies

$$f(0,0) = f_x(0,0) = 0.$$

Thus, Taylor expanding f about $(0,0)$ gives

$$\dot{x} = \mu f_\mu + \frac{x^2}{2} f_{xx} + x\mu f_{x\mu} + \frac{\mu^2}{2} f_{\mu\mu} + \cdots$$

where all derivatives are evaluated at $(0,0)$. Thus, locally near $(0,0)$,

$$\dot{x} = \left(\mu f_\mu + O(\mu^2)\right) + x\mu f_{x\mu} + \frac{x^2}{2} f_{xx} + O(|(x,\mu)|^3).$$

Generically this is of the form (4.5) and can therefore be reduced to the standard saddle-node bifurcation. However, there are some important special cases:

1. If the system is such that $f(0,\mu) = 0$ then $\partial_\mu^k f(0,0) = 0$ for all k and we have instead

$$\dot{x} = x\mu f_{x\mu} + \frac{x^2}{2} f_{xx} + O(|(x,\mu)|^3),$$

which is in the standard form for a transcritical bifurcation.

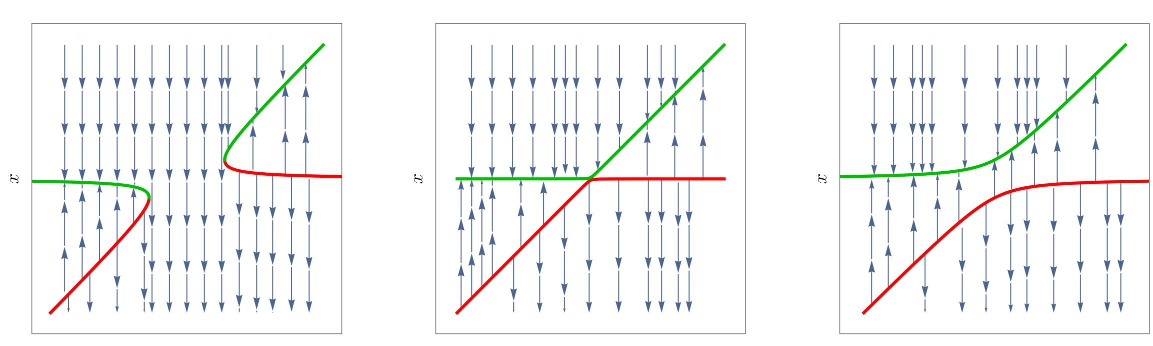
2. If the system has reflectional symmetry (i.e. the trajectories are invariant under the transformation $x \to -x$) then simple bifurcations are pitchforks. For the equations to have this symmetry f must be odd in x, and therefore the Taylor expansion instead gives

$$\dot{x} = x\left(\mu f_{x\mu} + O(\mu^2)\right) + x^3\left(\frac{1}{6} f_{xxx} + O(\mu)\right) + O(x^5),$$

which is the general form for a pitchfork bifurcation.

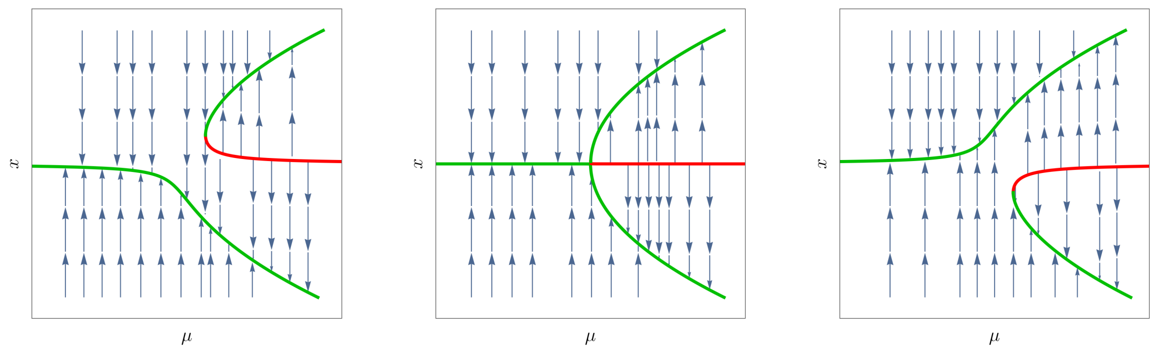
The saddle-node bifurcation is robust under small changes of parameters as shown above, but transcritical and pitchfork bifurcations depend on the vanishing of terms, and therefore change under small perturbations.

$$\dot{x} = \epsilon + \mu x - x^2$$



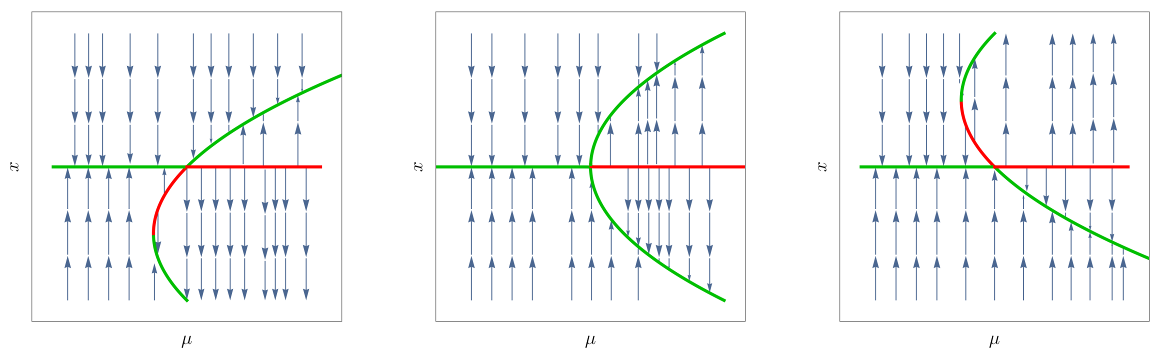
$\epsilon < 0$ $\epsilon = 0$ $\epsilon > 0$

$$\dot{x} = \epsilon + \mu x - x^3$$



$\epsilon < 0$ $\epsilon = 0$ $\epsilon > 0$

$$\dot{x} = \mu x + \epsilon x^2 - x^3$$



$\epsilon < 0$ $\epsilon = 0$ $\epsilon > 0$

Consider now the situation where $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ has a non-hyperbolic fixed point at the origin for some value μ_0 of μ. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $D_{\mathbf{x}}\mathbf{f}$ at the bifurcation point $(\mathbf{x}(\mu_0), \mu_0)$. There are two generic cases:

(i)

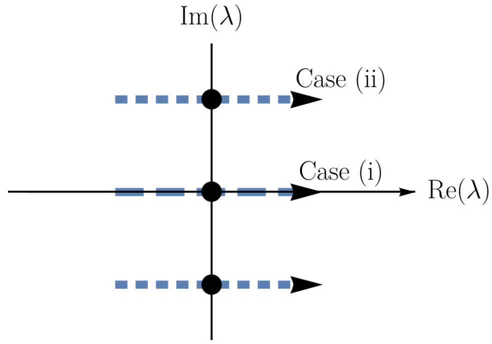
$$D_{\mathbf{x}}\mathbf{f} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right]$$

$\lambda_1 = 0$, $\mathrm{Re}(\lambda_j) \neq 0$, $j > 1$. This is a *simple* or *steady-state* bifurcation, and is essentially the same as the one dimensional examples we have seen. We will show this shortly when we discuss the extended centre manifold.

(ii)

$$D_{\mathbf{x}}\mathbf{f} = \left[\begin{array}{cc|c} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ \hline 0 & 0 & A \end{array}\right]$$

$\lambda_{1,2} = \pm i\omega$, $\mathrm{Re}(\lambda_j) \neq 0$, $j > 2$. This is a *Hopf* or *oscillatory* bifurcation, and it leads to the growth of oscillations. We will see an example later.



It is possible to have two or more zero eigenvalues:

$$D_{\mathbf{x}}\mathbf{f} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & A \end{array}\right] \qquad\qquad D_{\mathbf{x}}\mathbf{f} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & A \end{array}\right]$$

(double-zero bifurcation) (Takens-Bogdanov bifurcation)

but these are non seen generically as they need two parameters to take one special values. Note that there are extra technical requirements on the way the eigenvalues change with μ (e.g. for (i) we need $\mathrm{d}\lambda_1/\mathrm{d}\mu \neq 0$ at $\mu = \mu_0$).

4.2 The extended centre manifold

Consider the general one-parameter system (we are looking at co-dimension 1 bifurcations)

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}, \tilde{\mu}), \qquad \mathbf{z} \in \mathbb{R}^n, \quad \tilde{\mu} \in \mathbb{R} \tag{4.6}$$

Assume that for $\tilde{\mu} = \tilde{\mu}_c$, there is a non-hyperbolic fixed point $\mathbf{z}_c$ so that the matrix $M = D\mathbf{F}(\mathbf{z}_c)$ has eigenvalues with zero real part. We use the change of variables

$$\mathbf{z} = \mathbf{z}_c + C\tilde{\mathbf{z}}, \qquad \mu = \tilde{\mu} - \tilde{\mu}_c$$

where C is chosen such that

$$C^{-1}MC = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]. \tag{4.7}$$

where the matrices A and B of respective dimension n_c and $n_s + n_u$ are such that

$$\mathrm{Re}(\lambda) = 0 \;\; \forall\, \lambda \in \; \mathrm{Spec}(A), \qquad\qquad \mathrm{Re}(\lambda) \neq 0 \;\; \forall\, \lambda \in \; \mathrm{Spec}(B). \tag{4.8}$$

After the change of variables, the new system in the variable $\tilde{\mathbf{z}} = (\mathbf{x}, \mathbf{y})$ is

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}, \mu), \qquad && \mathbf{x} \in \mathbb{R}^{n_c} \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}, \mu) && \mathbf{y} \in \mathbb{R}^{n_s+n_u} \end{aligned} \tag{4.9}$$

and $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$ is a fixed point for $\mu = 0$. The **main idea** now is to extend the centre manifold to include the parameter.

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}, \mu), \qquad && \mathbf{x} \in \mathbb{R}^{n_c} \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}, \mu) && \mathbf{y} \in \mathbb{R}^{n_s+n_u} \\ \dot{\mu} &= 0 \end{aligned} \tag{4.10}$$

We can view this system as a dynamical system in the extended phase space in $m = n_s + n_u + n_c + 1$ dimensions.

Remark 4.5. When we first considered centre manifolds we were only interested in fixed points without unstable manifolds. This time the unstable manifold is not empty and the vector $\mathbf{y}$ denotes variables both in the stable and the unstable manifolds.

Remark 4.6. The centre manifold has now dimension $n_c + 1$ and is parameterised by the vector $(\mathbf{x}, \mu)$.

We can now proceed as before and look for a center manifold of the form

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mu). \tag{4.11}$$

Once this is known, we can write the dynamics on the extended centre manifold as:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}, \mu), \mu), \qquad \dot{\mu} = 0. \tag{4.12}$$

This equation captures the relevant part of the bifurcation. In the case (i) above $n_c = 1$ and equation (4.12) is one-dimensional as promised. In the case (ii) $n_c = 2$ and two-dimensional.

Example 4.7. Consider the system

$$\begin{aligned}\dot{x} &= \mu(x+y) - (x+y)^2 \\ \dot{y} &= -y - \mu(x+y) + (x+y)^2\end{aligned}$$

The fixed points are $(\mu, 0)$ and $(0,0)$. At the origin

$$M = \begin{bmatrix} \mu & \mu \\ -\mu & -1-\mu \end{bmatrix} \qquad \Rightarrow \qquad \lambda_1 = \frac{-1+\sqrt{1+4\mu}}{2}, \quad \lambda_2 = \frac{-1-\sqrt{1+4\mu}}{2}.$$

As expected (because the two fixed points coincide) $\lambda_1 = 0$ when $\mu = 0$. The eigenvector $\mathbf{w}_1$ when $(x, y, \mu) = (0,0,0)$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so that the system is already in the form (4.7) (phew!) if we write the generalised vector in the order (x, μ, y) (there is a second zero eigenvalue associated with $\dot{\mu} = 0$). Thus

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ \mu \\ y \end{bmatrix} + \begin{bmatrix} \mu(x+y) - (x+y)^2 \\ 0 \\ -\mu(x+y) + (x+y)^2 \end{bmatrix}$$

The centre manifold is given by $y = h(x, \mu)$. Writing $h = h_{20}x^2 + h_{11}x\mu + h_{02}\mu^2 + \cdots$ gives

$$Dh = [2h_{20}x + h_{11}\mu,\ h_{11}x + 2h_{02}\mu] + \cdots.$$

$$\begin{aligned} D\mathbf{h}(\mathbf{x})\left(A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))\right) &= [2h_{20}x + h_{11}\mu,\ h_{11}x + 2h_{02}\mu] \begin{bmatrix} \mu(x+y) - (x+y)^2 \\ 0 \end{bmatrix} + \cdots \\ &= (2h_{20}x + h_{11}\mu)\left(\mu(x+y) - (x+y)^2\right) + \cdots \end{aligned}$$

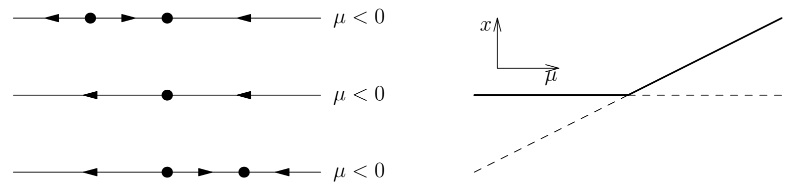


Figure 4.1: The dynamics on the centre manifold

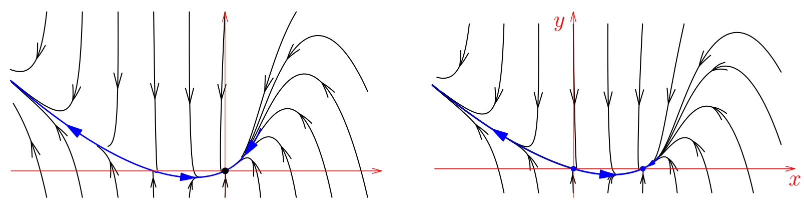


Figure 4.2: The dynamics in the full space (x, y) for $\mu < 0$ (left) and $\mu > 0$ (right).

$$\begin{aligned}
B\mathbf{h}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) &= -\left(h_{20}x^2 + h_{11}x\mu + h_{02}\mu^2\right) \\
&\quad - \mu(x + \left(h_{20}x^2 + h_{11}x\mu + h_{02}\mu^2 + \cdots\right)) + (x + \left(h_{20}x^2 + h_{11}x\mu + h_{02}\mu^2 + \cdots\right))^2 \\
&= -h_{20}x^2 - h_{11}x\mu - h_{02}\mu^2 - \mu x + x^2 + \cdots
\end{aligned}$$

Equating powers of x^2, $x\mu$ and μ^2 gives

$$h_{20} = 1, \qquad h_{11} = -1, \qquad h_{02} = 0.$$

Thus the cenre manifold is given locally by

$$y = x^2 - x\mu + O(x^3, x^2\mu, x\mu^2, \mu^3).$$

The dynamics on the centre manifold is then given by

$$\dot{x} = \mu x - x^2 + O(x\mu^2, x^2\mu, x^3, \mu^3).$$

Thus the bifurcation is transcritical.

Remark 4.7. Note that if the equilibrium point depends on the bifurcation parameter μ it may be necessary to include a term linear in the μ in the centre manifold:

$$h = h_{01}\mu + h_{20}x^2 + h_{11}x\mu + h_{02}\mu^2 + \cdots.$$

If you are not sure it is best to include this term. If it was not needed the coefficient h_{01} will turn out to be zero.

Example 4.8. Consider the system

$$\begin{aligned}
\dot{x} &= \mu x - xy \\
\dot{y} &= -y + x^2
\end{aligned} \tag{4.13}$$

Clearly $(x, y) = (0, 0)$ is an equilibrium point. At this point

$$D_{\mathbf{x}}\mathbf{f} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \qquad \Rightarrow \qquad \det(D_{\mathbf{x}}\mathbf{f}) = 0 \text{ when } \mu = 0.$$

Thus there is a bifurcation when $\mu = 0$. The eigenvalues and eigenvectors are

$$\lambda_1 = 0, \quad \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -1, \quad \boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus the centre subspace is $y = 0$. Expanding the extended centre manifold as

$$y = h_{01}\mu + h_{20}x^2 + h_{11}\mu x + h_{02}\mu^2 + \cdots$$

gives

$$\begin{aligned} \dot{y} &= -h + x^2 = -(h_{01}\mu + h_{20}x^2 + h_{11}\mu x + h_{02}\mu^2 + \cdots) + x^2 \\ &= h'(x)\dot{x} = (2h_{20}x + h_{11}\mu + \cdots)(\mu x - x(h_{01}\mu + h_{20}x^2 + h_{11}\mu x + h_{02}\mu^2 + \cdots)). \end{aligned}$$

Equating coefficients of powers of x and μ gives

$$h_{01} = 0, \qquad -h_{20} + 1 = 0, \qquad h_{11} = 0, \qquad h_{02} = 0.$$

Thus the centre manifold is

$$y = x^2 + \cdots$$

and the dynamics on the centre manifold is given by

$$\dot{x} = \mu x - x^3 + \cdots .$$

Thus the bifurcation is a pitchfork bifurcation.

