Geometric Group Theory

Cornelia Druţu

University of Oxford

Part C course HT 2023, Oxford

Quasi-isometry

Definition

Let $f : X \to Y$ be a map between metric spaces.

1. f is an (L, A)-quasi-isometric embedding, for $L \geq 1$, $A \geq 0$ if for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A.$$

   It is called a quasi-isometry if moreover for every $y \in Y$, there exists $x \in X$ such that $d(y, f(x)) \leq A$.

2. If $I \subseteq \mathbb{R}$ is an interval, then an (L, A)-quasi-isometric embedding $\gamma : I \to X$ is called an (L, A)-quasi-geodesic.

3. If there exists a quasi-isometry $f : X \to Y$ between two metric spaces then we say that X and Y are quasi-isometric.

Quasi-isometry

Example

If T_n is the n-valent tree, then $T_n \sim T_3$ for all $n \in \mathbb{N}$.

The following theorem is our main source of quasi-isometries.

Theorem (Milnor–Švarc)

Suppose G acts by isometries on a metric space X such that

1.
   a. *X is geodesic;*
   b. *X is proper (closed balls are compact);*
2. *the action is*
   a. *properly discontinuous: i.e. given a compact $K \subseteq X$, the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite;*
   b. *cocompact: i.e. there exists a compact $K' \subseteq X$ such that $GK' = X$;*

   then G is finitely generated and every orbit map $G \to X$, $g \mapsto g \cdot x_0$ is a quasi-isometry when G is endowed with a word metric.

Proof is non-examinable

Quasi-isometry

Corollary

Suppose G is a finitely generated group with some word metric.

1. *If $H \leq G$ is a finite index subgroup then H is quasi-isometric to G.*
2. *If $N \lhd G$ is a finite normal subgroup then G is quasi-isometric to G/N.*
3. *Suppose M is a compact Riemannian manifold. Then $\pi_1(M)$ is quasi-isometric to the universal cover $\tilde{M}$.*

Hyperbolic space

Definition

Let X be a geodesic metric space. Given $A \subseteq X$ and $r > 0$, the r-(closed) neighbourhood of A in X is the subset

$$\mathcal{N}_r(A) = \{x \in X : d(x, A) \leq r\} \subseteq X.$$

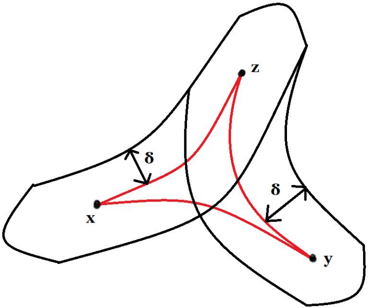
Let $x, y, z \in X$. A geodesic triangle $[x, y, z]$ in X is the union of three geodesic paths $[x, y], [y, z], [z, x]$:

$$[x, y, z] = [x, y] \cup [y, z] \cup [z, x]$$

We say that a geodesic triangle $[x, y, z]$ is δ-slim for some $\delta \geq 0$ if each side is within a δ-neighbourhood of the other two sides: for example $[x, y] \subseteq \mathcal{N}_\delta([y, z] \cup [z, x])$.

Slim triangles. Hyperbolic spaces

We say that a geodesic triangle $[x, y, z]$ is δ-slim for some $\delta \geq 0$ if each side is within a δ-neighbourhood of the other two sides: for example $[x, y] \subseteq \mathcal{N}_\delta([y, z] \cup [z, x])$.



We say that X is δ-hyperbolic if every geodesic triangle is δ-slim.

Thin triangles. Hyperbolic spaces

Equivalently:

If $\Delta = [x, y, z]$ is a triangle then:

- there exists a tripod $T_\Delta = [x', y'] \cup [y', z'] \cup [x', z']$;
- there exists an onto map $f_\Delta : \Delta \to T_\Delta$ which restricts to an isometry from each side $[x, y], [y, z], [x, z]$ to the corresponding side $[x', y'], [y', z'], [x', z']$ in T_Δ.

