

Vector fields

Notation: I like to write all vectors as column vectors: so use $(u, v)^T$ to mean $\begin{pmatrix} u \\ v \end{pmatrix}$. This is a good practice to keep especially when you start to consider vector calculus and the nabla operations.

Let Ω be a bounded convex domain in two dimensions, with a piecewise smooth boundary denoted by $\delta\Omega$. Consider the functional:

$$J(u, v) = \int_{\Omega} F(x, y, u(x, y), u_x(x, y), u_y(x, y), v(x, y), v_x(x, y), v_y(x, y)) dx dy,$$

defined for the C^2 vector field $(u(x, y), v(x, y))^T$ on Ω , satisfying boundary conditions where $(u, v)^T$ is given and continuous on the boundary, $\delta\Omega$. Find a pair of partial differential equations that the extremal vector field $(u, v)^T$ must satisfy.

Solution

Suppose that $(u, v)^T$ is an extremal.

Consider $J(u + \eta, v + \psi)$ where both η and ψ vanish on the boundary, $\delta\Omega$, then

$$J(u + \eta, v + \psi) = \int_{\Omega} F dx dy + \int_{\Omega} F_u \eta + F_{u_x} \eta_x + F_{u_y} \eta_y + F_v \psi + F_{v_x} \psi_x + F_{v_y} \psi_y dx dy,$$

Here F and its partial derivatives are all evaluated at

$$(x, y, u(x, y), u_x(x, y), u_y(x, y), v(x, y), v_x(x, y), v_y(x, y)).$$

We will write $(F_{u_x}, F_{u_y})^T$ to denote the column vector field over Ω .

The second integral must vanish at an extremal. It is equal to

$$\begin{aligned} & \int_{\Omega} F_u \eta + F_v \psi + \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} \cdot \nabla \eta + \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} \cdot \nabla \psi dx dy, \\ &= \int_{\Omega} \left(F_u - \nabla \cdot \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} \right) \eta + \left(F_v - \nabla \cdot \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} \right) \psi dx dy, \end{aligned}$$

This is true by the Divergence theorem (Green's theorem), since we have the identities

$$\nabla \cdot \left(\eta \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} \right) = \eta \nabla \cdot \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} + \nabla \eta \cdot \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}.$$

So

$$\begin{aligned} & \int_{\delta\Omega} \nabla \left(\eta \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} \right) \cdot \mathbf{n} dS + \int_{\Omega} \left(F_u - \nabla \cdot \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} \right) \eta dx dy \\ &= \int_{\Omega} \eta F_u + \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} \cdot \nabla \eta dx dy \end{aligned}$$

and the integrand within the boundary integral (from the Divergence theorem) vanishes due to the boundary condition on η . Similar for v and ψ .

Hence we have a pair of scalar partial differential equations

$$F_u - \nabla \cdot \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} = 0 \quad F_v - \nabla \cdot \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} = 0 \quad (x, y)^T \in \Omega.$$