5 Local analysis of maps

5.1 Mappings

We are interested in *iterative maps*, characterised by discrete iterations of the form

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n), \tag{5.1}$$

where $\mathbf{x} \in E \subseteq \mathbb{R}^m$. Equivalently, we write

$$\mathbf{x} \mapsto \mathbf{G}(\mathbf{x}), \tag{5.2}$$

We note that

$$\mathbf{x}_1 = \mathbf{G}(\mathbf{x}_0), \quad \mathbf{x}_2 = \mathbf{G}(\mathbf{x}_1) = \mathbf{G}^{(2)}(\mathbf{x}_0), \quad \dots, \quad \mathbf{x}_n = \mathbf{G}^{(n)}(\mathbf{x}_0). \tag{5.3}$$

where $\mathbf{G}^{(n)}(\mathbf{x}_0) = \mathbf{G}(\mathbf{G}(\dots \mathbf{G}(\mathbf{x}_0)))$. If $\mathbf{G}^{-1}$ exists then the orbits are unique (i.e. no two different starting points can give the same finishing point) and we can go either forwards or backwards. Otherwise, more generally, we can look at systems for which only forward dynamics is defined. A point $\mathbf{x}_0$ is a fixed point for the system if it is mapped onto itself:

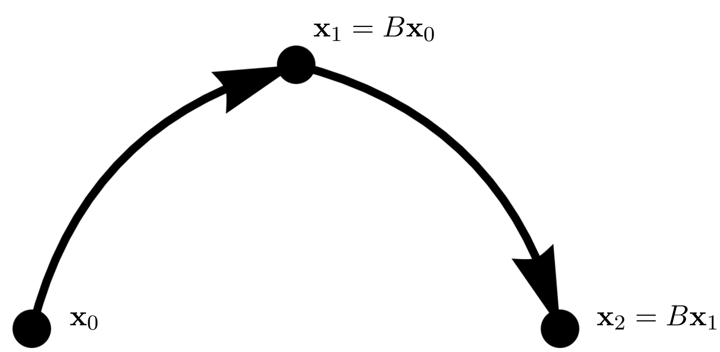
$$\mathbf{x}_0 = \mathbf{G}(\mathbf{x}_0). \tag{5.4}$$

5.1.1 Linear maps

In the linear case:

$$\mathbf{x}_{n+1} = B\mathbf{x}_n, \qquad B \in \mathcal{M}_n(\mathbb{R}), \quad n \in \mathbb{Z}^+, \quad \mathbf{x}_0 \in \mathbb{R}^n. \tag{5.5}$$

The map sends points to points. If $0 \notin \mathrm{Spec}(B)$, then B can be inverted and orbits are unique (i.e. no two different starting points can give the same finishing point).



The origin is always a fixed point. The stability of the origin is determined by the spectral properties of B. We write the eigenvalues and eigenvectors of B as $B\mathbf{w}_j = \lambda_j \mathbf{w}_j$ (for $j = 1, \dots, n$) where

$$\lambda_j = a_j + \mathrm{i}b_j, \quad a_j,\ b_j \in \mathbb{R}, \qquad \mathbf{w}_j = \mathbf{u}_j + \mathrm{i}\mathbf{v}_j, \quad \mathbf{u}_j,\ \mathbf{v}_j \in \mathbb{R}^n.$$

Definition 5.1. The *stable, unstable, centre linear subspaces* are defined respectively as

- $E^s = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j$ such that $|\lambda_j| < 1)$ (stable linear subspace)
- $E^c = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j$ such that $|\lambda_j| = 1)$ (centre linear subspace)
- $E^u = \mathrm{Span}(\mathbf{u}_j, \mathbf{v}_j$ such that $|\lambda_j| > 1)$ (unstable linear subspace)

The stable linear subspace defines contraction mappings: Let $\mathbf{x}_0 \in E^s$. Then there exists $\alpha < 1$, $c > 0$ such that

$$|\mathbf{x}_n| \leq c\alpha^n |\mathbf{x}_0| \tag{5.6}$$

There is a natural correspondence between linear flows and linear maps.
Every linear flow defines a linear map. Consider a linear flow with matrix A. Fix t and define $B = e^{tA}$, then

$$\varphi_t : \mathbf{x}_n \to B\mathbf{x}_n. \tag{5.7}$$

However, the converse is not true (can you give a counter-example?).

5.1.2 Stability of maps

A fixed point for a mapping is a point $\mathbf{x}_0 \in \mathbb{R}^m$, such that $\mathbf{G}(\mathbf{x}_0) = \mathbf{x}_0$.

Definition 5.2. A fixed point $\mathbf{x}_0 \in \mathbb{R}^n$ is *stable* if $\forall \epsilon > 0$, $\exists\, \delta > 0$ such that $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$, $\mathbf{G}^{(n)}(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0)$ for all $n \in \mathbb{Z}^+$.

Definition 5.3. A fixed point $\mathbf{x}_0 \in \mathbb{R}^m$ is *asymptotically stable* if it is stable and $\exists\, \delta > 0$ such that $\forall \mathbf{x} \in B_\delta(\mathbf{x}_0)$

$$\mathbf{G}^{(n)}(\mathbf{x}) \to \mathbf{x}_0 \text{ as } n \to \infty.$$

5.1.3 Stable and unstable manifolds

Theorem 5.4 (Stable and unstable manifold)**.** *For E be an open subset of $\mathbb{R}^m$ containing $\mathbf{x}_0$ consider the iterative map $\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n)$ with fixed point $\mathbf{x}_0$, where $\mathbf{G} : E \to E$ and $\mathbf{G}^{-1}$ exists on E. Suppose the linear stable subspace has dimension n_s and the linear unstable subspace has dimension n_u. Then*

- *there exists, in a neighbourhood of $\mathbf{x}_0$, an n_s-dimensional manifold $W^s_{loc}(\mathbf{x}_0)$ tangent to E^s such that $\mathbf{G}(W^s_{loc}) \subseteq W^s_{loc}$ and $\forall \mathbf{x} \in W^s_{loc}$, $\mathbf{G}^{(n)}(\mathbf{x}) \to \mathbf{x}_0$ as $n \to \infty$.*
- *there exists, in a neighbourhood of $\mathbf{x}_0$, a n_u-dimensional manifold $W^u_{loc}(\mathbf{x}_0)$ tangent to E^u such that $W^u_{loc} \subseteq \mathbf{G}(W^u_{loc})$ and $\forall \mathbf{x} \in W^u_{loc}$, $\mathbf{G}^{(n)}(\mathbf{x}) \to \mathbf{x}_0$ as $n \to -\infty$.*

Moreover, W^s_{loc} and W^u_{loc} are as smooth as $\mathbf{G}$.

By extension, we define the *stable and unstable manifold*:

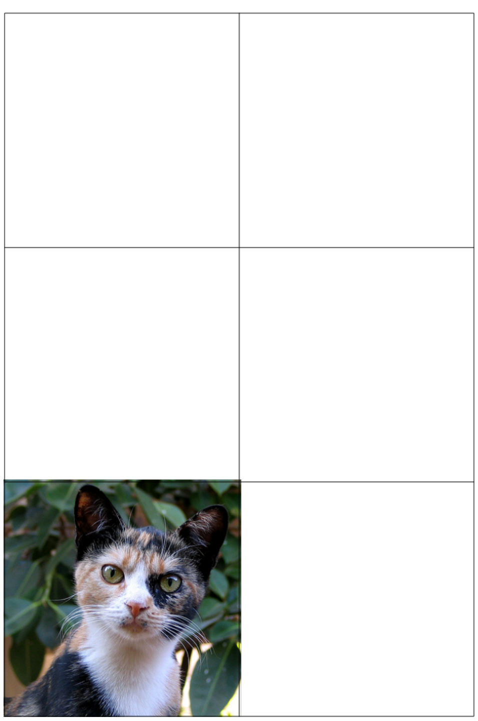
$$W^s(\mathbf{x}_0) = \bigcup_{n\leq 0} \mathbf{G}^{(n)}\left(W^s_{\mathrm{loc}}(\mathbf{x}_0)\right) \tag{5.8}$$

$$W^u(\mathbf{x}_0) = \bigcup_{n\geq 0} \mathbf{G}^{(n)}\left(W^u_{\mathrm{loc}}(\mathbf{x}_0)\right) \tag{5.9}$$

Remark 5.5. Stable and unstable manifolds are not trajectories but union of trajectories.

Example 5.1 (Cat map)**.** Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus (i.e. a point in T^2 is an equivalence class of points $(x, y) \in \mathbb{R}^2$ under the equivalence relation $(x, y) \sim (x+n, y+m)$ for n, $m \in \mathbb{Z}$). Any matrix B with integer entries and unit determinant (i.e. in $SL(2, \mathbb{Z})$) preserves equivalence classes in $\mathbb{R}^2$, so induces a map $B : T^2 \to T^2$. Consider the induced map given by

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$


unit square

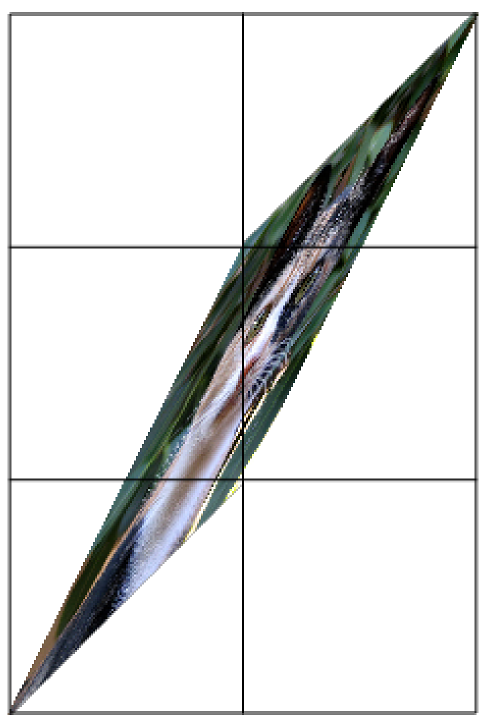
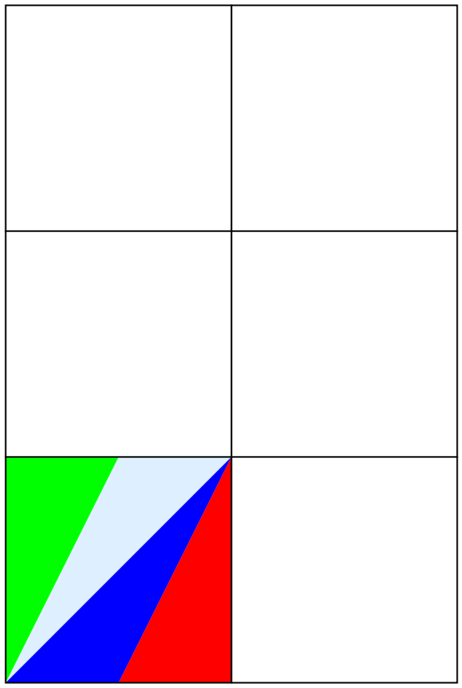
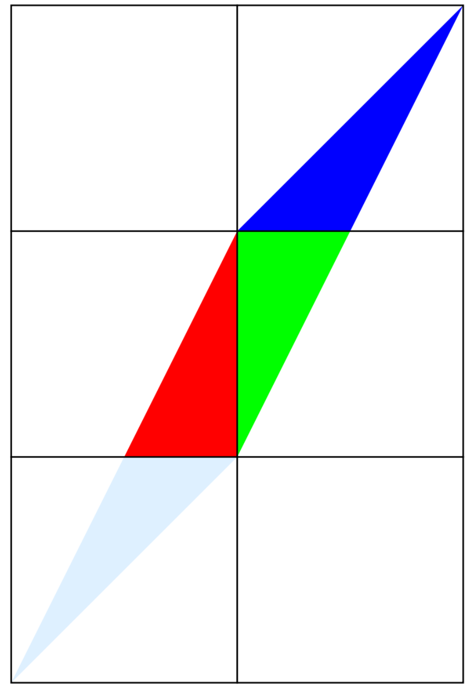
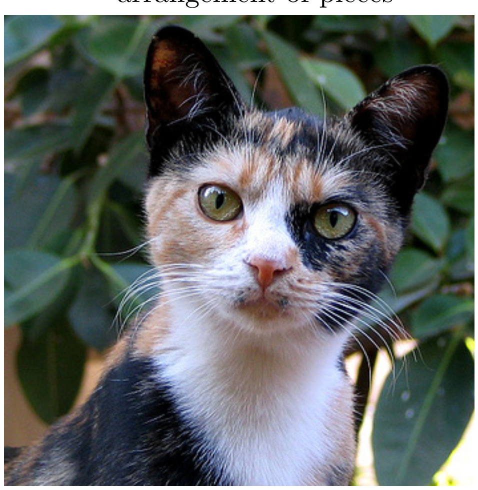


image under B

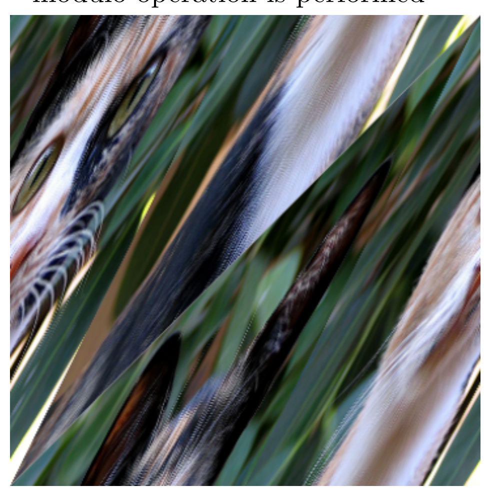




arrangement of pieces after modulo operation is performed



$\rightarrow$



Eigenvalues and eigenvectors:

$$\lambda_1 = \frac{1}{2}(3+\sqrt{5}) > 1 \qquad \lambda_2 = \frac{1}{2}(3-\sqrt{5}) < 1$$

$$\mathbf{w}_1 = \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Thus there is a one-dimensional stable manifold and a one-dimensional unstable manifold. The stable and unstable manifolds densely fill the torus (rational points are sent to rational points, while the slope eigenfunction is irrational, so the curve can never close).

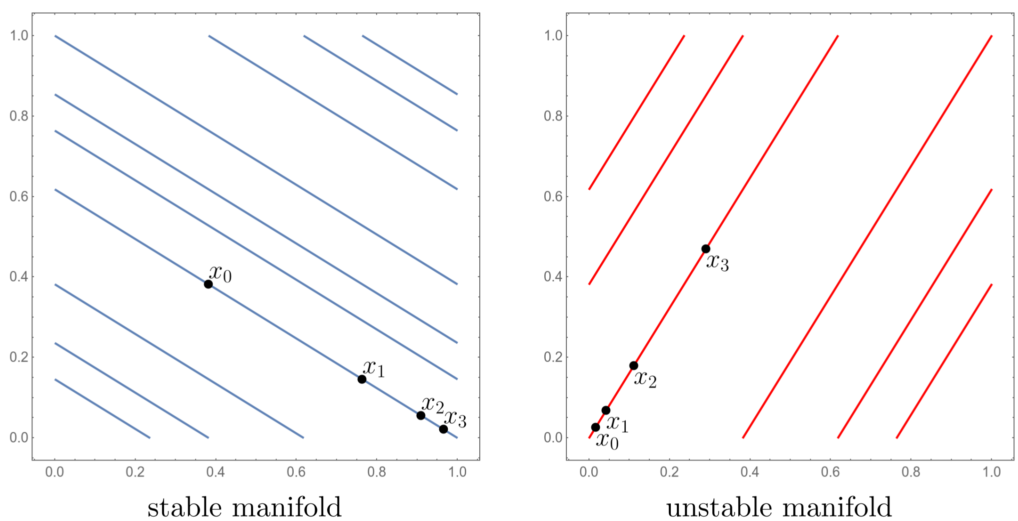
What happens to rational points?

$$\mathbf{x}_0 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

A periodic orbit of period 3.

number of orbits of period 2	:	4
number of orbits of period 3	:	15
number of orbits of period 4	:	44
number of orbits of period 5	:	120
number of orbits of period 6	:	319
number of orbits of period 7	:	840
...		

There are infinitely many periodic orbits. The set of such points also dense in T^2.



5.1.4 Stability of periodic orbits

For a continuous dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a periodic orbit Γ is a closed trajectory in phase space $E \subseteq \mathbb{R}^m$.

Let $d(\mathbf{x}, \Gamma)$ be the distance between a point $\mathbf{x}$ and Γ. Given a closed curve we can define a *neighbourhood of size* δ as the set of points

$$U_\delta(\Gamma) = \{\mathbf{x} \in E \mid d(\mathbf{x}, \Gamma) < \delta\} \tag{5.10}$$

Definition 5.6. A periodic orbit Γ is *Lyapunov stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\varphi_t(\mathbf{x}) \in U_\epsilon(\Gamma)$ for all $t \geq 0$ and $\mathbf{x} \in U_\delta$.

Definition 5.7. A periodic orbit Γ is *asymptotically stable* if it is Lyapunov stable and $\exists \delta > 0$ such that $d(\varphi_t(\mathbf{x}), \Gamma) \to 0$ as $t \to \infty$ for all $\mathbf{x} \in U_\delta$.

5.1.5 Poincaré map

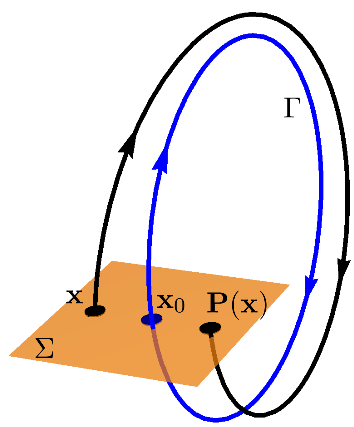
One way to study periodic orbits is via the so-called *Poincaré map*, which replaces the continuous flow by a lower-dimensional discrete map. The idea is quite simple: if Γ is a periodic orbit of the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{5.11}$$

through the point $\mathbf{x}_0$ and Σ is a hyperplane perpendicular to Γ at $\mathbf{x}_0$ then for any point $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$ the solution $\phi_t(\mathbf{x})$ of (5.11) through $\mathbf{x}$ at $t = 0$ will cross Σ again at a point $\mathbf{P}(\mathbf{x})$ near $\mathbf{x}_0$. The map

$$\mathbf{x} \mapsto \mathbf{P}(\mathbf{x}), \tag{5.12}$$

is called the Poincaré map.



