Geometric Group Theory

Cornelia Druțu

University of Oxford

Part C course HT 2023, Oxford

Quotations

Ralph Waldo Emerson: "Life is a journey, not a destination."

Donald Knuth: "It would be nice if we could design a virtual reality in Hyperbolic space, and meet each other there."

Graphs of groups and actions on trees

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions and such that

1. *The quotient graph $H \backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
   - a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
   - b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

Proof: For all $a \in V(Y)$, we define an equivalence relation on $\pi[a_0, a]$ by

$$|c_1| \sim |c_2| \iff |c_1| = |c_2|g \text{ for some } g \in G_a$$

Vertices of the tree:

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

$$V(T) = \bigsqcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

Every element of $\pi[a_0, a]/\sim$ has a unique representative corresponding to an S-reduced path of the form $(s_1, e_1, ..., s_n, e_n)$, $o(e_1) = a_0$, $t(e_n) = a$. Thus $V(T)$ can also be identified with S-reduced paths as above.

Edges of the tree: $\{(s_1, e_1, ..., s_n, e_n), (s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})\}$. Connectedness is obvious.

By our definition of edges, a cycle/circuit gives an S-reduced path with corresponding element $1 \in \pi[a_0, a]$ contradicting the uniqueness of the representation of a reduced path.

Action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a_0]$ on T: For all $h \in \pi[a_0, a_0]$ and for all $[g] \in V(T)$ (equivalence classes of $\pi[a_0, a]/\sim$) define the action

$$h \cdot [g] = [hg]$$

- If $[g_1], [g_2]$ are such that $h \cdot [g_1] = [g_2]$ then $a_1 = a_2$ where $g_i \in \pi[a_0, a_i]$.
- Conversely, if $[g_1], [g_2] \in \pi[a_0, a]$ then $h = g_2 g_1^{-1} \in \pi[a_0, a_0]$ and $h[g_1] = [g_2]$.

Thus $H \backslash V(T)$ can be identified with $V(Y)$. And likewise $H \backslash E(T)$ can be identified with $E(Y)$.

Graphs of groups and actions on trees

Stabilisers of vertices: For all $[v] \in V(T)$ with $v \in \pi[a_0, b]$,

$$h \in \mathrm{Stab}([v]) \iff hv \sim v \iff hv = vg_b \text{ for some } g_b \in G_b$$
$$\iff h = vg_b v^{-1} \text{ for some } g_b \in G_b$$

Thus $\mathrm{Stab}([v]) = vG_b v^{-1}$.

Stabilisers of edges: Every edge in $E(T)$ is of the form $\delta = [[v], [vge]]$, $v \in \pi[a_0, a], g \in G_a, \delta = [a, b]$. Then

$$\mathrm{Stab}(\delta) = \mathrm{Stab}(v) \cap \mathrm{Stab}(vge) = vG_a v^{-1} \cap (vge)G_b(vge)^{-1}$$
$$= vg(G_a \cap eG_b e^{-1})g^{-1}v^{-1} = vg(\alpha_{\bar{e}}(G_e))g^{-1}v^{-1}$$

We denote the tree thus obtained $\mathcal{T}(G, Y, a_0)$ and we call it the universal covering tree or the Bass–Serre tree of the graph of groups (G, Y). □

Graphs of groups and actions on trees

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions and such that

1. *The quotient graph $H \backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
   a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
   b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

Conversely, if a group Γ acts on a tree T with quotient Y then there exists a graph of groups (G, Y) such that $\Gamma \simeq \pi_1(G, Y, a_0)$. Indeed, suppose $\Gamma \curvearrowright T$, $Y = T/\Gamma$ and $p : T \to Y$. Let $X \subset S \subset T$ be such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and $p\big|_{\text{edges of } S}$ is 1-to-1.

Notation: If v is a vertex of Y and e is an edge of Y then let v^X be the vertex of X such that $p(v^X) = v$ and similarly let e^S be the edge of S such that $p(e^S) = e$.

A graph of groups with graph Y:

1. The map G:
   - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
   - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.

Graphs of groups and actions on trees

1. The map G:
   - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
   - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.
2. For each edge e, we define $\alpha_e : G_e \to G_{t(e)}$: For all $x \in V(S)$, define

$$g_x = \begin{cases} 1 & \text{if } x \in V(X) \\ \text{some } g_x \text{ such that } g_x x \in V(X) & \textit{otherwise}. \end{cases}$$

   Define $\alpha_e : G_e \to G_{t(e)}, \alpha_e(g) = g_{t(e)} g g_{t(e)}^{-1}$.

We can define a homomorphism $\varphi : F(G, Y) \to \Gamma$ by:

- $\forall a \in V(Y)$, $\varphi\big|_{G_a} = \mathrm{incl}_{G_a}$;
- $\forall e \in E(Y)$, $e = [y, x]$, $\varphi(e) = g_y g_x^{-1}$.

