Geometric Group Theory

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Inspirational quotations

G.H. Hardy: "A person's first duty, a young person's at any rate, is to be ambitious, and the noblest ambition is that of leaving behind something of permanent value."

Aristotle in "Metaphysics":

"The chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree."

Graphs of groups

Definition

The path group of the graph of groups (G, Y) is

$$F(G,Y) =$$
$$\langle \bigcup_{v\in V} G_v \cup E(Y) | \bar{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g), \forall e \in E(Y), g \in G_e\rangle .$$

If $G_v = \langle S_v | R_v\rangle$ then

$$F(G,Y) = \langle \bigcup_{v\in V} S_v \cup E(Y)| \bigcup_{v\in V(Y)} R_v, \bar{e} = e^{-1}, e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g)\rangle .$$

Graphs of groups

Remarks

1. *If all* $G_v = \{1\}$ *then* $F(G, Y) = F(E^+(Y))$.
2. *There exists an epimorphism* $F(G, Y) \to F(E^+(Y))$ *defined by sending each* G_v *to* $\{1\}$.
3. *If all* $G_e = 1$ *then*

$$F(G, Y) = *_{v \in V(Y)} G_v * F(E^+(Y)).$$

Definition

A path in (G, Y) is a sequence

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

such that $t(e_i) = o(e_{i+1})$ and $g_i \in G_{t(e_i)} = G_{o(e_{i+1})}$. If $v_0 = o(e_1)$, $v_n = t(e_n)$ then we call this a path from v_0 to v_n. We call

$$v_0, v_1 = t(e_1) = o(e_2), ..., v_i = t(e_i) = o(e_{i+1}), ..., v_n$$

its sequence of vertices. We define $|c|$ to be the element of the path group $g_0 e_1 g_1 ... e_n g_n$. If $a_0, a_1 \in V(Y)$ then we define

$$\pi[a_0, a_1] = \{|c| : c \text{ a path from } a_0 \text{ to } a_1\}$$

Graphs of groups

Remark

If $a_0, a_1, a_2 \in V(Y)$ and $\gamma \in \pi[a_0, a_1]$, $\delta \in \pi[a_1, a_2]$ then $\gamma\delta \in \pi[a_0, a_2]$.

Proposition

Let (G, Y) be a graph of groups and suppose $a_0 \in V(Y)$. The set $\pi[a_0, a_0]$ is a subgroup of $F(G, Y)$.

Proof.

If

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

is a path from a_0 to a_0 then

$$|c|^{-1} = g_n^{-1} \bar{e}_n g_{n-1}^{-1} ... \bar{e}_1 g_0^{-1} \in \pi[a_0, a_0]$$

□

Graphs of groups with basepoint

Proposition

Let (G, Y) be a graph of groups and suppose $a_0 \in V(Y)$. The set $\pi[a_0, a_0]$ is a subgroup of $F(G, Y)$.

We call this subgroup the fundamental group of the graph of groups (G, Y) with basepoint a_0 and denote it $\pi_1(G, Y, a_0)$.

Graphs of groups wrto a maximal subtree

Definition

Let (G, Y) be a graph of groups, and let T be a maximal subtree of Y. The fundamental group of (G, Y) with respect to T, denoted $\pi_1(G, Y, T)$, is

$$F(G, Y)/\langle\langle\{e : e \in T\}\rangle\rangle$$

Let $q : F(G, Y) \to \pi_1(G, Y, T)$ be the quotient map.

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ is an isomorphism to $\pi_1(G, Y, T)$.

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Proof: We define a homomorphism $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$ as follows.

$\forall a \in V(Y)$, there exists a unique geodesic path $e_1, e_2, ..., e_n$ in T from a_0 to a. Set

$$g_a := e_1...e_n \in F(G, Y)$$
$$g_{a_0} := 1$$

We first define $\hat{f} : F(G, Y) \to \pi_1(G, Y, a_0)$:

- $\forall g \in G_a$, set $\hat{f}(g) = g_a g g_a^{-1} \in \pi_1(G, Y, a_0)$
- $\forall e \in E^+(Y)$ with $o(e) = a, t(e) = b$, set $\hat{f}(e) = g_a e g_b^{-1} \in \pi_1(G, Y, a_0)$.