The idea can be generalised to an $(m-1)$-dimensional manifold Σ [where $\mathbf{x} \in \mathbb{R}^m$] which does not need to be perpendicular to Γ but it must not be tangential, i.e. it must be *transversal*, so that

$$\mathbf{n} \cdot \mathbf{f}(\mathbf{x}_0) > 0.$$

Remark 5.8. This intersection point always exists because if $\mathbf{x}_0$ lies on a perioric orbit and the transversality condition is satisfied then it can be shown that

$$\exists \delta > 0 \text{ such that } \forall \mathbf{x} \in B_\delta(\mathbf{x}_0)\ \exists T(\mathbf{x}) \text{ such that } \varphi_{T(\mathbf{x})}(\mathbf{x}) \in \Sigma.$$

Example 5.2.

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} r &= \left[1 + \left(\frac{1}{r_0^2} - 1\right) \mathrm{e}^{-2t}\right]^{-1/2} \\ \theta &= t + \theta_0 \end{aligned}$$

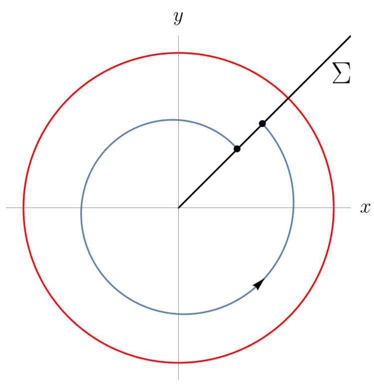
Choose Σ to be the line $\theta = \theta_0$. Then $\mathbf{p}_0 = (r_0, \theta_0) \in \Sigma$. Transversality is satisfied because $\mathbf{n} \cdot \mathbf{f} = 1$ for all r, θ (the normal is in the θ-direction, and $\dot{\theta} = 1$). We see that for any initial condition (r, θ_0) the next intersection occurs at $t = 2\pi$. Thus

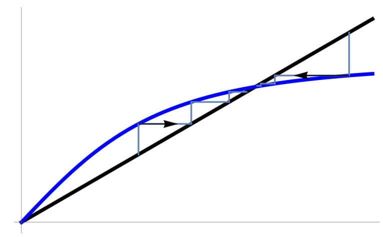
$$P(r) = \left[1 + \left(\frac{1}{r^2} - 1\right) \mathrm{e}^{-4\pi}\right]^{-1/2}.$$

There is a fixed point at $r = 1$ because $P(1) = 1$. This corresponds to the periodic orbit. We can study the stability of the periodic orbit by studying the stability of this fixed point. Since

$$P'(1) = \mathrm{e}^{-4\pi} < 1$$

the fixed point (and hence the periodic orbit) is stable.





6 Limit cycles and Hopf bifurcations

6.1 The Poincaré-Lindstedt method

The Poincaré-Lindstedt method is an asymptotic method to find the approximate behaviour of weakly nonlinear oscillations.

A canonical equation for oscillations of a conservative system is

$$\ddot{x} + x = \epsilon f(x), \tag{6.1}$$

where f is continuously differentiable with $f(0) = f'(0) = 0$ and ϵ is a parameter. When $\epsilon = 0$ equation (6.1) describes simple harmonic motion with period 2π (independent of the amplitude of the oscillation). When $0 < \epsilon \ll 1$ we anticipate that there are periodic solutions whose period depend on both ϵ and the amplitude of oscillation.

We might naïvely try an asymptotic expansion in powers of ϵ by expanding the solution as

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \qquad \text{as } \epsilon \to 0, \tag{6.2}$$

substituting this expansion into (6.1), equating coefficients of powers of ϵ and solving the resulting equations for x_0, x_1, etc. However, such an approach may give a solution which is not uniformly valid for all time as $\epsilon \to 0$. To see the difficulty we consider a simple linear example.

Example 6.1. Consider the equation

$$\ddot{x} + (1 + \epsilon)^2 x = 0, \qquad \text{with} \quad x(0) = 1, \quad \dot{x}(0) = 0. \tag{6.3}$$

Substituting the expansion (6.2) into (6.3) and equating coefficients of ϵ^0 gives

$$\ddot{x}_0 + x_0 = 0, \qquad \text{with} \quad x_0(0) = 1, \quad \dot{x}_0(0) = 0.$$

Thus

$$x_0(t) = \cos t.$$

Next, equating coefficients of ϵ, we find

$$\ddot{x}_1 + x_1 = -2x_0 = -2\cos t, \qquad \text{with} \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

Therefore

$$x_1(t) = -t \sin t.$$

Thus

$$x(t) = \cos t - \epsilon t \sin t + O(\epsilon^2) \text{ as } \epsilon \to 0. \tag{6.4}$$

We see that no matter how small ϵ is, $\epsilon x_1(t)$ is as large as $x_0(t)$ after along enough time (when $\epsilon t \approx 1$). So the expansion (6.4) may converge uniformly to the exact solution $x(t, \epsilon)$ on a given finite interval, but not if t belongs to an infinite interval.

In this simple linear example we can see exactly what is happening, because we can write down the exact solution explicitly:

$$x(t, \epsilon) = \cos((1 + \epsilon)t) = \cos(\epsilon t) \cos t - \sin(\epsilon t) \sin t.$$

Expanding $\cos \epsilon t$ and $\sin \epsilon t$ in a power series gives the solution (6.4). The small mismatch in frequency between the approximate solution and the exact solution means that eventually they get out of phase.

Poincaré recognised that the nonuniformity could be resolved by expanding the frequency of the periodic solution as well as the solution itself (and acknowledged Lindstedt's earlier use of the idea).

We introduce the method by means of an example.

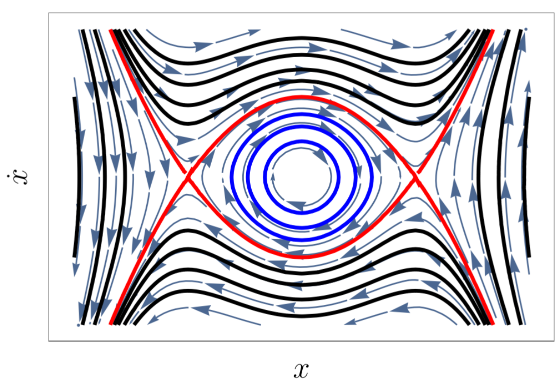
Example 6.2 (A nonlinear spring)**.** Consider the equation

$$\ddot{x} + x - \epsilon x^3 = 0. \tag{6.5}$$

The energy

$$E = \frac{\dot{x}^2}{2} + \frac{x^2}{2} - \frac{\epsilon x^4}{4}$$

is conserved during the motion. The point $x = \dot{x} = 0$ is a centre, while the points $\dot{x} = 0$, $x = \pm 1/\sqrt{\epsilon}$ are saddle points.



Motions of small enough amplitude correspond to closed curves, and so are periodic. We now describe these orbits using the Poincaré-Lindstedt method.

Define ω (which may depend on ϵ) to be the unknown frequency of a solution, so that the period is $2\pi/\omega$. Then define a new timescale $\tau = \omega t$. Then, in terms of τ, without approximation, we have

$$\omega^2 \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + x - \epsilon x^3 = 0. \tag{6.6}$$

Because of our definition of τ, the chosen periodic solution must have period 2π, i.e.

$$x(\tau + 2\pi, \epsilon) = x(\tau, \epsilon) \quad \text{for all } \tau. \tag{6.7}$$

By translation of time we may assume

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = 0 \qquad \text{at } t = 0. \tag{6.8}$$

We define the amplitude of the oscillation to be

$$a = x(0, \epsilon). \tag{6.9}$$

We now expand both the solution **and** the frequency ω in powers of ϵ as

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots, \tag{6.10}$$

$$x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots \qquad \text{as } \epsilon \to 0, \tag{6.11}$$

Substituting the expansions (6.10)-(6.11) into (6.6)-(6.9) and equating coefficients of ϵ^0 gives

$$\omega_0^2 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + x_0 = 0, \tag{6.12}$$

with

$$x_0(\tau + 2\pi) = x_0(\tau), \qquad x_0(0) = a, \qquad \frac{\mathrm{d}x_0}{\mathrm{d}\tau}(0) = 0. \tag{6.13}$$

The general solution of (6.12) is

$$x_0 = A_0 \cos(\tau/\omega_0) + B_0 \sin(\tau/\omega_0),$$

for some constants A_0 and B_0. Conditions (6.13) then give

$$\omega_0 = 1, \qquad A_0 = a, \qquad B_0 = 0.$$

Equating coefficients of ϵ we find that

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + x_1 = -2\omega_1 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + x_0^3 = \left(2\omega_1 a + \frac{3a^3}{4}\right) \cos\tau + \frac{a^3}{4} \cos 3\tau, \tag{6.14}$$

with

$$x_1(\tau + 2\pi) = x_1(\tau), \qquad x_1(0) = 0, \qquad \frac{\mathrm{d}x_1}{\mathrm{d}\tau}(0) = 0, \tag{6.15}$$

where we have used the identity

$$\cos^3\tau = \frac{3\cos\tau + \cos 3\tau}{4}.$$

If the term proportional to $\cos\tau$ on the right-hand side of (6.14) did not vanish, then there would be a secular term

$$\frac{1}{2}\left(2\omega_1 a + \frac{3a^3}{4}\right) \tau \sin\tau$$

in the particular integral of x_1, so x_1 could not satisfy the periodicity condition. Thus we must have

$$\omega_1 = -\frac{3a^2}{8}.$$

Then

$$x_1 = A_1 \cos\tau + B_1 \sin\tau - \frac{a^3}{32}\cos 3\tau,$$

for some constants A_1 and B_1. The initial conditions give

$$A_1 = \frac{a^3}{32}, \qquad B_1 = 0.$$

This gives the uniformly valid approximation

$$\begin{aligned} x(t,\epsilon) &= a\cos\omega t + \frac{\epsilon a^3}{32}(\cos\omega t - \cos 3\omega t) + O(\epsilon^2 a^5), \\ \omega &= 1 - \frac{3\epsilon a^2}{8} + O(\epsilon^2 a^4) \qquad \text{as } \epsilon \to 0. \end{aligned}$$

We may continue and find higher-order approximations in a similar way, giving ω_2, x_2, etc., but already we have found the leading-order correction to simple harmoic motion. We see that the period of oscillation does indeed depend on both ϵ and a.

The vanishing of the coefficient of $\cos\tau$ on the right-hand side of (6.14) is an example of the Fredholm Alternative. The (self-adjoint) homogeneous periodic problem has the non-trivial solutions $\cos\tau$ and $\sin\tau$, so there will be a solution to the non-homogeneous problem if and only if the right-hand side is orthogonal to both these functions. This can be seen directly by multiplying (6.14) by $\cos\tau$ and integrating over $(0, 2\pi)$ to give, after integrating twice by parts

$$\begin{aligned} \int_0^{2\pi}\left(\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + x_1\right)\cos\tau\,\mathrm{d}\tau &= \int_0^{2\pi}(-x_1 + x_1)\cos\tau\,\mathrm{d}\tau + \left[\frac{\mathrm{d}x_1}{\mathrm{d}\tau}\cos\tau - x_1\sin\tau\right]_0^{2\pi} = 0 \\ &= \int_0^{2\pi}\left(2\omega_1 a + \frac{3a^3}{4}\right)\cos^2\tau + \frac{a^3}{4}\cos 3\tau\,\cos\tau\,\mathrm{d}\tau \\ &= \frac{1}{2}\left(2\omega_1 a + \frac{3a^3}{4}\right). \end{aligned}$$

Remark 6.1. Instead of translating time to fix $\mathrm{d}x/\mathrm{d}\tau(0) = 0$, $x(0) = a$, we could just write the leading order solution as

$$x_0 = a\cos(\tau + \Phi).$$

Note that this "amplitude-phase" representation is almost always better (i.e. the algebra is easier) than the equivalent "cos-sin" representation

$$x_0 = A\cos\tau + B\sin\tau.$$

6.2 The Hopf bifurcation

There is another generic bifurcation with one parameter. It happens when the eigenvalues at the bifurcation are imaginary. Recall that we can bring a system to its canonical form

$$\begin{aligned} &\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}, \mu), \qquad \mathbf{x} \in \mathbb{R}^{n_c} \\ &\dot{\mathbf{y}} = B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}, \mu), \qquad \mathbf{y} \in \mathbb{R}^{n_s + n_u} \\ &\dot{\mu} = 0 \end{aligned} \tag{6.16}$$

We have studied the case where A is of dimension 1 and vanishes at the bifurcation. Next, we study the case where

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}. \tag{6.17}$$

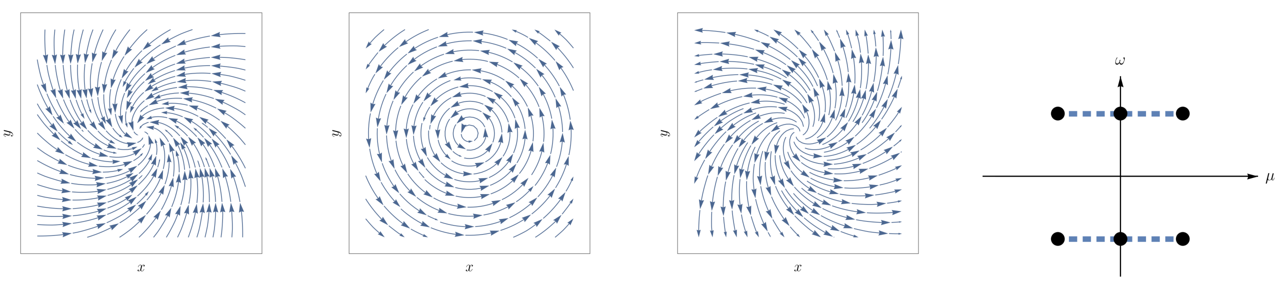
On the center manifold, at the bifurcation, the dynamics of the linear part (with $\mathbf{x} = (x, y)$) is

$$\begin{aligned} \dot{x} &= -\omega y \\ \dot{y} &= \omega x. \end{aligned} \tag{6.18}$$

To obtain the behaviour of the system close to the bifurcation (*unfolding*), we consider the generic perturbation around the linear system: Close to the bifurcation, the system *unfolds* to

$$\begin{aligned} \dot{x} &= \mu x - \omega y + f(x, y, \mu) \\ \dot{y} &= \omega x + \mu y + g(x, y, \mu). \end{aligned} \tag{6.19}$$

$$\det \begin{bmatrix} \mu - \lambda & -\omega \\ \omega & \mu - \lambda \end{bmatrix} = (\lambda - \mu)^2 + \omega^2 = 0 \qquad \Rightarrow \qquad \lambda = \mu \pm \mathrm{i}\omega.$$



$\mu < 0$ $\mu = 0$ $\mu > 0$

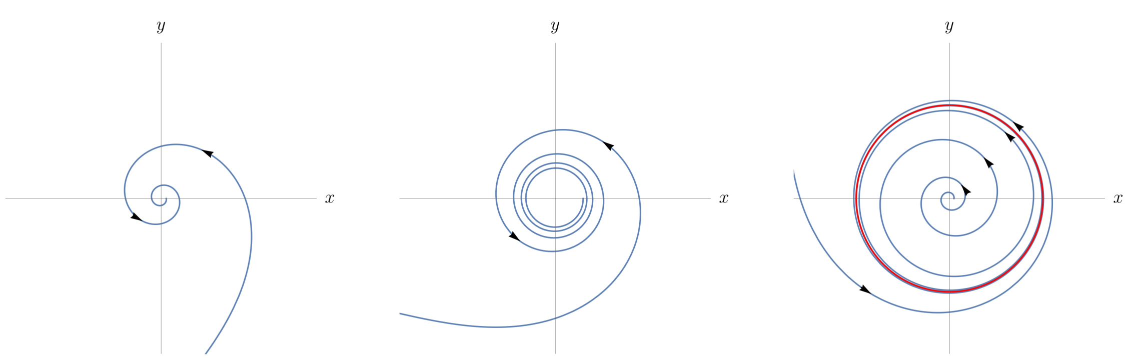
Example 6.3. Consider the typical example of a Hopf bifurcation

$$\begin{aligned} \dot{x} &= \mu x - \omega y - x(x^2 + y^2) \\ \dot{y} &= \omega x + \mu y - y(x^2 + y^2). \end{aligned} \tag{6.20}$$

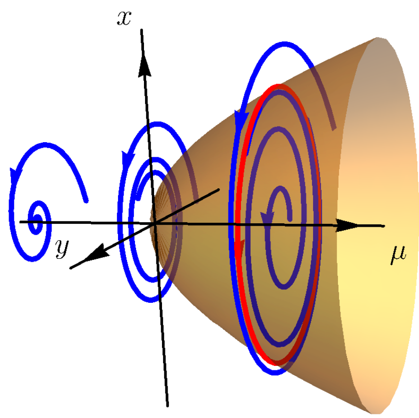
In polar coordinates, it reads

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega. \end{aligned} \tag{6.21}$$

Compare to (**??**).