We can define a homomorphism $\varphi : F(G, Y) \to \Gamma$ by:

- $\forall a \in V(Y)$, $\varphi\big|_{G_a} = \mathrm{incl}_{G_a}$;
- $\forall e \in E(Y)$, $e = [y, x]$, $\varphi(e) = g_y g_x^{-1}$.

It satisfies the relations:

$$\varphi(e\alpha_e(g)e^{-1}) = (g_y g_x^{-1})(g_x g g_x^{-1})(g_x g_y^{-1}) = g_y g g_y^{-1} = \varphi(\alpha_{\bar{e}}(g))$$

Also, $\forall e \in p(X)$, $\varphi(e) = 1$. Hence, φ defines a homomorphism

$$\bar{\varphi} : \pi_1(G, Y, p(X)) \simeq \pi_1(G, Y, a_0) \to \Gamma$$

Hence, φ defines a homomorphism

$$\bar{\varphi} : \pi_1(G, Y, p(X)) \simeq \pi_1(G, Y, a_0) \to \Gamma$$

Theorem

The homomorphism $\bar{\varphi}$ is an isomorphism. If $\tilde{T}$ is the universal covering tree of (G, Y) then there exists a graph isomorphism $f : \tilde{T} \to T$ such that $\forall g \in \pi_1(G, Y, a_0)$, $\forall v \in V(\tilde{T})$,

$$f(g \cdot v) = \bar{\varphi}(g) \cdot f(v)$$

Proof: Not provided and non-examinable.

Subgroups

Theorem

Let $\Gamma = \pi_1(G, Y, a_0)$. If $B \leq \Gamma$ then there exists (H, Z) a graph of groups such that $B = \pi_1(H, Z, b_0)$ and

- *for all $v \in V(Z)$, $H_v \leq gG_a g^{-1}$ for some $a \in V(Y)$, $g \in \Gamma$;*
- *for all $e \in E(Z)$, $H_e \leq \gamma G_y \gamma^{-1}$, for some $y \in E(Y)$, $\gamma \in \Gamma$.*

Proof.

Γ acts on a tree T with quotient a graph of groups (G, Y). The subgroup B acts on T, $\mathrm{Stab}_B(v) \leq \mathrm{Stab}_\Gamma(v)$ for all $v \in V(T)$ and $\mathrm{Stab}_B(e) \leq \mathrm{Stab}_\Gamma(e)$ for all $e \in E(T)$. □

Subgroups

Theorem (Kurosh)

Suppose $G = G_1 * ... * G_n$. *If* $H \leq G$ *then*

$$H = (*_{i \in I} H_i) * F$$

where I *is finite or countable,* F *is a free group and the* H_i *are subgroups of conjugates of* G_j.

Unique decomposition I

We say that G is indecomposable if $G \neq A * B$.

Theorem (Grushko)

Suppose G is finitely generated. There exists indecomposable $G_1, \dots, G_k$ such that

$$G = G_1 * \dots * G_k * F_n$$

Moreover, if there exist other indecomposable $H_1, \dots, H_m$ such that

$$G = H_1 * \dots * H_m * F_r$$

then $m = k$, $r = n$ and, after reordering, H_i is conjugate to G_i for all i.

Unique decomposition II

Theorem (Dunwoody)

Suppose Γ is finitely presented. Then Γ can be written as $\pi_1(G, Y, a_0)$ where (G, Y) is a finite graph of groups such that all edge groups are finite and all the G_v do not split over finite groups.

Quasi-isometry

Definition

Let $f : X \to Y$ be a map between metric spaces.

1. We say that f is an (L, A)-quasi-isometric embedding if for some constants $L \geq 1$, $A \geq 0$ and for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A$$

   It is called a quasi-isometry if moreover we have that for all $y \in Y$, there exists some $x \in X$ such that $d(y, f(x)) \leq A$.

2. If $I \subseteq \mathbb{R}$ is an interval, then an (L, A)-quasi-isometric embedding $\gamma : I \to X$ is called an (L, A)-quasi-geodesic.
3. If there exists a quasi-isometry $f : X \to Y$ between two metric spaces then we say that X and Y are quasi-isometric.

Quasi-isometry

Examples

1. *$\mathbb{Z}^2$ and $\mathbb{R}^2$ are quasi-isometric.*
2. *If G is a finitely generated group with finite generating sets S, S' then the Cayley graphs $\Gamma(S, G)$, $\Gamma(S', G)$ are quasi-isometric.*
3. *If T_n is the n-valent tree, then $T_n \sim T_3$ for all $n \in \mathbb{N}$.*

The following theorem implies the first example above and is our main source of quasi-isometries.