Further Partial Differential Equations (2023)
Problem Sheet 4

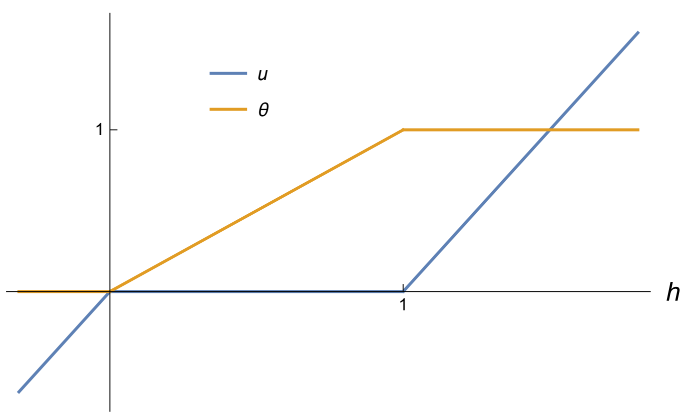

Figure 1: Normalised temperature u and liquid fraction θ versus enthalpy h.

1. **Enthalpy for mushy layers**

   Show that the free boundary problem (2.31) may be posed as

$$\frac{\partial h}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q,$$

   where $h = \mathrm{St}u + \theta$ is the (dimensionless) *enthalpy*. Deduce that u is a piecewise linear function of h, as indicated in Figure 1.

   Solution

   This is obtained straightforwardly by substituting in.

2. **Unsteady electropainting**

Consider the unsteady version of the model problem depicted in Figure 2.9, i.e., with the conditions on $y = 0$ replaced by

$$\frac{\partial \phi}{\partial y} = \frac{\phi}{h}, \quad \frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta \qquad y = 0, \ |x| < c, \tag{1}$$

$$\phi = 0 \qquad y = 0, \ |x| > c, \tag{2}$$

where now $c = c(t)$.

(a) By considering the set-up at $t = 0$, show how the boundary conditions (1) simplify and hence find the solution for ϕ at $t = 0$ using the method of images or otherwise.

(b) By substituting this solution into (1) find the early time behaviour for h and thus show that painting commences provided $\delta < 1/\pi$, in which case the layer initially grows over a half-width $c_0 = \sqrt{1/(\delta\pi) - 1}$.

Solution

The unsteady problem is described by

$$\nabla^2 \phi = 0 \tag{3}$$

with

$$\frac{\partial \phi}{\partial y} = \frac{\phi}{h}, \qquad \frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta, \qquad y = 0, \quad |x| < c, \tag{4}$$

$$\phi = 0 \qquad y = 0 \quad |x| > c, \tag{5}$$

$$\phi \sim -\frac{1}{4\pi} \log\left(x^2 + (y-1)^2\right) \qquad \text{as} \quad (x, y) \to (0, 1). \tag{6}$$

(a) At $t = 0$, $h = 0$ so (4) gives $\phi = 0$ and so we have

$$\nabla^2 \phi = 0 \tag{7}$$

with

$$\phi = 0 \qquad y = 0, \tag{8}$$

$$\phi \sim -\frac{1}{4\pi} \log\left(x^2 + (y-1)^2\right) \qquad \text{as} \quad (x, y) \to (0, 1). \tag{9}$$

The solution to this problem is

$$\phi = \frac{1}{4\pi} \log\left(\frac{x^2 + (y+1)^2}{x^2 + (y-1)^2}\right), \tag{10}$$

using the method of images.

(b) So the growth is initially given by

$$\frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta \tag{11}$$

$$= \frac{1}{\pi(1 + x^2)} - \delta, \tag{12}$$

and so

$$h(x, t) \sim \left(\frac{1}{\pi(1 + x^2)} - \delta\right) t. \tag{13}$$

This is valid provided $h \geq 0$ so

$$\frac{1}{\pi(1 + x^2)} \geq \delta \qquad \Rightarrow \qquad |x| \leq \sqrt{\frac{1}{\delta\pi} - 1} \tag{14}$$

as required.

3. **One-dimensional welding**

(a) Derive the dimensionless one-dimensional welding problem (2.31).

(b) Show that the normalised heating coefficient is given by

$$q = \frac{a^2 J^2}{\sigma k(T_{\mathrm{m}} - T_0)} = \frac{\sigma V^2}{k(T_{\mathrm{m}} - T_0)},$$

where V is the applied voltage. Assuming that we require $q = O(1)$ to melt the plate, roughly how high must the voltage be to achieve melting?