$\mu < 0$ $\mu = 0$ $\mu > 0$



In polar coordinates, the general form of a Hopf bifurcation is

$$\begin{aligned}\dot{r} &= d\mu r + ar^3 \\ \dot{\theta} &= \omega + c\mu + br^2,\end{aligned} \tag{6.22}$$

where a, b, c, d are parameters that depend on the vector field at the bifurcation. The parameters c and d can be found from a linear analysis: if $\lambda(\mu)$ is the eigenvalue such that $\lambda(0) = \mathrm{i}\omega$, then

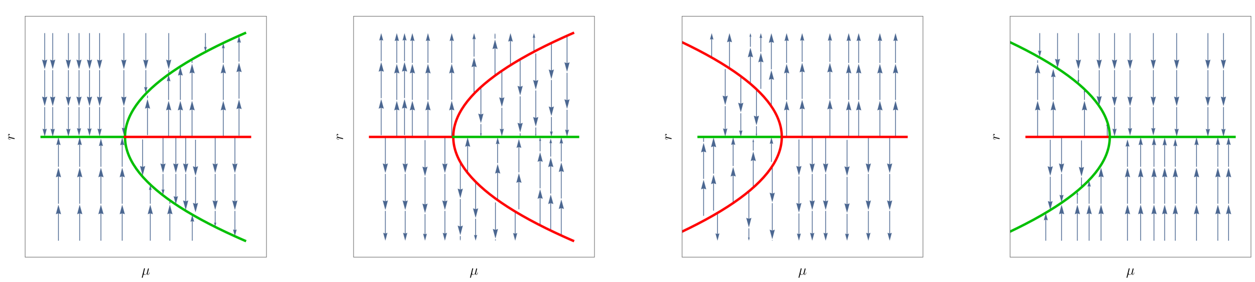
$$d = \frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{Re}(\lambda(\mu)), \qquad c = \frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{Im}(\lambda(\mu)).$$

A Hopf bifurcation is a bifurcation from a fixed point to a limit cycle. For $d\mu/a < 0$ the radius of the limit cycle is

$$r = \sqrt{-\frac{d\mu}{a}}$$

and the period is

$$T = \frac{2\pi}{\omega + c\mu + br^2} = \frac{2\pi}{\omega + c\mu - bd\mu/a} \approx \frac{2\pi}{\omega}\left(1 + \frac{\mu}{\omega a}(ac + bd)\right).$$



$d > 0,\ a < 0$ $\quad$ $d < 0,\ a > 0$ $\quad$ $d > 0,\ a > 0$ $\quad$ $d < 0,\ a < 0$

Theorem 6.2 (Hopf '42)**.** *Let*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu), \qquad \mathbf{x} \in \mathbb{R}^n, \quad \mu \in \mathbb{R},$$

and with $\mathbf{f} \in C^4(E \times J)$ *for open subsets* $E \subseteq \mathbb{R}^n$ *and* $J \subseteq R$. *Let* $(\mathbf{x}_0, \mu_0) \in E \times J$ *be such that* $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0, \mu_0)$ *has a single pair of eigenvalues* $\pm \mathrm{i}\omega$ $(\omega \in \mathbb{R})$ *and no other eigenvalues with zero real part. Then there exists a smooth curve of equilibria* $(\mathbf{x}(\mu), \mu)$ *with* $\mathbf{x}(\mu_0) = \mathbf{x}_0$. *The eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$ *of* $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}(\mu), \mu)$ *vary smoothly with* μ *and are such that* $\lambda(\mu_0) = \mathrm{i}\omega$, $\bar{\lambda}(\mu_0) = -\mathrm{i}\omega$.

If, moreover,

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \operatorname{Re}(\lambda(\mu))\big|_{\mu=\mu_0} = d \neq 0,$$

then there exists a unique two-plus-one dimenensional centre manifold W^c *in* $E \times J$ *passing through* $(\mathbf{x}_0, \mu)$ *and a smooth change of coordinates such that on the centre manifold the system is transformed to the normal form*

$$\begin{aligned}
\dot{x} &= (d\tilde{\mu} + a(x^2 + y^2))x - (\omega + c\tilde{\mu} + b(x^2 + y^2))y, \\
\dot{y} &= (d\tilde{\mu} + a(x^2 + y^2))y + (\omega + c\tilde{\mu} + b(x^2 + y^2))x,
\end{aligned}$$

in a neighbourhood of the origin, where $\tilde{\mu} = \mu - \mu_0$.

If $a \neq 0$ then W^c is a paraboloid at $(\mathbf{x}_0, \mu_0)$ and for $d > 0$

$$\begin{aligned}
a < 0 &\Rightarrow \text{ stable limit cycle for } \mu > \mu_0, \\
a > 0 &\Rightarrow \text{ unstable limit cycle for } \mu < \mu_0.
\end{aligned}$$

while for $d < 0$

$$\begin{aligned}
a < 0 &\Rightarrow \text{ stable limit cycle for } \mu < \mu_0, \\
a > 0 &\Rightarrow \text{ unstable limit cycle for } \mu > \mu_0.
\end{aligned}$$

6.2.1 Normal Form Transformations for Hopf Bifurcations

Consider the 2D system

$$\dot{x} = f(x, y; \mu), \qquad \dot{y} = g(x, y; \mu).$$

Suppose the linearisation at a fixed point (x_0, y_0) shows that a pair of complex eigenvalues cross the imaginary axis $\operatorname{Re}(\lambda) = 0$ at a bifurcation point $\mu = \mu_0$. In order to put the system into the normal form for a Hopf bifurcation, i.e.

$$\begin{aligned}
\dot{r} &= \mu r + ar^3 + O(r^5), \\
\dot{\theta} &= \omega + br^2 + O(r^4),
\end{aligned}$$

or, equivalently

$$\dot{z} = (\mu + \mathrm{i}\omega)z + (a + \mathrm{i}b)|z|^2 z + O(z^5)$$

it is in general necessary to do the following (see Glendinning pp. 227-243):

1. Shift coordinates by writing $(\tilde{x}, \tilde{y}) = (x - x_0(\mu), y - y_0(\mu))$ and $\tilde{\mu} = \mu - \mu_0$ so that the fixed point is at the origin $\tilde{x} = \tilde{y} = 0$ for all $\tilde{\mu}$ and the bifurcation is at $\tilde{\mu} = 0$.

2. .If necessary, rescale $\tilde{\mu}$ so that the eigenvalues are $\lambda = \mu \pm \mathrm{i}\omega$. Make a linear change of basis so that the Jacobian is in the Jordan Normal Form

$$\begin{bmatrix} \operatorname{Re}(\lambda) & -\operatorname{Im}(\lambda) \\ \operatorname{Im}(\lambda) & \operatorname{Re}(\lambda) \end{bmatrix}.$$

3. Drop the tildes and write (x, y) as the single complex-valued variable $z = x + \mathrm{i}y$. Set $\mu = 0$. Then, Taylor expanding the right-hand sides we see that the ODEs take the form
$$\dot{z} = \mathrm{i}\omega z + a_1 z^2 + a_2 z\bar{z} + a_3 \bar{z}^2 + O(z^3).$$
It turns out that **all** these quadratic terms can be removed by making a suitable choice of the coefficients α_i in a near-identity change of coordinates $z = w + \alpha_1 w^2 + \alpha_2 w\bar{w} + \alpha_3 \bar{w}^2$. [Note: the algebra can be done by differentiating the inverse $w = z - \alpha_1 z^2 - \alpha_2 z\bar{z} - \alpha_3 \bar{z}^2 + O(z^3)$ and then substituting for $\dot{z}$ and $\dot{\bar{z}}$.]

4. Now we may attempt to eliminate all the cubic terms in $\dot{w} = \mathrm{i}\omega w + b_1 w^3 + b_2 w^2\bar{w} + b_3 w\bar{w}^2 + b_4 \bar{w}^3 + O(w^4)$ by a suitable choice of the coefficients β_i in another near-identity map
$$w = Z + \beta_1 Z^3 + \beta_2 Z^2 \bar{Z} + \beta_3 Z\bar{Z}^2 + \beta_4 \bar{Z}^3.$$
It turns out that the $b_2 w^2 \bar{w}$ term cannot be eliminated!

5. Continuing with more near-identity transformations, it is possible to eliminate all the quartic terms to show that the next term remaining in the normal form is quintic.

The chief point of these steps is to find the sign of a in the normal form, and hence decide whether the bifurcation is subcritical or supercritical. If steps 1 and 2 have already been done so that the system is in the form
$$\begin{aligned} \dot{x} &= \mu x - \omega y + f(x, y), \\ \dot{y} &= \omega x + \mu y + g(x, y) \end{aligned}$$
then
$$a = \frac{1}{16\omega} \left((f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy})\omega + f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy} \right).$$

Example 6.4.
$$\begin{aligned} \dot{x} &= (1 + \mu)x - 4y + x^2 - 2xy, \\ \dot{y} &= 2x - 4\mu y - y^2 - x^2. \end{aligned}$$
For the equilibrium point at the origin we have
$$D\mathbf{f}(0, 0) = \begin{bmatrix} 1 + \mu & -4 \\ 2 & -4\mu \end{bmatrix}. \tag{6.23}$$
There is a bifurcation at $\mu = 1/3$ with matrix
$$D = \begin{bmatrix} 4/3 & -4 \\ 2 & -4/3 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - \frac{16}{9} + 8 = 0 \qquad \lambda = \pm \mathrm{i}\frac{2\sqrt{14}}{3}, \qquad \mathbf{v} = \begin{bmatrix} 2 \pm \mathrm{i}\sqrt{14} \\ 3 \end{bmatrix}.$$

1. Compute d. First find λ for general μ:
$$\lambda = \frac{1}{2}\left(1 - 3\mu \pm \sqrt{-31 + 10\mu + 25\mu^2}\right) \qquad \Rightarrow \qquad d = \mathrm{Re}(\partial_\mu \lambda)\big|_{\mu=1/3} = -\frac{3}{2} \neq 0.$$

2. Change axes. Let

$$P = \begin{bmatrix} 2 & -\sqrt{14} \\ 3 & 0 \end{bmatrix} \qquad \Rightarrow \qquad P^{-1}\,D\,P = \begin{bmatrix} 0 & -\frac{2\sqrt{14}}{3} \\ \frac{2\sqrt{14}}{3} & 0 \end{bmatrix}.$$

Then set

$$\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \Rightarrow \qquad \begin{aligned} \dot{u} &= -\frac{2\sqrt{14}}{3} v + f(u,v), \\ \dot{v} &= \frac{2\sqrt{14}}{3} u + g(u,v), \end{aligned}$$

where

$$\begin{aligned} \begin{bmatrix} f \\ g \end{bmatrix} &= P^{-1} \begin{bmatrix} x^2 - 2xy \\ -y^2 - x^2 \end{bmatrix} = \frac{1}{3\sqrt{14}} \begin{bmatrix} 0 & \sqrt{14} \\ -3 & 2 \end{bmatrix} \begin{bmatrix} (2u - \sqrt{14}v)^2 - 2(2u - \sqrt{14}v)(3u) \\ -9u^2 - (2u - \sqrt{14}v)^2 \end{bmatrix} \\ &= \frac{1}{3\sqrt{14}} \begin{bmatrix} \sqrt{14}(-13u^2 + 4\sqrt{14}\,uv - 14v^2) \\ -3(-8u^2 + 2\sqrt{14}uv + 14v^2) + 2(-13u^2 + 4\sqrt{14}\,uv - 14v^2) \end{bmatrix} \\ &= \frac{1}{3\sqrt{14}} \begin{bmatrix} \sqrt{14}(-13u^2 + 4\sqrt{14}\,uv - 14v^2) \\ -2u^2 + 2\sqrt{14}uv - 70v^2 \end{bmatrix} \end{aligned}$$

3. Use the formula for a:

$$\begin{aligned} a &= \frac{1}{16\omega}\left((f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv})\omega + f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\right) \\ &= \frac{3}{32\sqrt{14}} \frac{1}{9 \times 14} \Big(4 \times 14(-26\sqrt{14} - 28\sqrt{14}) - 2\sqrt{14}(-4 - 140) - 26\sqrt{14} \times 4 + 28\sqrt{14} \times 140\Big) \\ &= \frac{45}{56} \end{aligned}$$

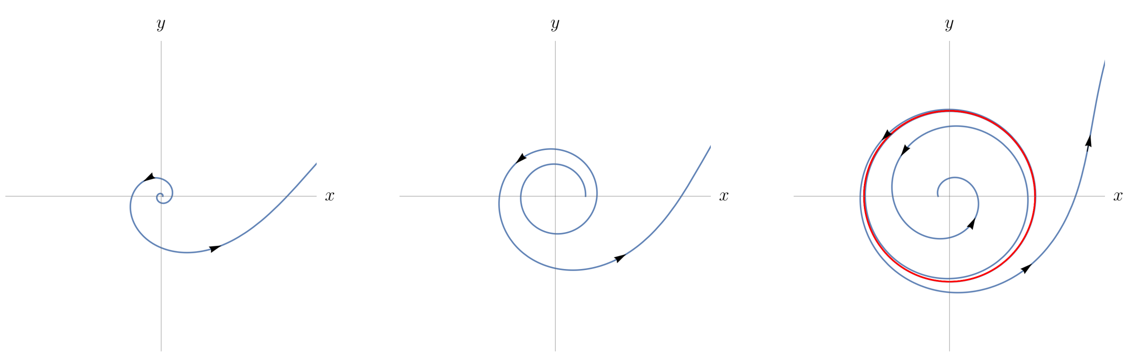
Thus

$$a = \frac{45}{56} > 0, \qquad d = -\frac{3}{2} < 0, \qquad \omega = \frac{2\sqrt{14}}{3} > 0,$$

and the normal form is

$$\dot{r} = d\tilde{\mu}r + ar^3, \qquad \tilde{\mu} = \mu - 1/3.$$

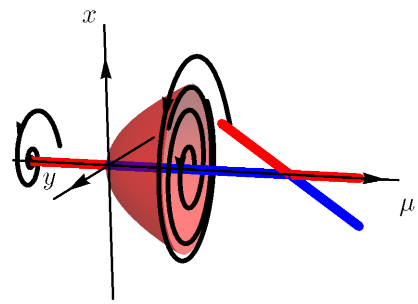
$\omega > 0 \Rightarrow$ anticlockwise rotation



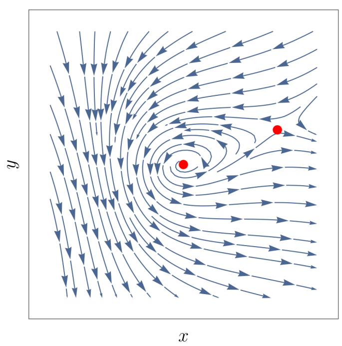
$\tilde{\mu} < 0$, $d\tilde{\mu} > 0$
$\Rightarrow r = 0$ unstable

$\tilde{\mu} = 0$, $\dot{r} = ar^3$, $a > 0$
$\Rightarrow r = 0$ unstable

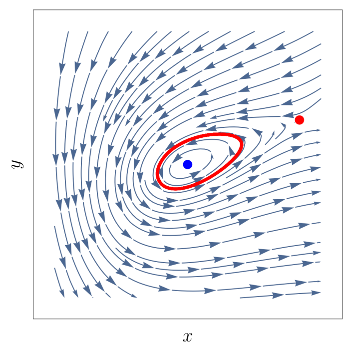
$\tilde{\mu} > 0$
$r = 0$ stable
$r = \sqrt{-\mu d/a}$ unstable



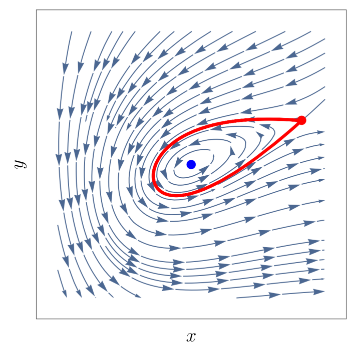
We have seen there is a Hopf bifurcation at $\mu = 1/3$ and there is also a transcritical bifurcation at $\mu = 1$.



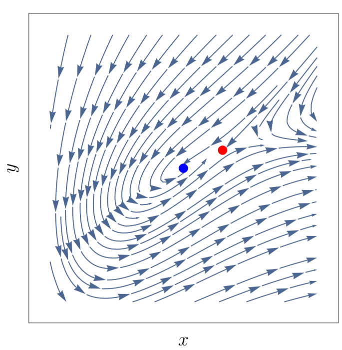
$\mu = 0$



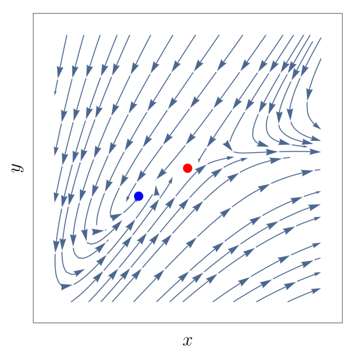
$\mu = 0.34$



$\mu \approx 0.351$
homoclinic connection



$\mu = 0.8$



$\mu = 1.2$

We can study bifurcation of periodic orbits by studying bifurcations of the Poincaré map.

6.3 Local bifurcation of maps

Consider the mapping

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n), \tag{6.24}$$

where $\mathbf{x} \in E \subseteq \mathbb{R}^m$. Assume that $\mathbf{x}_0$ is a fixed point (i.e. $\mathbf{G}(\mathbf{x}_0) = \mathbf{x}_0$).
This fixed point is asymptotically stable if $|\lambda| < 1$ for all $\lambda \in \mathrm{Spec}(D\mathbf{G}(\mathbf{x}_0))$.
The fixed point is unstable if there $\exists\ \lambda \in \mathrm{Spec}(D\mathbf{G}(\mathbf{x}_0))$ s.t. $|\lambda| > 1$.
So bifurcation will occur when an eigenvalue is on the unit complex circle.

We consider a mapping with one parameter μ

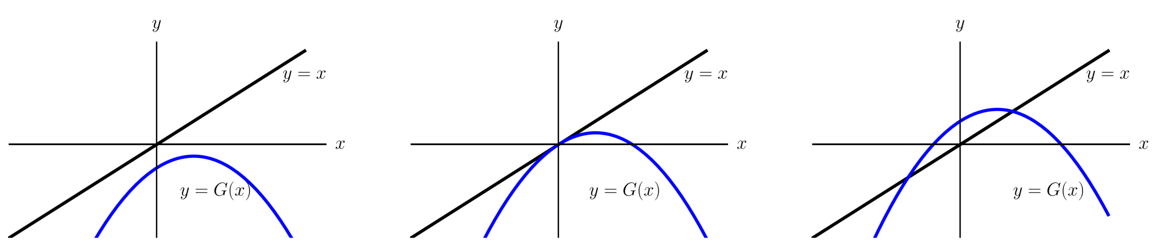
$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n, \mu), \tag{6.25}$$

where $\mathbf{x} \in E \subseteq \mathbb{R}^m$. Assume that $\mathbf{x}_0 = \mathbf{x}_0(\mu)$ is a fixed point. We are interested in the case where one of the eigenvalues crosses the unit disk. This gives three possibilities at the bifurcation: either (I) $\lambda = 1$, (II) $\lambda = -1$ or (III) $\lambda \neq \bar{\lambda}$ with $|\lambda| = 1$.