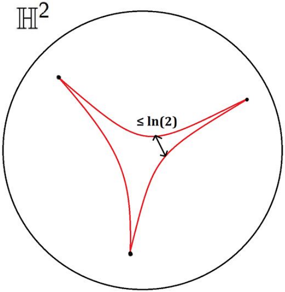
We say that Δ is δ-thin if for every $t \in T_\Delta = [x', y', z']$, $\mathrm{diam}(f_\Delta^{-1}(t)) \leq \delta$.

We say that X is δ-hyperbolic if every geodesic triangle is δ-thin.

Examples of δ-hyperbolic spaces

Examples

1. *Any tree is* 0*-hyperbolic.*
2. *Any metric space* X *with finite diameter is* δ*-hyperbolic (take* δ *to be the diameter of* X*).*
3. $\mathbb{R}^2$ *is not hyperbolic.*
4. $\mathbb{H}^2$ *is* $\ln(2)$*-hyperbolic; more generally,* $\mathbb{H}^n$ *with* $n \geq 2$*:*



δ-hyperbolic spaces

Proposition (Morse lemma)

Let X be a δ-hyperbolic metric space. For any $\lambda \geq 1$ and $\mu \geq 0$, there exists some $M = M(\lambda, \mu)$ such that if

- $\alpha : [u, v] \to X$ *is a (λ, μ)-quasigeodesic with endpoints $x = \alpha(u)$, $y = \alpha(v)$;*
- $\gamma = [x, y]$ *is a geodesic with the same endpoints as α;*

then $\alpha \subseteq \mathcal{N}_M(\gamma)$ and $\gamma \subseteq \mathcal{N}_M(\alpha)$.

Corollary

Let X, Y be geodesic metric spaces. If X is δ-hyperbolic and Y is quasi-isometric to X then Y is δ' hyperbolic for some $\delta' \geq 0$.

δ-hyperbolic spaces

Corollary

Let X, Y be geodesic metric spaces. If X is δ-hyperbolic and Y is quasi-isometric to X then Y is δ' hyperbolic for some $\delta' \geq 0$.

Proof.

Let $f : Y \to X$ be a (L, A)-quasi-isometry. For all geodesic triangles Δ in Y, $f(\Delta)$ is a triangle in X with quasigeodesic edges. By Morse Lemma, there exists a geodesic triangle Δ' such that

$$f(\Delta) \subseteq \mathcal{N}_M(\Delta')$$

Since Δ' is δ-slim, $f(\Delta)$ is $(\delta + 2M)$-slim and so Δ is δ'-slim where $\delta' = \delta'(\delta, M, L, A)$. □

Hyperbolic groups

Definition

A finitely generated group G is hyperbolic if some (equivalently, every) Cayley graph is hyperbolic.

Examples

1. *F_k is hyperbolic.*
2. *If $G \curvearrowright \mathbb{H}^n$ by isometries properly discontinuously and cocompactly, then G is hyperbolic.*
3. *Random groups (among finitely presented groups).*

Dehn presentations

Definition

A group G has a Dehn presentation if there exists a finite presentation $G = \langle S | R \rangle$ such that every $w \in F(S)$ with $w =_G 1$ contains more than half of a word in R.

Lemma

Groups with Dehn presentations have solvable word problem.

Procedure: Check if $w \in F(S)$ contains more than half of a word in R.

- If the answer is no, then $w \neq 1$ in G.
- If the answer is yes, then $w = aub$ where $r = uv$ and $|u| > \frac{1}{2}|r| > |v|$. So in G, $w = \underbrace{av^{-1}b}_{w'}$ and $|w'| < |w|$.

The procedure terminates after finitely many steps. □

Hyperbolic groups have Dehn presentations

Theorem

A hyperbolic group has a Dehn presentation. Hence, it is finitely presented and has solvable word problem.