Solution

(a) The dimensional problem is

$$
\begin{aligned}
\rho c \frac{\partial T}{\partial t} &= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{J^2}{\sigma} && 0 \le x \le a, \\
\frac{\partial T}{\partial x} &= 0 && \text{on} \quad x = 0, t > 0, \\
T &= T_0 (< T_{\mathrm{m}}) && \text{on} \quad x = a, t > 0, \\
T &= T_0 (< T_{\mathrm{m}}) && 0 < x < a, t = 0.
\end{aligned}
$$

Non-dimensionalize via

$$
\begin{aligned}
T &= T_{\mathrm{m}} + (T_m - T_0)\, u, \\
x &= a x', \\
t &= \left(\frac{\rho L a^2}{k(T_{\mathrm{m}} - T0)}\right) t'.
\end{aligned}
$$

This gives the dimensionless problem (2.31) with

$$
q = \frac{J^2 a^2}{k\sigma(T_{\mathrm{m}} - T_0)}.
$$

$J =$ current per unit area $= I/A$.
$V = IR$.
$R = a/\sigma A$ where a is the length of the material.
So $J = V\sigma/a$. So

$$
q = \frac{V^2 \sigma}{k(T_{\mathrm{m}} - T_0)}.
$$

We need $q = O(1)$ for a chance to melt the plate, so we need

$$
V \gtrsim \sqrt{\frac{k(T_{\mathrm{m}} - T_0)}{\sigma}}.
$$

4. **OPTIONAL (will not be marked) One-dimensional welding**

(a) Derive the dimensionless one-dimensional welding problem (2.31).

(b) Show that the normalised heating coefficient is given by

$$q = \frac{a^2 J^2}{\sigma k (T_\mathrm{m} - T_0)} = \frac{\sigma V^2}{k(T_\mathrm{m} - T_0)},$$

where V is the applied voltage. Assuming that we require $q = O(1)$ to melt the plate, roughly how high must the voltage be to achieve melting?

(c) Consider the dimensionless one-dimensional welding problem (2.31). Show that, before melting occurs, the solution is given by

$$u(x,t) = -1 + \frac{q}{2}\left(1 - x^2\right) + \sum_{n=0}^{\infty} c_n \cos\left[\left(n + \tfrac{1}{2}\right)\pi x\right] \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 \pi^2 t/\mathrm{St}} \tag{15}$$

and use Fourier series to evaluate the constants c_n.

(d) Deduce that the sample will eventually melt provided $q > 2$, at a time t_m that satisfies

$$q = \left(\frac{1}{2} - 2\sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 \pi^2 t_\mathrm{m}/\mathrm{St}}}{\left(n + \frac{1}{2}\right)^3 \pi^3}\right)^{-1}. \tag{16}$$

(e) Show that the leading-order asymptotic dependence of equation (16) between t_m/St and q is

$$\frac{t_\mathrm{m}}{\mathrm{St}} \sim \frac{1}{q} \qquad \text{as} \quad t_\mathrm{m}/\mathrm{St} \to 0,$$
$$\frac{t_\mathrm{m}}{\mathrm{St}} \sim \frac{4}{\pi^2} \log\left(\frac{64}{\pi^3(q-2)}\right) \qquad \text{as} \quad t_\mathrm{m}/\mathrm{St} \to \infty.$$

(Hint: for the second limit, split up the summation (16) *into* $0 \le n \le m$ *and* $m \le n < \infty$ *where* $m^2 t_\mathrm{m}/St \ll 1$ *and* $m \gg 1$.*)*

(f) For $t > t_\mathrm{m}$, consider the free boundary problem (2.31). Explain why $s_2(t) = 0$ until $t = t_\mathrm{m} + 1/q$.

(g) Now consider the limit $\mathrm{St} \to 0$. Show that the plate will have melted entirely to a depth $x = 1 - \sqrt{2/q}$ (so the mush has disappeared) after a time $t_\mathrm{c} \sim t_\mathrm{m} + 1/q + O(\mathrm{St})$.

(h) Show that the subsequent leading-order behaviour of the solid–liquid free boundary $x = s(t)$ is governed by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{q}{2}(1 + s) - \frac{1}{1 - s}, \qquad s(t_\mathrm{c}) = 1 - \sqrt{\frac{2}{q}}.$$

(i) Deduce that the solid ahead of the free boundary is not superheated, and that the system approaches a steady state with the plate melted to a depth $x = \sqrt{1 - 2/q}$.