6.3.1 Case I: $\lambda = 1$ at bifurcation

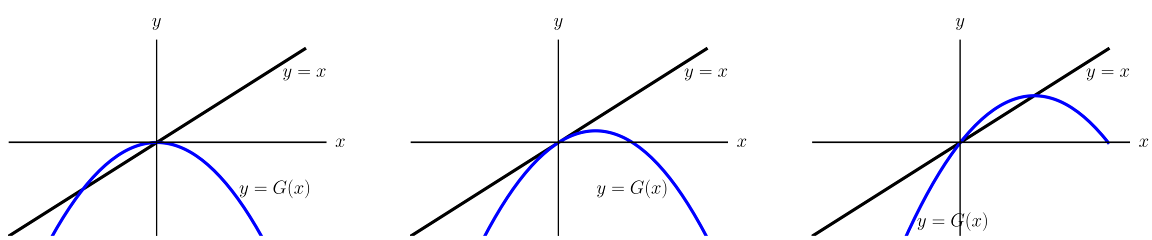
This case is similar to the cases obtained for vector fields, namely we have locally

$x \mapsto x + \mu - x^2$	*saddle-node bifurcation*
$x \mapsto x + \mu x - x^2$	*transcritical bifurcation*
$x \mapsto x + \mu x - x^3$	*pitchfork bifurcation.*



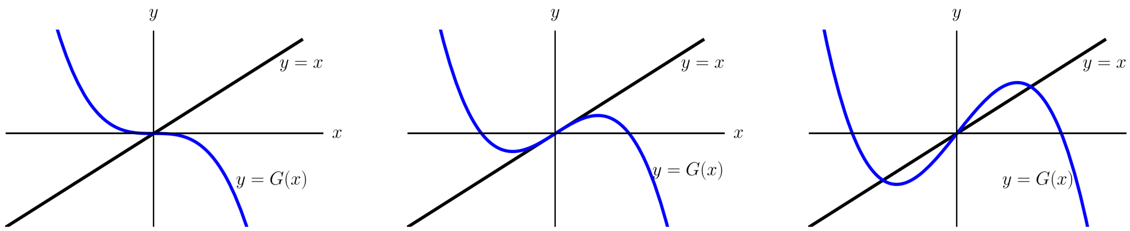
$\mu = -1$ $\mu = 0$ $\mu = 1$

Saddle-node bifurcation



$\mu = -1$ $\mu = 0$ $\mu = 1$

Transcritical bifurcation



$\mu = -1$ $\mu = 0$ $\mu = 1$

Pitchfork bifurcation

Example 6.5. Consider

$$x_{n+1} = \mu \sin x_n.$$

There is a fixed point at $x = 0$, with $D\mathbf{G}(0) = \mu \cos 0 = \mu$. For a one-dimensional map the eigenvalue is just $\lambda = D\mathbf{G}(0) = \mu$, so that there is a bifurcation when $\mu = 1$. Setting $\mu = 1 + \tilde{\mu}$ and Taylor expanding we find that locally

$$x_{n+1} = (1 + \tilde{\mu}) \sin x_n = x_n + \tilde{\mu} x_n - \frac{x_n^3}{6} + \cdots .$$

We see that the bifurcation is a pitchfork bifurcation. The non-zero fixed points are present when $\tilde{\mu} > 0$. The fixed point $x = 0$ is stable for $\tilde{\mu} < 0$ (because $\lambda = \mu < 1$) and unstable for $\tilde{\mu} > 0$. Thus we expect that the non-trivial fixed points are stable, and a check of the eigenvalue shows that this is indeed the case, since linearising about $x = (6\tilde{\mu})^{1/2}$ gives

$$\lambda = 1 + \tilde{\mu} - \frac{x^2}{2} = 1 + \tilde{\mu} - 3\tilde{\mu} = 1 - 2\tilde{\mu} < 1 \qquad \text{when} \qquad \tilde{\mu} > 0.$$

Thus we have a supercritical pitchfork bifurcation.

Example 6.6. Consider

$$\begin{aligned} x_{n+1} &= \mu x_n + x_n y_n \\ y_{n+1} &= \frac{y_n}{2} - x_n^2 \end{aligned}$$

There is a fixed point at $(x, y) = (0, 0)$, with

$$D\mathbf{G}(0,0) = \begin{bmatrix} \mu & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = \mu, \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = \frac{1}{2}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

There is a bifurcation at $\mu = 1$ with

$$E_c = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \qquad E_s = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle.$$

Let $\mu = 1 + \tilde{\mu}$. The centre manifold locally is

$$y = h(x) = ax^2 + bx\tilde{\mu} + c\tilde{\mu}^2 + \cdots .$$

Then

$$y_{n+1} = \frac{y_n}{2} - x_n^2 = \frac{h(x_n)}{2} - x_n^2 = \frac{ax_n^2 + bx_n\tilde{\mu} + c\tilde{\mu}^2}{2} - x_n^2 + \cdots .$$

But also

$$y_{n+1} = h(x_{n+1}) = h\left((1+\tilde{\mu})x_n + x_n y_n\right) = ax_n^2 + bx_n\tilde{\mu} + c\tilde{\mu}^2 + \cdots .$$

[Note that we cannot use the chain rule for discrete maps, but we can still Taylor expand, observing $(1+\tilde{\mu})x_n + x_n y_n = x_n + \cdots$]. Equating coefficients of x_n^2, $\tilde{\mu}x_n$ and $\tilde{\mu}^2$ gives

$$a = \frac{a}{2} - 1, \quad b = \frac{b}{2}, \quad c = \frac{c}{2}, \qquad \Rightarrow \qquad a = -2, \quad b = 0, \quad c = 0.$$

the equation for x now gives, on the centre manifold,

$$x_{n+1} = x_n + \tilde{\mu}x_n + x_n h(x_n) = x_n + \tilde{\mu}x_n - 2x_n^3 + \cdots .$$

Again we have a supercritical pitchfork bifurcation.

Remark 6.3. If the problem is simple enough, a direct analysis of the steady states can help identify the bifurcation. In this case the steady states satisfy

$$x = \mu x + xy, \qquad y = \frac{y}{2} - x^2.$$

Thus

$$y = -2x^2, \qquad x(\mu - 1 - 2x^2) = 0 \qquad \Rightarrow \qquad x = 0, \quad x = \pm\left(\frac{\mu-1}{2}\right)^{1/2}.$$

We see the emergence of two new steady states (in addition to $x = 0$) when $\mu > 1$, which is the hallmark of a pitchfork bifurcation.

Example 6.7 (Bifurcation of periodic orbit)**.** Consider

$$\begin{aligned} \dot{x} &= -y - x(\mu - (x^2+y^2-1)^2) \\ \dot{y} &= x - y(\mu - (x^2+y^2-1)^2) \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \dot{r} &= -r\left(\mu - (r^2-1)^2\right) \\ \dot{\theta} &= 1 \end{aligned}$$

We see $(0,0)$ is a fixed point, and there are periodic orbits at

$$(r^2-1)^2 = \mu \qquad \Rightarrow \qquad r = r_\pm = \sqrt{1 \pm \mu^{1/2}}, \qquad \mu \geq 0.$$

Define

$$\Gamma = \left\{(r,\theta) \in \mathbb{R} \times S^1 \mid \theta = 0\right\}$$

and consider the Poincaré map from $P : \Gamma \mapsto \Gamma$,

$$r \mapsto P(r,\mu).$$

The bifurcation point is $\mu = 0$ with the fixed point (of the map—i.e. a periodic orbit really) $r = 1$. Then expanding locally

$$\begin{aligned} 1+\delta \mapsto P(1+\delta,\mu) \quad &\sim \quad P(1,0) + \delta P_r(1,0) + \frac{\delta^2}{2}P_{rr}(1,0) + \mu P_\mu(1,0) + \cdots \\ &= \quad 1 + \delta + \frac{\delta^2}{2}P_{rr}(1,0) + \mu P_\mu(1,0) + \cdots \end{aligned}$$

since $P(1,0)=1$ ($r=1$ is the fixed point when $\mu=0$) and $P_r(1,0)=1$ (since $P_r(1,0)=\lambda=1$ at the bifurcation). Thus, locally,

$$\delta \mapsto \delta + \frac{\delta^2}{2}P_{rr}(1,0) + \mu P_\mu(1,0)$$

and we expect a saddle-node bifurcation if these terms are all non-zero. How do we find P_{rr} and P_μ? For a system in which

$$\dot{r} = f(r,\theta,\mu),$$

has a fixed point $f(1,\theta,0)=0$ we are interested in orbits close to $r=1$ for values of μ close to zero. Suppose we start from $r=1+\delta$. Let $r(t)=r_0+R(t)$ with $R(0)=\delta$. Then

$$\begin{aligned}\dot{R} &= f(1,\theta,0) + f_r(1,\theta,0)R + \frac{R^2}{2}f_{rr}(1,\theta,0) + \mu f_\mu(1,\theta,0) + \cdots \\ &= \frac{R^2}{2}f_{rr}(1,\theta,0) + \mu f_\mu(1,\theta,0) + \cdots \end{aligned} \tag{6.26}$$

where $f_r(1,\theta,0)=0$ because we are at the bifurcation point. The Poincaré map takes $R(0)=\delta$ to $R(2\pi)$. We are interested in $P_{\delta\delta}(0,0)$ and $P_\mu(0,0)$. We could just solve the nonlinear ode (6.26), evaluate at $t=2\pi$ and then differentiate the answer. But it is much easier to differentiate and then solve. Thus consider $R=R(t,\delta,\mu)$. Differentiating (6.26) with respect to μ and evaluating at $\mu=0$, $\delta=0$ gives

$$\dot{R}_\mu = RR_\mu f_{rr}(1,\theta,0) + f_\mu(1,\theta,0) = f_\mu(1,\theta,0), \qquad R_\mu(0) = \frac{\partial\delta}{\partial\mu} = 0.$$

since $R(t,0,0)=0$ (the solution to (6.26) with $\mu=0$ and $R(0)=0$ is just $R(t)\equiv 0$). Thus

$$P_\mu(1,0) = R_\mu(2\pi,0,0) = \int_0^{2\pi} f_\mu(1,\theta,0)\,\mathrm{d}t.$$

Similarly

$$\begin{aligned}\dot{R}_\delta &= RR_\delta f_{rr}(1,\theta,0) = 0, & R_\delta(0) &= 1,\\ \dot{R}_{\delta\delta} &= RR_{\delta\delta} f_{rr}(1,\theta,0) + R_\delta^2 f_{rr}(1,\theta,0) = R_\delta^2 f_{rr}(1,\theta,0), & R_{\delta\delta}(0) &= 0,\end{aligned}$$

Thus

$$R_\delta \equiv 1, \qquad R_{\delta\delta} = \int_0^t f_{rr}(1,\theta,0)\,\mathrm{d}t.$$

Thus

$$P_r(1,0) = 1, \text{(as expected)} \qquad P_{rr}(1,0) = R_{\delta\delta}(2\pi) = \int_0^{2\pi} f_{rr}(1,\theta,0)\,\mathrm{d}t.$$

Thus locally the map is

$$\delta \mapsto \delta + \frac{\delta^2}{2}\int_0^{2\pi} f_{rr}(1,\theta,0)\,\mathrm{d}t + \mu\int_0^{2\pi} f_\mu(1,\theta,0)\,\mathrm{d}t.$$

In our case $f_{rr}(1,0)=8$, $f_\mu(1,0)=-1$ and the local form is

$$\delta \mapsto \delta + 8\pi\delta^2 - 2\pi\mu,$$

and we see a saddle-node bifurcation with the emergence of two fixed points for $\mu > 0$.

We can check the stability of the bifurcating branches by computing the eigenvalues along them. For the system

$$\dot{r} = f(r), \qquad \dot{\theta} = 1$$

with fixed point $f(r_0) = 0$ we consider orbits close to $r = r_0$. Suppose we start from $r = r_0 + \delta$. Set $r(t) = r_0 + R(t)$ as before:

$$\dot{R} = f'(r_0)R + O(\delta^2) \qquad \Rightarrow \qquad R(t) = \delta \mathrm{e}^{f'(r_0)t}.$$

Note that since we are not at a bifurcation $f'(r_0) \neq 0$. Thus, after $t \to t + 2\pi$, $\delta \to \delta \mathrm{e}^{2\pi f'(r_0)}$. Therefore

$$\lambda = \mathrm{e}^{2\pi f'(r_0)}.$$

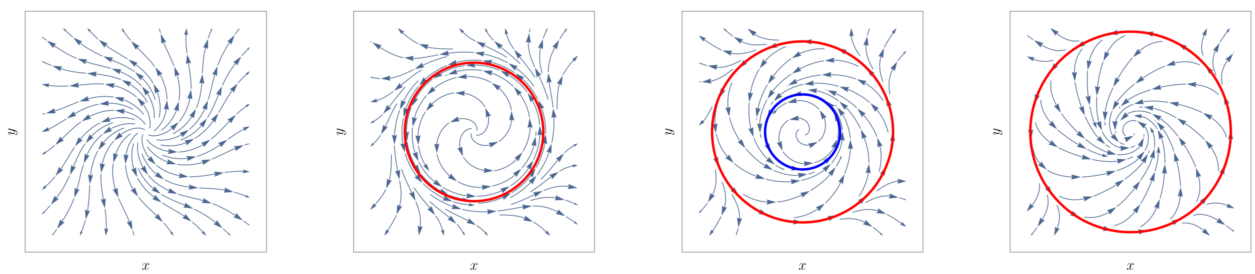
In our case

$$f'(r_\pm) = -(\mu - (r_\pm^2 - 1)^2) + r_\pm 2(r_\pm^2 - 1)2r_\pm = 4r_\pm^2(r_\pm^2 - 1) = \pm 4\mu^{1/2}(1 \pm \mu^{1/2}).$$

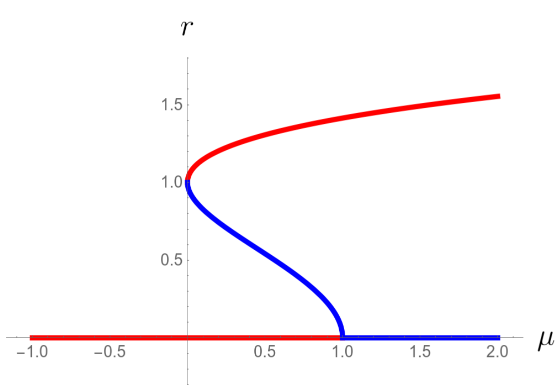
Thus

$$\lambda_\pm = \mathrm{e}^{\pm 8\pi\mu^{1/2}(1 \pm \mu^{1/2})}.$$

At the bifurcation point $\mu = 0$, $\lambda_\pm = 1$, while for $\mu > 0$, $\lambda_+ > 1$ and $\lambda_- < 1$, so that r_+ is unstable and r_- is stable.



$\tilde{\mu} < 0$ $\qquad$ $\tilde{\mu} = 0$ $\qquad$ $0 < \tilde{\mu} < 1$ $\qquad$ $\tilde{\mu} > 1$



Example 6.8 (Slightly less trivial bifurcation of periodic orbit)**.** For the previous example we used the machinery of the Poincaré map, but we could have just used the equation for r. To illustrate the power of the Poincaré map more fully let us consider the example

$$\begin{aligned} \dot{x} &= -y - x(\mu(1+x) - (x^2+y^2-1)^2) \\ \dot{y} &= x - y(\mu(1+x) - (x^2+y^2-1)^2) \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \dot{r} &= -r\left(\mu(1 + r\cos\theta) - (r^2-1)^2\right) \\ \dot{\theta} &= 1 \end{aligned}$$

Now the equations for r and θ do not decouple. However, we see that there is still a periodic orbit $r = 1$ (twice) when $\mu = 0$. The fact that $r = 1$ is a double root of the right-hand side of the r-equation when $\mu = 0$ leads us to believe this will still be a bifurcation point. As before define

$$\Gamma = \left\{(r, \theta) \in \mathbb{R} \times S^1 \mid \theta = 0\right\}$$

and consider the Poincaré map from $P : \Gamma \mapsto \Gamma$,

$$r \mapsto P(r, \mu).$$

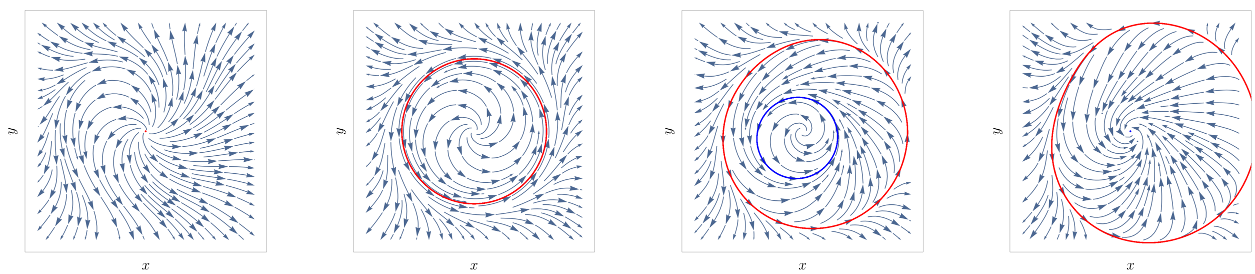
The bifurcation point is $\mu = 0$ and the fixed point is $r = 1$. This time

$$f_{rr}(1, \theta, 0) = 8, \qquad f_\mu(1, \theta, 0) = -(1 + \cos t),$$

so that the local form is in fact the same as before, namely,

$$\delta \mapsto \delta + 8\pi\delta^2 - 2\pi\mu$$

and we see exactly the same saddle-node bifurcation with the emergence of two fixed points for $\mu > 0$.