Graphs of groups

The definition of $\hat{f}$ is consistent with the relations $e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g)$ since if $e = [P, Q]$:

$$\begin{aligned}\hat{f}(e\alpha_e(g)\bar{e}) &= (g_P e g_Q^{-1})(g_Q \alpha_e(g) g_Q^{-1})(g_Q \bar{e} g_P^{-1}) \\ &= g_P(e\alpha_e(g)e^{-1})g_P^{-1} \\ &= g_P \alpha_{\bar{e}}(g) g_P^{-1}\end{aligned}$$

So $\hat{f}$ is defined on $F(G, Y)$. Now, for every $e = [P, Q] \in T$, $\hat{f}(e) = g_P e g_Q^{-1} = 1$.

Hence $\hat{f}$ defines $f : \pi_1(G, Y, T) \to \pi_1(G, Y, a_0)$.

Graphs of groups

Consider $q \circ f : \pi_1(G, Y, T) \to \pi_1(G, Y, T)$. For all $g \in G_a$,

$$q \circ f(g) = q(g_a g g_a^{-1}) = g.$$

For all $e \notin T$,

$$q \circ f(e) = q(g_P e g_Q^{-1}) = e.$$

Hence $q \circ f = \mathrm{id}$.

Consider now $f \circ q : \pi_1(G, Y, a_0) \to \pi_1(G, Y, a_0)$.

If $g_0 e_1 ... e_n g_n$ arbitrary in $\pi_1(G, Y, a_0)$ and $e_i = [P_{i-1}, P_i]$,

$$\begin{aligned} f \circ q(g_0 e_1 ... e_n g_n) &= g_0 (e_1 g_{P_1}^{-1})(g_{P_1} g_1 g_{P_1}^{-1})(g_{P_1} e_2 g_{P_2}^{-1}) ... g_{P_{n-1}}^{-1}(g_{P_{n-1}} e_n) g_n \\ &= g_0 e_1 ... e_n g_n \end{aligned}$$

NB If $e_i \in T$ then $f \circ q(g_{i-1} e_i g_i) = f(g_{i-1} g_i) = g_{P_{i-1}} g_{i-1} g_{P_{i-1}}^{-1} g_{P_i} g_i g_{P_i}^{-1} = g_{P_{i-1}} g_{i-1} e_i g_i g_{P_i}^{-1}$. □

Proposition

$q\big|_{\pi_1(G,Y,a_0)}$ *is an isomorphism to* $\pi_1(G, Y, T)$.

Corollary

The fundamental group $\pi_1(G, Y, a_0)$ *of the graph of groups* (G, Y) *does not depend on the choice of basepoint* a_0.

Corollary

The quotient $\pi_1(G, Y, T)$ *of the path group does not depend on the choice of the tree* T.

Reduced paths of graphs of groups

Definition

Let (G, Y) be a graph of groups. A path $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ is reduced if

1. $g_0 \neq 1$ if $n = 0$;
2. If $e_{i+1} = \bar{e}_i$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

We say that $g_0 e_1 ... e_n g_n$ is a reduced word.

Recall that $|c|$ is the element in $F(G, Y)$ represented by a path c.

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

Reduced paths of graphs of groups

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Proof

First assume that Y is finite. We will argue by induction on the number of edges in Y. If there are no edges, then the theorem holds. So assume the theorem is true for graphs with n edges, and suppose that Y has $n + 1$ edges.

Reduced paths of graphs of groups

Case 1: $Y = Y' \cup \{e\}$, $o(e) \in V(Y')$, $v = t(e) \notin V(Y')$. Then

$$F(G, Y) = (F(G, Y') * G_v)*_{\alpha_e(G_e)}$$

and a reduced word containing e corresponds to a reduced word in the HNN extension that is $\neq 1$.

Case 2: $Y = Y' \cup \{e\}$, $\{o(e), t(e)\} \subseteq V(Y')$. Then

$$F(G, Y) = F(G, Y')*_{\alpha_e(G_e)}$$

and the comment above holds again.