Geometric Group Theory

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Reduced paths of graphs of groups

Definition

Let (G, Y) be a graph of groups. A path $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ is reduced if

1. $g_0 \neq 1$ if $n = 0$;
2. If $e_{i+1} = \bar{e}_i$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

We say that $g_0 e_1 ... e_n g_n$ is a reduced word.

Recall that $|c|$ is the element in $F(G, Y)$ represented by a path c.

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

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Proof

First assume that Y is finite. We will argue by induction on the number of edges in Y. If there are no edges, then the theorem holds. So assume the theorem is true for graphs with n edges, and suppose that Y has $n+1$ edges.

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Case 1: $Y = Y' \cup \{e\}$, $o(e) \in V(Y')$, $v = t(e) \notin V(Y')$. Then

$$F(G, Y) = (F(G, Y') * G_v)*_{\alpha_e(G_e)}$$

with stable letter e. A reduced word containing e corresponds to a reduced word in the HNN extension that is $\neq 1$.

Case 2: $Y = Y' \cup \{e\}$, $\{o(e), t(e)\} \subseteq V(Y')$. Then

$$F(G, Y) = F(G, Y')*_{\alpha_e(G_e)}$$

and the comment above applies again.

Now suppose that Y is infinite. Any reduced path c involves finitely many orbits of vertices and edges and so c lies within a finite subgraph Y_1 of Y.

c is a reduced path in $F(G, Y_1)$ and so $c \neq 1$ in $F(G, Y_1)$. □

Reduced paths of graphs of groups

Theorem

If c is a reduced path then $|c| \neq 1$ in $F(G, Y)$. In particular, $G_v \hookrightarrow F(G, Y)$ is injective for every $v \in V(Y)$.

Corollary

For every $v \in V(Y)$, the homomorphism $G_v \to \pi_1(G, Y, T)$ is injective.

Proof.

$G_v \to F(G, Y)$ is injective and $\pi : \pi_1(G, Y, v) \to \pi_1(G, Y, T)$ is an isomorphism. $\square$

Graphs of groups

One can easily see that

1. If Y has 2 vertices and one edge then
$$\pi_1(G, Y, T) = G_u *_{G_e} G_v .$$

2. If Y has 1 vertex and 1 edge with stable letter 'e' then
$$\pi_1(G, Y, T) = G_v *_{\alpha_e(G_e)}$$
and $\theta : \alpha_e(G_e) \to \alpha_{\bar{e}}(G_e) \in G_v$, $\theta(g) = \alpha_{\bar{e}} \circ \alpha_e^{-1}$.

3. If $Y = Y' \cup \{e\}$ and $t(e) = v \notin Y'$ then
$$\pi_1(G, Y, T) = \pi_1(G, Y', T') *_{G_e} G_v .$$

4. If $Y = Y' \cup \{e\}$ and $v = t(e) \in Y'$ then
$$\pi_1(G, Y, T) = \pi_1(G, Y', T) *_{\alpha_e(G_e)} .$$

Reduced words of graphs of groups

We will find a choice of representatives for elements in $F(G, Y)$, where (G, Y) is a graph of groups. For each edge $e \in E(Y)$, pick a set S_e of left coset representatives of $\alpha_{\bar{e}}(G_e)$ in $G_{o(e)}$, with $1 \in S_e$.

Definition

An S-reduced path is a path $(s_1, e_1, ..., s_n, e_n, g)$ with

- $s_i \in S_{e_i} \ \forall i$;
- $s_i \neq 1$ if $e_i = \bar{e}_{i-1}$;
- $g \in G_{t(e_n)}$.

Lemma

Given $a, b \in V(Y)$, every element in $\pi[a, b]$ is represented by a unique S-reduced path.

Reduced words of graphs of groups

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Proof

Existence: Let $\gamma \in \pi[a, b]$ and consider the path $c = (g_0, e_1, g_1, e_2, \dots, g_{n-1}, e_n, g_n)$ such that $t(e_i) = o(e_{i+1})$, $g_i \in G_{t(e_i)} = G_{o(e_{i+1})}$ and $\gamma = |c|$.

We will prove by induction on n that γ can be represented by an S-reduced path. For $n = 0$ it is obvious. For $n = 1$,

$$\gamma = g_0 e_1 g_1 = s_0 \alpha_{\bar{e}_1}(h_0) e_1 g_1 = s_0 e_1 \alpha_{e_1}(h_0) g_1 = s_0 e_1 g_1'$$

A similar argument holds for the inductive step.

Reduced words of graphs of groups

Uniqueness: Consider two reduced paths

$$c = (s_1, e_1, ..., s_n, e_n, g)$$
$$c' = (\sigma_1, \eta_1, ..., \sigma_k, \eta_k, \gamma)$$

such that $|c| = |c'|$. Then

$$\gamma^{-1}\eta_k^{-1}\sigma_k^{-1}...\eta_1^{-1}\sigma_1^{-1}s_1 e_1...s_n e_n g = 1$$

We will prove that $c = c'$ by induction on the length. The above word cannot be reduced hence $\eta_1^{-1} = e_1^{-1}$ and $\sigma_1^{-1}s_1 \in \alpha_{\bar{e}_1}(G_{e_1})$. So $\sigma_1 = s_1$. And so we can apply the inductive assumption. $\square$

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions and such that

1. *The quotient graph $H\backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
    - a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
    - b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

Proof: For all $a \in V(Y)$, we define an equivalence relation on $\pi[a_0, a]$ by

$$|c_1| \sim |c_2| \iff |c_1| = |c_2| g \text{ for some } g \in G_a$$

Vertices of the tree:

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

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By the lemma, every element of $\pi[a_0, a]/\sim$ has a unique representative corresponding to an S-reduced path of the form $(s_1, e_1, ..., s_n, e_n)$, $o(e_1) = a_0,\ t(e_n) = a$. Thus $V(T)$ can also be identified with S-reduced paths as above.

Edges of the tree: $\{(s_1, e_1, ..., s_n, e_n), (s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})\}$.
Connectedness is obvious.

By our definition of edges, a cycle/circuit gives an S-reduced path with corresponding element $1 \in \pi[a_0, a]$ contradicting the uniqueness of the representation of a reduced path.

Graphs of groups and actions on trees

Action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a_0]$ on T: For all $h \in \pi[a_0, a_0]$ and for all $[g] \in V(T)$ (equivalence classes of $\pi[a_0, a]/\sim$) define the action

$$h \cdot [g] = [hg]$$

- If $[g_1], [g_2]$ are such that $h \cdot [g_1] = [g_2]$ then $a_1 = a_2$ where $g_i \in \pi[a_0, a_i]$.
- Conversely, if $[g_1], [g_2] \in \pi[a_0, a]$ then $h = g_2 g_1^{-1} \in \pi[a_0, a_0]$ and $h[g_1] = [g_2]$.

Thus $H \backslash V(T)$ can be identified with $V(Y)$. And likewise $H \backslash E(T)$ can be identified with $E(Y)$.