$\tilde{\mu} < 0$ $\quad$ $\tilde{\mu} = 0$ $\quad$ $0 < \tilde{\mu} < 1$ $\quad$ $\tilde{\mu} > 1$

6.3.2 Case II: $\lambda = -1$ at bifurcation. Period doubling

Consider

$$x \mapsto f(x, \mu) = -x - \mu x + x^3 \tag{6.27}$$

Then

$$f(0, \mu) = 0, \quad f_x(0, \mu) = -1 - \mu.$$

There is a pitchfork bifurcation $\lambda = 1$ at $\mu = -2$. At $\mu = 0$ there is a period doubling bifurcation with $\lambda = -1$. Fixed points are given by

$$x = -x - \mu x + x^3 \quad \Rightarrow \qquad x = 0, \quad x = \pm\sqrt{2 + \mu}.$$

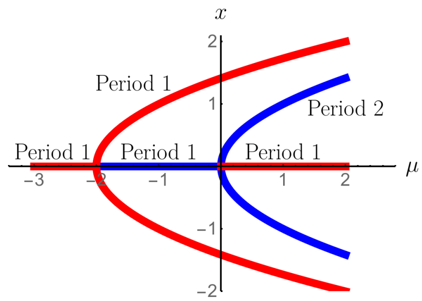
Consider applying the map twice:

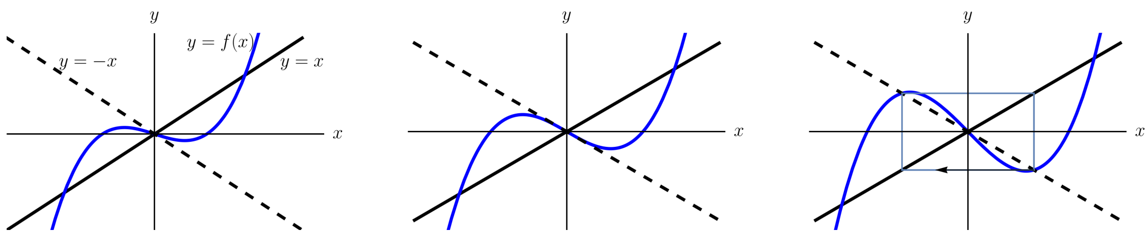
$$x \mapsto f(f(x, \mu), \mu) = f^{(2)}(x, \mu).$$

Then

$$\begin{aligned} f^{(2)} &= -(1 + \mu)(-(1 + \mu)x + x^3) + (-(1 + \mu)x + x^3)^3 \sim (1 + \mu)^2 x - ((1 + \mu) + (1 + \mu)^3)x^3 + O(x^4) \\ &\sim x + 2\mu x - 2x^3 + \cdots \end{aligned}$$

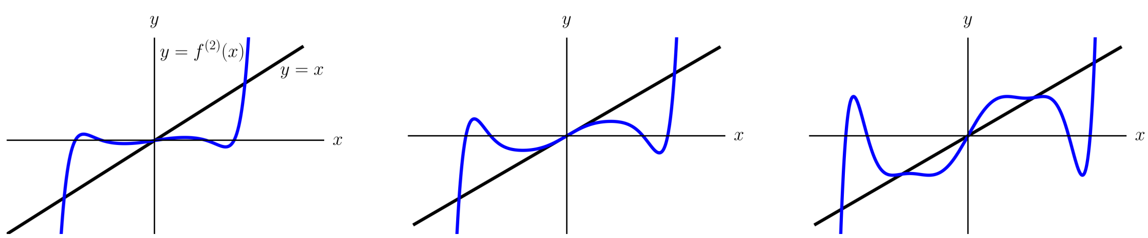
We see that $\mu = 0$ is a pitchfork bifurcation for $f^{(2)}$, with two additional fixed points $x = \pm\sqrt{\mu}$ which exist for $\mu > 0$. These are not fixed points of f, but periodic orbits of period 2.





$\mu = -0.5$ $\mu = 0$ $\mu = 0.75$

The map f



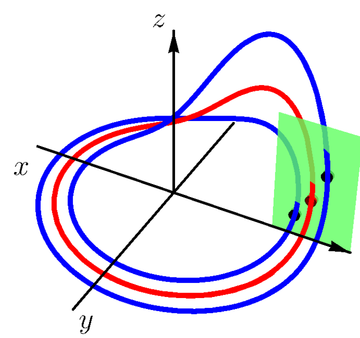
$\mu = -0.5$ $\mu = 0$ $\mu = 0.75$

The map $f^{(2)}$

When we have a period doubling bifurcation in the Poincaré map of a limit cycle, topologically the trajectories lie on a Möbius strip. The example below is from the Rössler system

$$
\begin{aligned}
\dot{x} &= -y - z, \\
\dot{y} &= x + by, \\
\dot{z} &= b + z(x - a)
\end{aligned}
$$

with $a = 3.1$ and $b = 0.2$.



An important example of period-doubling cascade leading to chaotic dynamics is the *logistic map*

$$x \mapsto \mu x(1-x) \tag{6.28}$$

7 Global bifurcations, Homoclinic chaos, Melnikov's method

7.1 A paradigm

Consider the Duffing oscillator

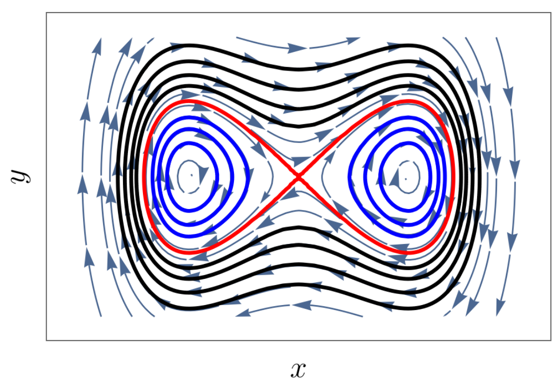
$$\ddot{x} = x - x^3 - \delta\dot{x} + \gamma\cos(t) \tag{7.1}$$

For $\delta = \gamma = 0$ there are homoclinic orbits. When $\delta > 0$ these homoclinic connections are broken. When $\gamma > 0$, $\delta > 0$ we get chaos. For what values? What does chaos mean?

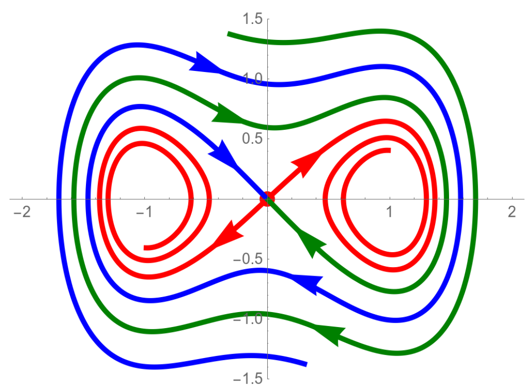
Similar questions arise for any conservative system

$$\ddot{x} = -\frac{\partial V}{\partial x}$$

in which the force may be written as the gradient of a potential with multiple wells.



$\delta = \gamma = 0$



$\gamma = 0$, $\delta > 0$

7.2 The problem

Consider the first-order system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon\mathbf{g}(\mathbf{x}, t) \qquad \text{where} \quad \mathbf{x} \in E \subseteq \mathbb{R}^n, \tag{7.2}$$

and assume that $\mathbf{g}$ is periodic in t ($\exists\, T > 0$ such that $\mathbf{g}(\mathbf{x}, t + T) = \mathbf{g}(\mathbf{x}, t)$).

Assuming we know the dynamics of the system when $\varepsilon = 0$ and that it supports periodic and homoclinic orbits:

- What happens when $\varepsilon > 0$?
- Are there still periodic orbits?
- Are there homoclinic orbits?
- Are there new orbits?

Main idea Our construction will be in four steps of increasing complexity

Step 1: Bernoulli shift (the simplest dynamical system with chaos)

Step 2: Smale's horseshoe (a geometric construction)

Step 3: Homoclinic chaos in ODEs

Step 4: Melnikov's method (an explicit method to detect chaos)

7.3 Step 1: Bernoulli Shift

To define a dynamical system we need:
- A *phase space* Σ.
- The *dynamics* on Σ (how elements of Σ are mapped to other elements).

1. The phase space.

For the Bernoulli shift we define Σ as the set of bi-infinite sequence of 0 and 1:

$$s \in \Sigma : \qquad s = \{\ldots, s_{-n}, \ldots, s_{-1} | s_0, s_1, \ldots, s_n, \ldots\}, \tag{7.3}$$

where s_i is equal to 0 or 1 for $i \in \mathbb{Z}$. To define *distance* on Σ, take two elements s, $s' \in \Sigma$ and define

$$d(s, s') = \sum_{i \in \mathbb{Z}} \frac{|s_i - s'_i|}{2^{|i|}} \tag{7.4}$$

Two elements are close if their central blocks agree.

2. The dynamics on Σ

Define the shift map $\sigma : \Sigma \mapsto \Sigma$ as follows. If

$$s = \{\ldots, s_{-n}, \ldots, s_{-2}, s_{-1} | s_0, s_1, \ldots, s_n, \ldots\},$$

then

$$\sigma(s) = \{\ldots, s_{-n}, \ldots, s_{-1}, s_0 | s_1, \ldots, s_n, \ldots\}.$$

Equivalently

$$(\sigma(s))_i = s_{i+1}. \tag{7.5}$$

Question: What are the orbits of σ on Σ?

Theorem 7.1. *The shift map has:*

1. *a countable infinity of periodic orbits, and periodic orbits of arbitrary period;*
2. *an uncountable infinity of non-periodic orbits;*

3. *a dense orbit.*

Definition 7.2. A *dense orbit* for the shift map is a particular orbit $s_d \in \Sigma$ such that for any $s \in \Sigma$ and $\epsilon > 0$, $\exists n \in \mathbb{N}$ such that $d(\sigma^n(s_d), s) < \epsilon$.

Proof.

1. All periodic sequences are periodic orbits:

$$\{\ldots, 1010|1010\ldots\}$$
$$\{\ldots, 100100|100100\ldots\}$$

   Clearly the period can have arbitrary length.

2. We can map and $s \in \Sigma$ to

$$S = 0.s_0 s_1 s_{-1} s_2 s_{-2} \cdots .$$

   This is the binary coding of a real number $S \in [0,1]$ Since the irrational numbers form an uncountable set, so do the non-periodic orbits in Σ.

3. To create a dense orbit we must find $s_d \in \Sigma$ such that for any $s \in \Sigma$, $\epsilon > 0$ $\exists n \in \mathbb{N}$ such that $d(\sigma^n(s_d), s) < \epsilon$. We create s_d by taking the concatenation of all possible finite sequences of length n, for all $n = 1, 2, \ldots$:

$$s_d = \{0\cdots 0|\underbrace{1\,0}_{n=1}\underbrace{00\;01\;10\;11}_{n=2}\underbrace{000\;001\;010\;011\;100\;101\;110\;111}_{n=3}\cdots\}$$

   Now for given $\epsilon > 0$ there exists k such that

$$\sum_{|i|>k} \frac{1}{2^{|i|}} < \epsilon.$$

   For any $s \in S$, the middle sequence $s_{-k} \ldots s_{-1}|s_0 \ldots s_k$ is in s_d somewhere (since all finite sequences are in s_d). If we choose n to shift this sequence to the middle block, then

$$d(s, \sigma^n(s_d)) = \sum_{i\in\mathbb{Z}} \frac{|s_i - \sigma^n(s_d)_i|}{2^{|i|}} = \sum_{|i|>k} \frac{|s_i - \sigma^n(s_d)_i|}{2^{|i|}} \le \sum_{|i|>k} \frac{1}{2^{|i|}} \le \epsilon.$$

□

7.3.1 Sensitive dependence to initial conditions

Two important notions in dynamical systems.
Let Λ be an invariant compact set for an invertible iterative map $f : \mathcal{M} \to \mathcal{M}$.

Definition 7.3. f has *sensitivity to initial conditions* on Λ if $\exists\epsilon > 0$ such that for any $p \in \Lambda$ and any neighbourhood U of p, there exists $p' \in U$ and $n \in \mathbb{N}$ such that $|f^n(p) - f^n(p')| > \epsilon$.

Definition 7.4. f is *topologically transitive* on Λ if for any open sets $U, V \subseteq \Lambda$ then $\exists\, n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

Together they lead to the notion of chaos:

Definition 7.5. Let Λ be an invariant compact set for an invertible iterative map $f : \mathcal{M} \to \mathcal{M}$. Then f is *chaotic* on Λ if it has sensitivity to initial conditions on Λ **and** is topologically transitive on Λ.

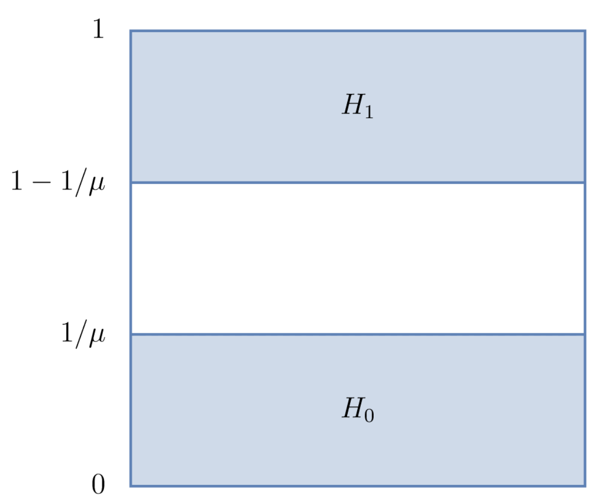
Theorem 7.6. *The shift map is chaotic on* Σ.

Proof. Consider a sequence s and all sequences in a neighbourhood U (i.e. all sequences with the same central block of size k). However large k is, choose a sequence $s' \in U$ which differs from s in position $N > k$. Then $d(\sigma^N(s), \sigma^N(s')) \geq 1$ so that we have sensitivity to initial conditions (choose any $0 < \epsilon < 1$). Moreover, since Σ has no isolated point then the existence of a dense orbit can be shown to imply topological transitivity. □

7.4 Step 2: Smale's horseshoe

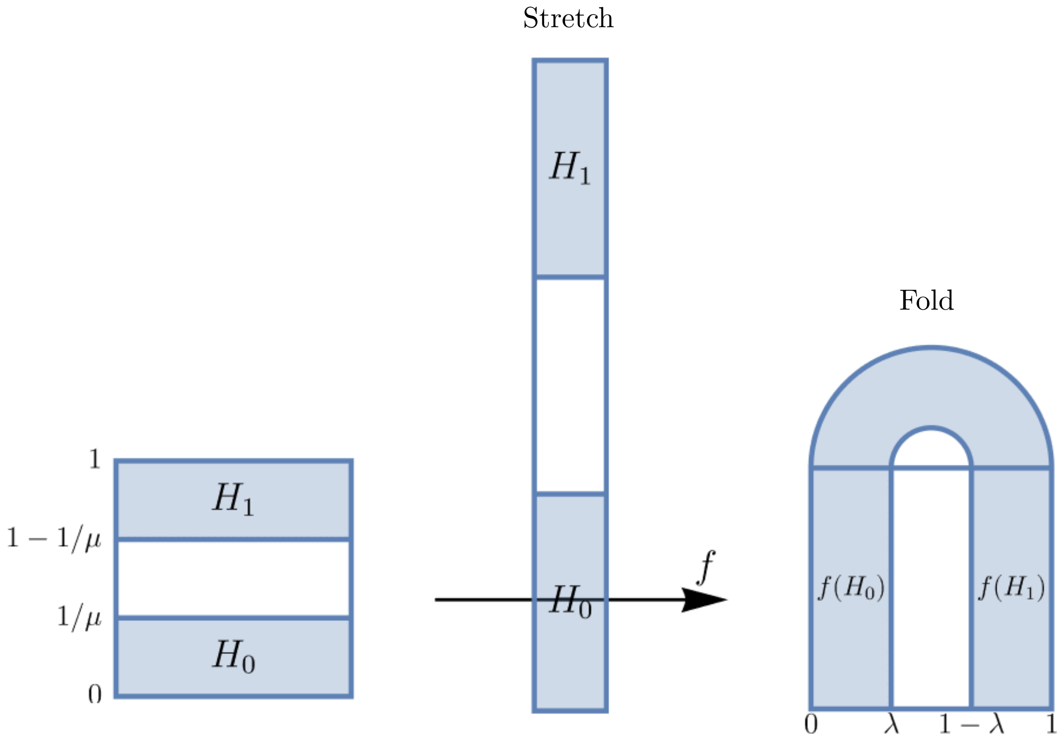
First define two rectangular regions in the unit square:

$$\begin{aligned} H_0 &= \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1/\mu\}, \\ H_1 &= \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 1 - 1/\mu \leq y \leq 1\}. \end{aligned}$$

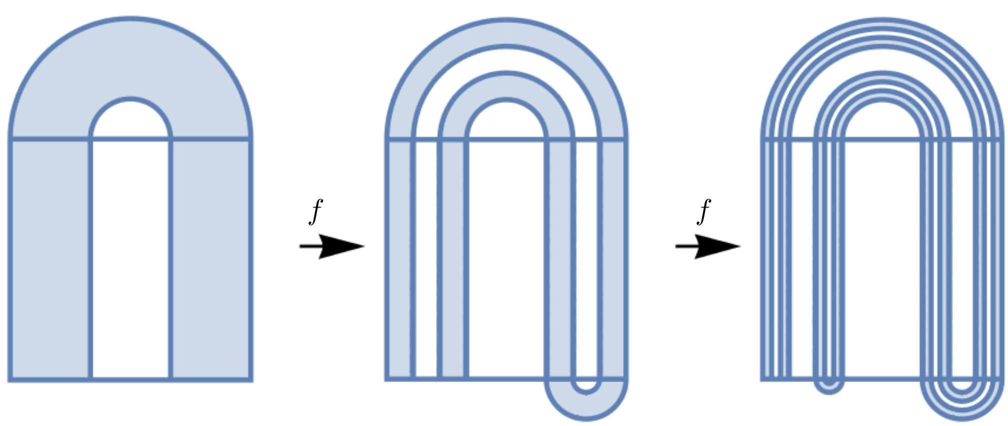


Second, define a map of these rectangles into themselves:

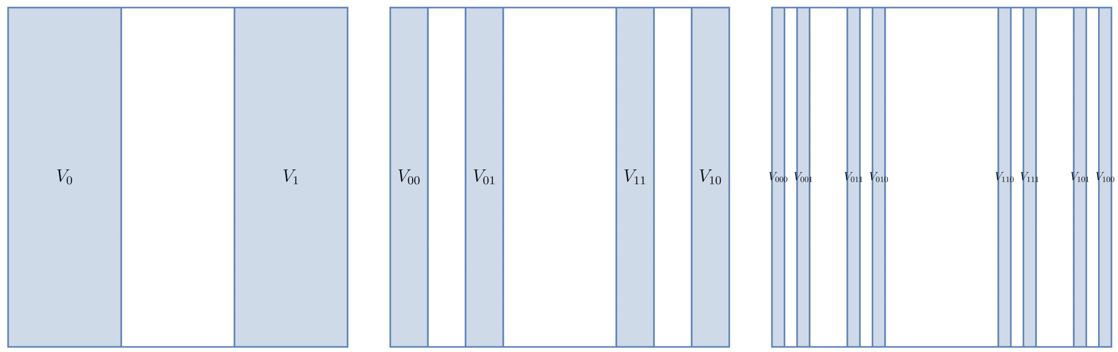
$$\begin{aligned} H_0 : \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \\ H_1 : \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \begin{bmatrix} -\lambda & 0 \\ 0 & -\mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ \mu \end{bmatrix}. \end{aligned}$$



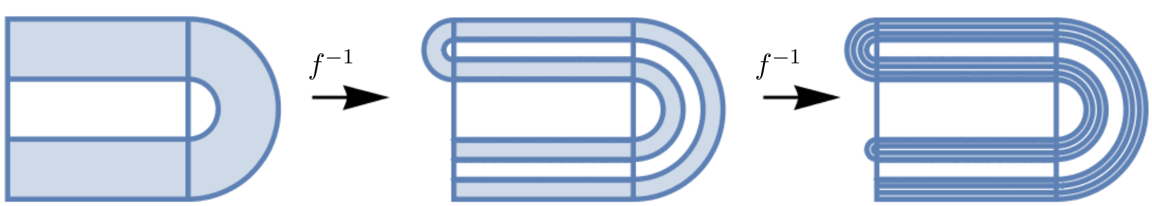
Third, repeat the operation.