Proof

There exists some $\delta \geq 0$ such that $\mathrm{Cay}(G, S)$ has δ-thin geodesic triangles. WLOG assume that $\delta \in \mathbb{N}$. Consider

$$R = \{w \in F(S) : |w| \leq 10\delta, w =_G 1\}$$

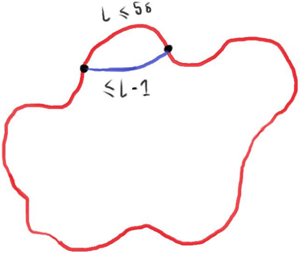
Claim: $\langle S|R\rangle$ is a Dehn presentation.

Hyperbolic groups have Dehn presentations

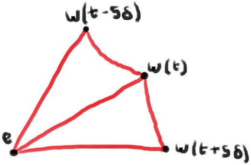
$$R = \{w \in F(S) : |w| \leq 10\delta, w =_G 1\}$$

Claim: $\langle S|R\rangle$ is a Dehn presentation.

Take $w = 1$ in G. It labels a closed path in $\mathrm{Cay}(G, S)$ of length n. Let $w(0) = e, w(1), ..., w(n-1)$ be the vertices of this path. If there exists a subpath of length $\leq 5\delta$ which is not geodesic then we are done.



Otherwise, take $w(t)$ such that $d(e, w(t))$ is maximal. Consider the geodesic triangles of vertices $[e, w(t), w(t-5\delta)]$ and $[e, w(t), w(t+5\delta)]$.



We have that $d(w(t \pm 5\delta), e) \leq d(w(t), e)$. Therefore, since both the triangles are δ-thin,

$$d(w(t-2\delta), w(t+2\delta)) \leq 2\delta$$

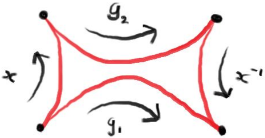
and so $w\big|_{[t-2\delta, t+2\delta]}$ is not geodesic. Contradiction. □

Lemma

Let $G = \langle S|R\rangle$ be a δ-hyperbolic group. If $g_1, g_2 \in G$ are conjugate then $g_1 = xg_2x^{-1}$ for some x with $|x| \leq (2|S|)^{2\delta+|g_1|} + |g_2|$.

Proof

Let x be of minimal length such that $g_1 = xg_2x^{-1}$.



The bound follows from minimality and thinness of the quadrilateral. □

Hyperbolic groups

Lemma

Let $G = \langle S|R \rangle$ be a δ-hyperbolic group. If $g_1, g_2 \in G$ are conjugate then $g_1 = xg_2x^{-1}$ for some x with $|x| \leq (2|S|)^{2\delta+|g_1|} + |g_2|$.

Corollary

The conjugacy problem is solvable for hyperbolic groups.

Proof.

Given $w_1, w_2 \in F(S)$, check whether $w_2 = xw_1x^{-1}$ in G for all $x \in F(S)$ with $|x| \leq (2|S|)^{2\delta+|w_1|} + |w_2|$. Can be done, because solvable word problem. □

Theorem (Sela–Guirardel–Dahmani)

The isomorphism problem is solvable for hyperbolic groups.

More results

Theorem

Let G be an infinite hyperbolic group which is not virtually $\mathbb{Z}$. Then G contains a free subgroup of rank 2.

Theorem

Let G be a hyperbolic group and let $g_1, ..., g_n \in G$. Then there is some $N > 0$ such that the group $\langle g_1^N, ..., g_n^N \rangle$ is free.

Theorem (Sela)

Torsion-free hyperbolic groups are Hopf.

Open questions

There are a number of open questions about hyperbolic groups:

- Are all hyperbolic groups residually finite?
- Let G be hyperbolic. Does G have a torsion-free subgroup of finite index?
- M. Gromov has conjectured that if G is torsion-free hyperbolic then G has finitely many torsion-free finite extensions.