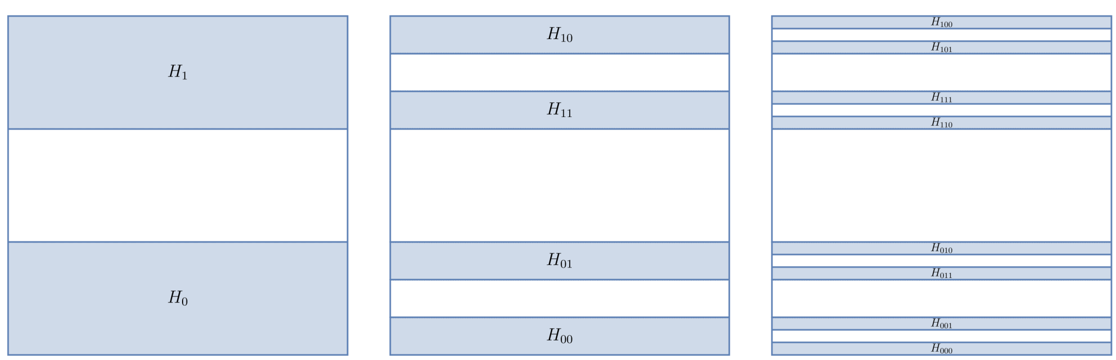


Fourth, introduce a coding: 101 means that it was right at the first iteration, left at the second, right at the third (read from the right).

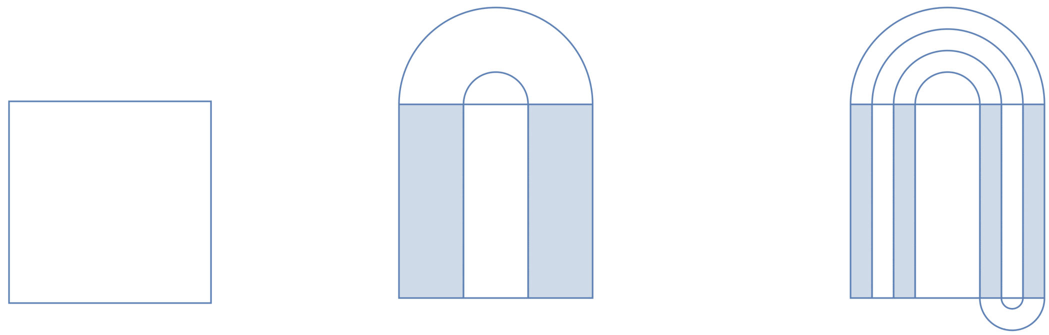


Fifth, do the same for the inverse map:

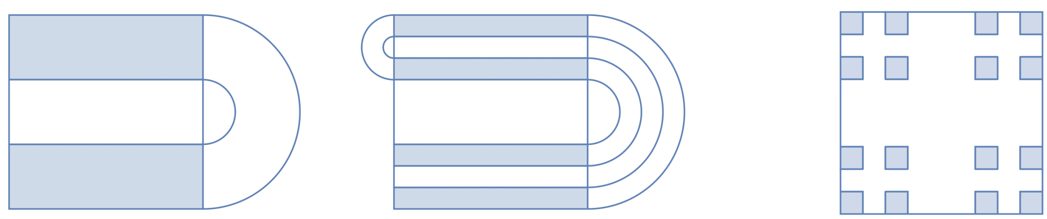




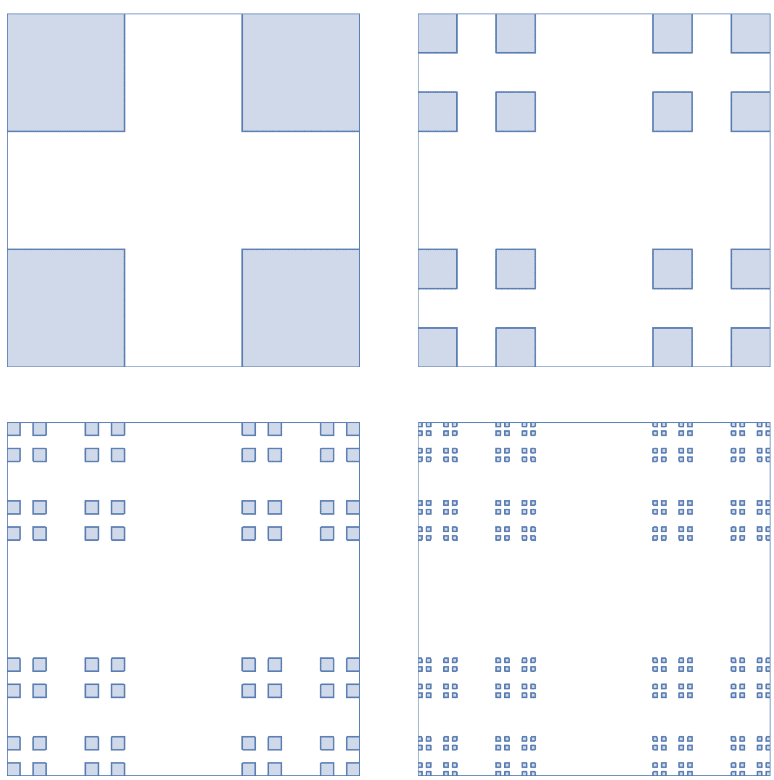
Sixth, take the intersection between the two sets



D $\qquad$ $D \cap f(D)$ $\qquad$ $D \cap f(D) \cap f^2(D)$



$D \cap f^{-1}D$ $\qquad$ $D \cap f{-}1(D) \cap f^{-2}(D)$ $\qquad$ $f^{-2}(D) \cap f^{-1}(D) \cap D \cap f(D) \cap f^2(D)$



$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(H_0 \cup H_1).$$

This is the intersection of a Cantor set of vertical lines and a Cantor set of horizontal lines. The dynamics on Λ is given by $f : \Lambda \mapsto \Lambda$. By construction Λ is an invariant set.

Each point in Λ can be coded by two binary sequences. The first sequence codes its horizontal position, the second sequence codes its vertical position. Therefore to each point p in Λ we can associated a bi-infinite sequence σ in Σ.

What about the dynamics? For the vertical encoding we read from the right: ...101 meant right at the first iteration, left at the second, right at the third, etc. Applying the map to such a point clearly just chops off the last digit and shifts the sequence to the right: the first iteration of the new point will be the second iteration of the old point. The same is true for the horizontal encoding with respect to the inverse map: applying f^{-1} just chops off the last digit and shifts the sequence to the right. Thus the forward map f must shift the horizontal sequence to the left and add a new digit at the end. Since $f(H_0) = V_0$ and $f(H_1) = V_1$ the new digit is exactly that which is chopped off the vertical encoding. Thus if we write vertical encoding backwards to read from left to right

$$s = \left\{ \underbrace{\ldots, s_{-n}, \ldots, s_{-2}, s_{-1}}_{\text{horizontal encoding}} \mid \underbrace{s_0, s_1, \ldots, s_n, \ldots}_{\text{vertical encoding}} \right\},$$

then f maps s to

$$\{\ldots, s_{-n}, \ldots, s_{-1}, s_0 | s_1, \ldots, s_n, \ldots\},$$

that is, the same as the shift map σ. If we label the map from Λ to Σ

$$h : \Lambda \to \Sigma$$

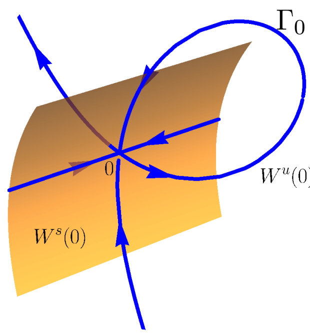
then h is a homeomorphism (1:1, onto, continuous with continuous inverse). Since there exists a homeomorphism h, it implies that the dynamics of f on Λ is *topologically conjugate* to the dynamics of σ on Σ.

Topological equivalence

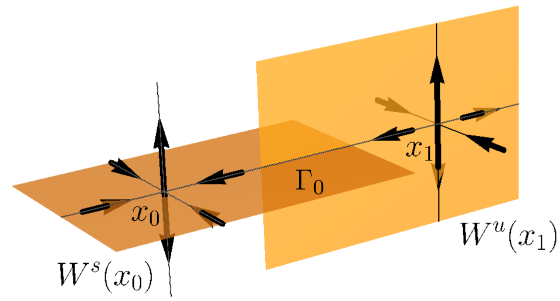
$$\begin{array}{ccc} \Lambda & \xrightarrow{f} & \Lambda \\ h\downarrow & & \downarrow h \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

To each orbit in Σ there is a corresponding orbit on Λ Therefore, the system has, a countable infinity of periodic orbits, an uncountable infinity of non-periodic orbit, a dense orbit, sensitivity dependence to initial conditions. We conclude that f on Λ is chaotic.

7.5 Step 3: Transverse homoclinic points



The stable and unstable manifolds from the origin intersect tangentially on the homoclinic orbit Γ_0.



The stable manifold from $\mathbf{x}_0$ and the unstable manifold from $\mathbf{x}_1$ intersect transversely on the heteroclinic orbit Γ_0.

A transverse homoclinic point is a point at which the stable and unstable manifolds from a hyperbolic fixed point intersect transversly. It is not possible for a dynamical system to have a transverse homoclinic orbit since

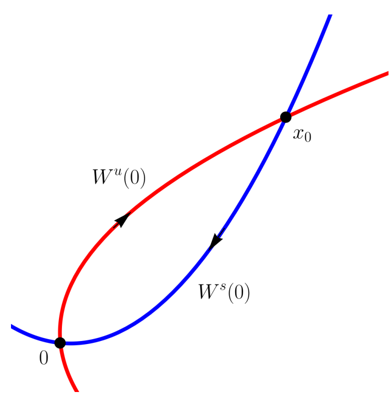
$$\dim W^s(\mathbf{x}_0) + \dim W^u(\mathbf{x}_0) \leq n$$

whereas transversality requires

$$\dim W^s(\mathbf{x}_0) + \dim W^u(\mathbf{x}_0) > n.$$

However, the Poincaré map associated with a dynamical system can have a transverse homoclinic orbit.

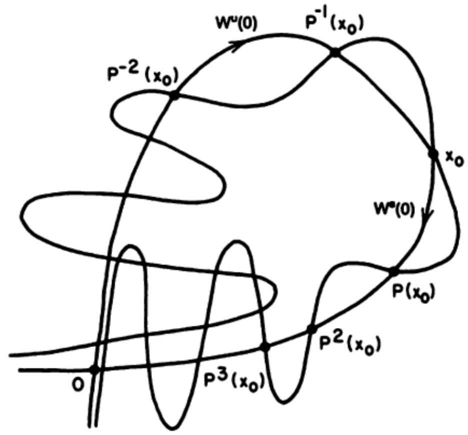
Consider a C^1-map $P : \mathbb{R}^n \mapsto \mathbb{R}^n$ and suppose that 0 is a hyperbolic fixed point (i.e. no centre manifold). Suppose that the stable and unstable manifolds intersect transversally at a point $\mathbf{x}_0$.



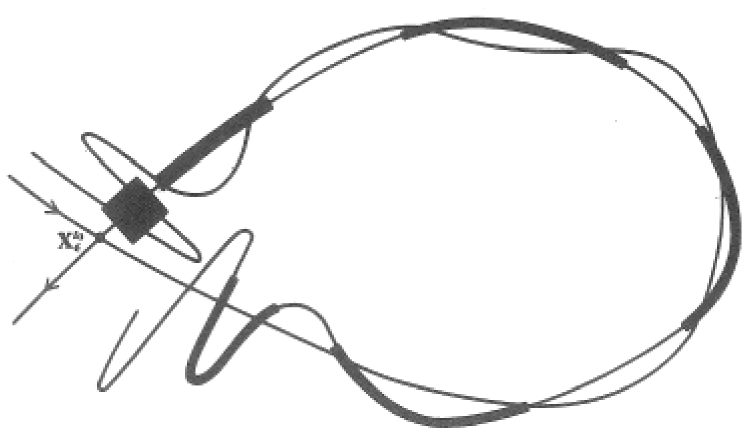
Since $W^s(\mathbf{0})$ and $W^u(\mathbf{0})$ are invariant under P, iterates of $\mathbf{x}_0$ under P and P^{-1} must also lie in $W^s(\mathbf{0}) \cap W^u(\mathbf{0})$. Thus

$$\{\ldots, P^{-n}(\mathbf{x}_0), \ldots, P^{-1}(\mathbf{x}_0), \mathbf{x}_0, P(\mathbf{x}_0), \ldots, P^n(\mathbf{x}_0), \ldots\} \in W^s(\mathbf{0}) \cap W^u(\mathbf{0}).$$

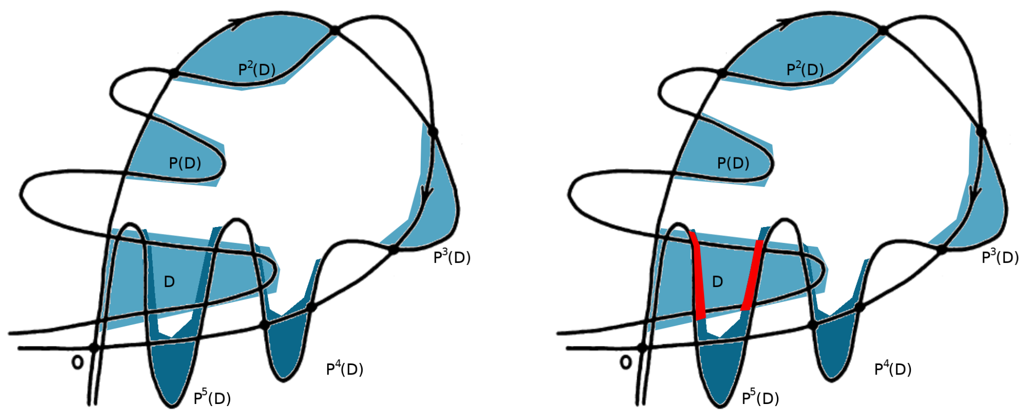
None of these points can coincide, since otherwise this would become a periodic orbit, while we know that $P^n(\mathbf{x}_0) \to \mathbf{0}$ as $n \to \infty$ [since $\mathbf{x}_0 \in W^s(\mathbf{0})$] and as $n \to -\infty$ [since $\mathbf{x}_0 \in W^u(\mathbf{0})$]. Thus the existence of one transverse homoclinic point implies the existence of an infinite number of homoclinic points. It can be shown that they are all transverse, and accumulate at $\mathbf{0}$. This leads to what is known as a "homoclinic tangle".



In a homoclinic tangle, a high enough iterate of P will lead to a horseshoe map. To see why this is so consider what happens to a small square near the critical point under iterates of the map, as illustrated below.



The square is stretched in the unstable direction and compressed in the stable direction. Then as the unstable manifold approaches the critical point again it is folded. In the following figure we see that the intersection between the domain D and $P^{(5)}(D)$ (highlighted red) resembles that shown for Smale's horseshoe map.



The formal argument above can be made rigorous:

Theorem 7.7 (The Smale-Birkhoff Homoclinic Theorem)**.** *Let $P : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a diffeomorphism such that P has a hyperbolic fixed point of saddle type, $\mathbf{p}$, and a transverse homoclinic point $\mathbf{q} \in W^s(\mathbf{p}) \cap W^u(\mathbf{p})$. Then there exists an integer N such that $F = P^N$ has a hyperbolic compact invariant Cantor set Λ on which F is topologically equivalent to a shift map on bi-infinite sequences of zeros and ones. The invariant set Λ*

(i) contains a countable set of periodic orbits of F of arbitrarily long periods;

(ii) contains an uncountable set of bounded nonperiodic orbits, and

(iii) contains a dense orbit.

7.6 Step 4: Melnikov's method

Melnikov's method gives us an analytical tool to determine the existence of transverse homoclinic points for the Poincaré map for a periodic orbit of a perturbed dynamical system. We consider the case of periodically perturbed Hamiltonian planar systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, t), \qquad \mathbf{x} = (x, y), \tag{7.6}$$

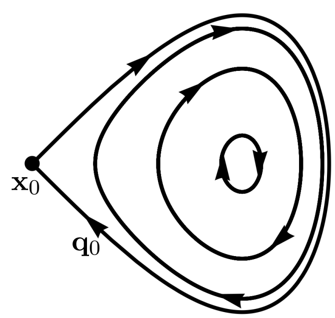
where g is periodic in t of period T and

(1) For $\epsilon = 0$ the system (7.6) has a homoclinic orbit

$$\mathbf{x} = \mathbf{q}_0(t) \qquad -\infty < t < \infty$$

at a hyperbolic saddle point $\mathbf{x}_0$.

(2) For $\epsilon = 0$ the system (7.6) has a continuous one-parameter family of periodic orbits $\mathbf{x} = \mathbf{q}_\alpha(t)$ of period T_α, $\alpha \in (-1, 0)$, in the interior of the homoclinic orbit, with $\lim_{\alpha \to 0} \mathbf{q}_\alpha = \mathbf{q}_0$ and $\lim_{\alpha \to 0} T_\alpha = \infty$.



Since the unperturbed system is Hamiltonian

$$\mathbf{f}(\mathbf{x}) = (f_1, f_2) = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x} \right),$$

for some function $H(x, y)$ (the Hamiltonian). We will embed the system in 3-dimensional phase space $(\mathbf{x}, \theta)$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, \theta), \qquad \dot{\theta} = 1. \tag{7.7}$$

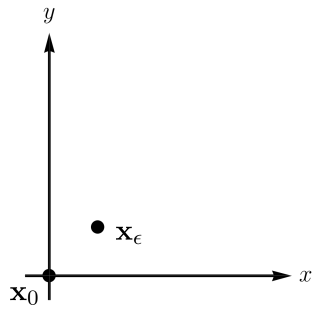
We define the Poincaré map

$$P_\epsilon^{t_0} : \Sigma^{t_0} \mapsto \Sigma^{t_0}$$

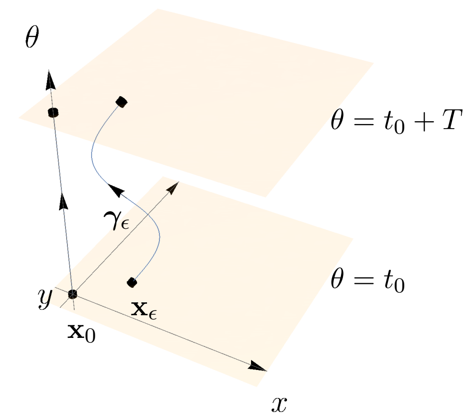
where $t_0 \in [0, T)$ is fixed and

$$\Sigma^{t_0} = \{(\mathbf{x}, \theta) : \theta = t_0\},$$

in the usual way, that is, given $\boldsymbol{\xi}$ we integrate (7.7) from t_0 to $t_0 + T$ with the initial condition $\mathbf{x} = \boldsymbol{\xi}$ at $t = t_0$. Then $P_\epsilon^{t_0}(\boldsymbol{\xi}) = \mathbf{x}(t_0 + T)$. Conditions (1) and (2) are enough to guarantee that the perturbed system (7.6) has a unique hyperbolic periodic orbit $\mathbf{x} = \boldsymbol{\gamma}_\epsilon(t)$ of period T and that $\boldsymbol{\gamma}_\epsilon(t) = \mathbf{x}_0 + O(\epsilon)$, that is, the orbit lies close to that of the unperturbed system. The Poincaré map $P_\epsilon^{t_0}$ has a unique hyperbolic fixed point of saddle type $\mathbf{x}_\epsilon$ which is close to that of the unperturbed system, i.e., $\mathbf{x}_\epsilon = \mathbf{x}_0 + O(\epsilon)$.

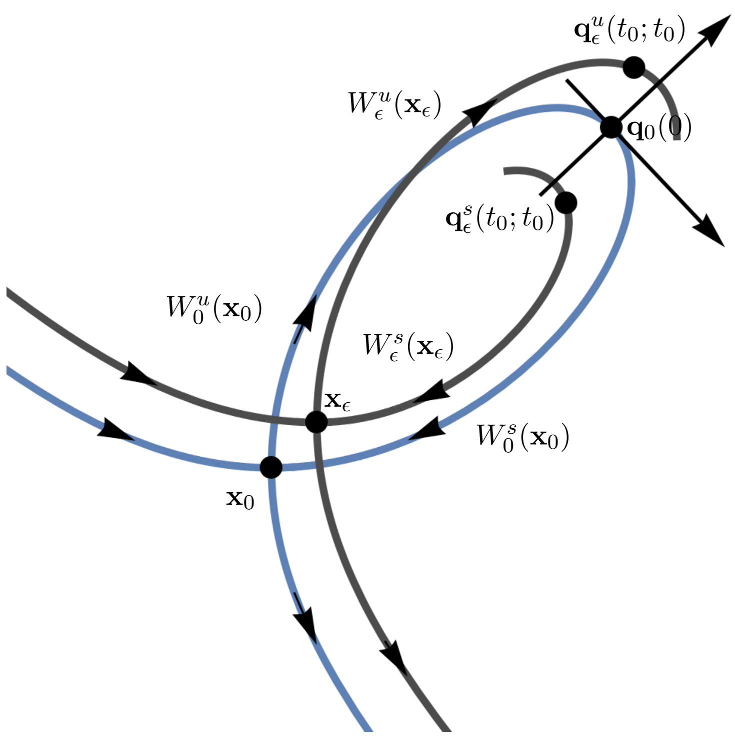


Fixed points of $P_\epsilon^{t_0}$ in the xy-plane



The periodic orbit $\boldsymbol{\gamma}_\epsilon$ of (7.7) and the periodic orbit $(\mathbf{x}_0, t)$ when $\epsilon = 0$.

Choose a point on the (unperturbed) homoclinic orbit. We can shift the origin of time so that this point is $\mathbf{q}_0(0)$. We aim to determine the distance between the stable and unstable manifolds of the map $P_\epsilon^{t_0}$ near $\mathbf{q}_0(0)$. Denote the orbits which satisfy (7.7) and lie in the stable and unstable manifolds in the three-dimensional phase space by $(\mathbf{q}_\epsilon^s(t; t_0), t)$ and $(\mathbf{q}_\epsilon^u(t; t_0), t)$ respectively.



When $\epsilon = 0$ we know that $(\mathbf{q}_0(t - t_0), t)$ lies in the stable and unstable manifolds of $P_\epsilon^{t_0}$. We expand about this solution to give

$$
\begin{aligned}
\mathbf{q}_\epsilon^s(t; t_0) &= \mathbf{q}_0(t - t_0) + \epsilon \mathbf{q}_1^s(t - t_0) + \cdots \qquad \text{for } t \geq t_0, \\
\mathbf{q}_\epsilon^u(t; t_0) &= \mathbf{q}_0(t - t_0) + \epsilon \mathbf{q}_1^u(t - t_0) + \cdots \qquad \text{for } t \leq t_0.
\end{aligned}
$$

The limit on the range of t is due to the fact that solutions approach the stable manifold exponentially for forward time, but diverge from it exponentially for backward time (and vice versa for the unstable manifold). In principle we can find $\mathbf{q}_1^s$ and $\mathbf{q}_1^u$ by regular perturbation theory. Substituting into (7.7) and using the fact that $\mathbf{q}_0$ satisfies the equation when $\epsilon = 0$ gives

$$\dot{\mathbf{q}}_1^s(t;t_0) = \mathbf{J}(\mathbf{q}_0(t-t_0))\mathbf{q}_1^s(t;t_0) + \mathbf{g}(\mathbf{q}_0(t-t_0),t) \qquad \text{for } t \geq 0, \tag{7.8}$$

where $\mathbf{J}$ is the Jacobian matrix of $\mathbf{f}$, with

$$\mathbf{q}_1^s \to \mathbf{x}_1 \qquad \text{as } t \to \infty,$$

where

$$\mathbf{x}_\epsilon = \mathbf{x}_0 + \epsilon \mathbf{x}_1 + \cdots .$$

Similarly

$$\dot{\mathbf{q}}_1^u(t;t_0) = \mathbf{J}(\mathbf{q}_0(t-t_0))\mathbf{q}_1^u(t;t_0) + \mathbf{g}(\mathbf{q}_0(t-t_0),t) \qquad \text{for } t \leq 0, \tag{7.9}$$

with

$$\mathbf{q}_1^u \to \mathbf{x}_1 \qquad \text{as } t \to -\infty.$$

Since for small ϵ $W^u(\mathbf{x}_\epsilon)$ and $W^s(\mathbf{x}_\epsilon)$ are almost tangential to the homoclinic orbit at the point $\mathbf{q}_0(0)$ we can measure the distance between $W^u(\mathbf{x}_\epsilon)$ and $W^s(\mathbf{x}_\epsilon)$ near to the point $\mathbf{q}_0(0)$ in the direction normal to the homoclinic orbit. We define the displacement

$$\mathbf{d}(t_0) = \mathbf{q}_\epsilon^u(t_0;t_0) - \mathbf{q}_\epsilon^s(t_0;t_0) = \epsilon\left(\mathbf{q}_1^u(t_0;t_0) - \mathbf{q}_1^s(t_0;t_0)\right) + \cdots .$$

If $\mathbf{f} = (f_1, f_2)$ then the outward normal vector is $\mathbf{n} = (-f_2, f_1)/|\mathbf{f}|$, so the distance between the two manifolds at $\mathbf{q}_0(0)$ is

$$D(t_0) = \mathbf{d}\cdot\mathbf{n} = \frac{\epsilon \mathbf{f}(\mathbf{q}_0(0)) \wedge (\mathbf{q}_1^u(t_0;t_0) - \mathbf{q}_1^s(t_0;t_0))}{|\mathbf{f}(\mathbf{q}_0(0))|} + \cdots . \tag{7.10}$$

Rather than solve for $\mathbf{q}_1^u$ and $\mathbf{q}_1^u$ and the substitute into (7.10) we are going to use (7.8) and (7.9) to get a differential equation for D which we can then solve. To this end define

$$\Delta^s(t;t_0) = \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{q}_1^s(t;t_0).$$

Differentiating gives

$$\begin{aligned}
\dot{\Delta}^s(t;t_0) &= \mathbf{J}(\mathbf{q}_0(t-t_0))\mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{q}_1^s(t;t_0) + \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \dot{\mathbf{q}}_1^s(t;t_0)\\
&= \mathbf{J}(\mathbf{q}_0(t-t_0))\mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{q}_1^s(t;t_0) + \\
&\quad\ \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge (\mathbf{J}(\mathbf{q}_0(t-t_0))\mathbf{q}_1^s(t;t_0) + \mathbf{g}(\mathbf{q}_0(t-t_0),t))\\
&= \text{trace}(\mathbf{J}(\mathbf{q}_0(t-t_0)))\left[\mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{q}_1^s(t;t_0)\right] + \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t-t_0),t)\\
&= \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t-t_0),t)
\end{aligned}$$

since

$$(\mathbf{J}\mathbf{a}) \wedge \mathbf{b} + \mathbf{a} \wedge (\mathbf{J}\mathbf{b}) = (\text{trace } \mathbf{J})(\mathbf{a} \wedge \mathbf{b})$$

for any $\mathbf{a}$ and $\mathbf{b}$ and in in our case

$$\text{trace } \mathbf{J} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = \frac{\partial^2 H}{\partial x \partial y} - \frac{\partial^2 H}{\partial y \partial x} = 0$$

because the system is Hamiltonian. Therefore

$$\Delta^s(t;t_0) = \Delta^s(\infty;t_0) - \int_t^\infty \mathbf{f}(\mathbf{q}_0(t'-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t'-t_0),t')\,\mathrm{d}t',$$

so that

$$\Delta^s(t_0;t_0) = \Delta^s(\infty;t_0) - \int_{t_0}^\infty \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t-t_0),t)\,\mathrm{d}t.$$

But $\Delta^s(\infty;t_0) = 0$ because

$$\mathbf{f}(\mathbf{q}_0(t-t_0)) \to \mathbf{f}(\mathbf{x}_0) = \mathbf{0} \qquad \text{as } t \to \infty,$$

while $\mathbf{q}_1^s$ is bounded. Similarly

$$\Delta^u(t_0;t_0) = \int_{-\infty}^{t_0} \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t-t_0),t)\,\mathrm{d}t.$$

Thus (7.10) becomes

$$D(t_0) = \frac{\epsilon M(t_0)}{|\mathbf{f}(\mathbf{q}_0(0))|},$$

where the Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t-t_0),t)\,\mathrm{d}t. \tag{7.11}$$

If M has a simple zero at a point $t_0 = \tau$ then so does D, so that the stable and unstable manifolds of the Poincaré map P_ϵ^τ intersect transversally at the point $\mathbf{q}_0(0)$. What about the Poincaré maps for other values of t_0? Since the system is autonomous when $\epsilon = 0$ changing t_0 is equivalent to changing the origin of time in $\mathbf{q}_0$, so corresponds to moving the point $\mathbf{q}_0(0)$ around the homoclinic orbit.

Theorem 7.8 (Melnikov '63)**.** *Under assumptions (1) and (2), if the Melnikov function $M(t_0)$ defined by (7.11) has a simple zero in $[0,T]$ then for all sufficiently small $\epsilon \neq 0$ the stable and unstable manifolds $W^s(\mathbf{x}_\epsilon)$ and $W^u(\mathbf{x}_\epsilon)$ of the Poincaré map P_ϵ intersect transversally, i.e. P_ϵ has a transverse homoclinic point. If $M(t_0) > 0$ (or < 0) for all t_0 then $W^s(\mathbf{x}_\epsilon) \cap W^u(\mathbf{x}_\epsilon) = \emptyset$.*

We note that there is a generalisation of this theorem to non-Hamiltonian systems with a slightly different M.

Example 7.1 (Duffing's equation)**.** Let us return to our motivational example:

$$\ddot{x} = x - x^3 - \delta\dot{x} + \gamma\cos(t).$$

Suppose δ and γ are small. To quantify this we set $\delta \to \epsilon\delta$, $\gamma \to \epsilon\gamma$. We write $y = \dot{x}$ to put the equation in the form of a first order system:

$$\dot{x} = y, \qquad \dot{y} = x - x^3 - \epsilon\delta y + \epsilon\gamma\cos(t).$$

This is in the required form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x},t)$ with

$$\mathbf{f} = \begin{bmatrix} y \\ x - x^3 \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ \gamma\cos t - \delta y \end{bmatrix}.$$

We see $\mathbf{g}$ is periodic in t with period $T = 2\pi$. When $\epsilon = 0$ the system is Hamiltonian with

$$H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}; \qquad \mathbf{f} = \left[\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right]^T.$$

Since H is conserved during the motion, it follows that the homoclinic orbits have $H = 0$, so that

$$\dot{x} = y = \pm x\left(1 - \frac{x^2}{2}\right)^{1/2}.$$

The solutions are

$$(x_0(t), y_0(t)) = \mathbf{q}_0(t) = \pm(\sqrt{2}\,\mathrm{sech}\,t, -\sqrt{2}\,\mathrm{sech}\,t\tanh t).$$

Then, for the right-hand homoclinic orbit (the plus sign),

$$\begin{aligned}
M(t_0) &= \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{q}_0(t-t_0)) \wedge \mathbf{g}(\mathbf{q}_0(t-t_0), t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} y_0(t-t_0)\left[\gamma\cos t - \delta y_0(t-t_0)\right]\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} y_0(s)\left[\gamma\cos(s+t_0) - \delta y_0(s)\right]\,\mathrm{d}s \\
&= -\int_{-\infty}^{\infty} \sqrt{2}\,\mathrm{sech}\,s\tanh s\left[\gamma\cos(s+t_0) + \delta\sqrt{2}\,\mathrm{sech}\,s\tanh s\right]\,\mathrm{d}s \\
&= -\int_{-\infty}^{\infty} \sqrt{2}\,\mathrm{sech}\,s\tanh s\left[\gamma\cos s\cos t_0 - \gamma\sin s\sin t_0 + \delta\sqrt{2}\,\mathrm{sech}\,s\tanh s\right]\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \sqrt{2}\,\mathrm{sech}\,s\tanh s\left[\gamma\sin s\sin t_0 - \delta\sqrt{2}\,\mathrm{sech}\,s\tanh s\right]\,\mathrm{d}s \\
&= \sqrt{2}\gamma\sin t_0\int_{-\infty}^{\infty}\mathrm{sech}\,s\tanh s\sin s\,\mathrm{d}s - 2\delta\int_{-\infty}^{\infty}\mathrm{sech}^2 s\tanh^2 s\,\mathrm{d}s \\
&= \sqrt{2}\gamma\sin t_0\pi\,\mathrm{sech}(\pi/2) - 4\delta/3
\end{aligned}$$

on making the substitution $s = t - t_0$. Thus if

$$\sqrt{2}\gamma\pi\,\mathrm{sech}(\pi/2) > 4\delta/3$$

then M has simple zeros and there is chaos for small ϵ, while if

$$\sqrt{2}\gamma\pi\,\mathrm{sech}(\pi/2) < 4\delta/3$$

then $M < 0$ for all t_0 and there is no homoclinic tangle.