Solution

(a) The dimensional problem is

$$\begin{aligned}
\rho c\frac{\partial T}{\partial t} &= \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{J^2}{\sigma} && 0\le x\le a,\\
\frac{\partial T}{\partial x} &= 0 && \text{on}\quad x=0, t>0,\\
T &= T_0(<T_{\mathrm{m}}) && \text{on}\quad x=a, t>0,\\
T &= T_0(<T_{\mathrm{m}}) && 0<x<a, t=0.
\end{aligned}$$

Non-dimensionalize via

$$\begin{aligned}
T &= T_{\mathrm{m}}+(T_m-T_0)\,u,\\
x &= ax',\\
t &= \left(\frac{\rho L a^2}{k(T_{\mathrm{m}}-T_0)}\right)t'.
\end{aligned}$$

This gives the dimensionless problem (2.31) with

$$q=\frac{J^2a^2}{k\sigma(T_{\mathrm{m}}-T_0)}.$$

$J=$ current per unit area $=I/A$.
$V=IR$.
$R=a/\sigma A$ where a is the length of the material.
So $J=V\sigma/a$.

(b) From (a) we have

$$q=\frac{V^2\sigma}{k(T_{\mathrm{m}}-T_0)}.$$

We need $q=O(1)$ for a chance to melt the plate, so we need

$$V\gtrsim\sqrt{\frac{k(T_{\mathrm{m}}-T_0)}{\sigma}}.$$

(c) A particular solution to (2.31) is $u_p=-1+q/2(1-x^2)$. We then seek a solution $u=u_p+v$ where v satisfies

$$\begin{aligned}
\mathrm{St}\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2}, && 0\le x\le 1, && (17)\\
\frac{\partial v}{\partial x} &= 0 && \text{on}\quad x=0, && (18)\\
v &= 0 && \text{on}\quad x=1, && (19)\\
v &= -\frac{q}{2}\left(1-x^2\right) && \text{at}\quad t=0. && (20)
\end{aligned}$$

Separation of variables gives the general homogeneous solution to this problem as

$$v(x,t)=\sum_{n=0}^{\infty}c_n\cos((n+1/2)\pi x)\exp\left(-(n+1/2)^2\pi^2 t/\mathrm{St}\right)$$

where

$$c_n=-q\int_0^1(1-x^2)\cos((n+1/2)\pi x)=-\frac{2q(-1)^n}{(n+1/2)^3\pi^3}.$$

is obtained by multiplying $v(x,t)$ by $\cos((m+1/2)\pi x)$ and integrating using the initial condition (20).

(d) The sample will melt if $u = 0$. The first place that this happens will be at $x = 0$. Here,

$$u = \frac{q}{2} - 1 - \sum_{n=0}^{\infty} \frac{2q(-1)^n}{(n+1/2)^3\pi^3} \exp(-(n+1/2)^2\pi^2 t/\mathrm{St}) \tag{21}$$

As $t \to \infty$, $u \to q/2 - 1$ so we certainly need $q > 2$. Setting $u = 0$ in (21) and rearranging gives (16).

(e) When $t_{\mathrm{m}}/\mathrm{St} \gg 1$ we retain only the first term in the exponential, which gives

$$\frac{1}{q} = \frac{1}{2} - \frac{16}{\pi^2} \exp\left(-\pi^2 t_{\mathrm{m}}/4\mathrm{St}\right), \tag{22}$$

which may be rearranged to give

$$\frac{t_{\mathrm{m}}}{\mathrm{St}} \sim \frac{4}{\pi^2} \log\left(\frac{32q}{\pi^3(q-2)}\right) \sim \frac{4}{\pi^2} \log\left(\frac{64}{\pi^3(q-2)}\right) \qquad \text{as} \quad t_{\mathrm{m}}/\mathrm{St} \to \infty \tag{23}$$

since $q \sim 2$ as $t_{\mathrm{m}}/\mathrm{St} \to \infty$. When $t_{\mathrm{m}}/\mathrm{St} \ll 1$ we split up the summation into $0 \le n \le m$ and $m \le n < \infty$ where $m^2 t_{\mathrm{m}}/St \ll 1$ and $m \gg 1$. Then in the first summation we can expand the exponential while we can neglect the second summation since it is $O(1/m^3)$. This gives

$$\frac{1}{q} = \frac{1}{2} - 2\sum_{n=0}^{m} \frac{(-1)^n}{(n+1/2)^3\pi^3} + 2\sum_{n=0}^{m} \frac{(-1)^n (n+1/2)^2\pi^2}{(n+1/2)^3\pi^3} \frac{t_{\mathrm{m}}}{\mathrm{St}}. \tag{24}$$

Taking the limit as $m \to \infty$ gives

$$\frac{1}{q} = \frac{1}{2} - 2 \times \frac{1}{4} + 2 \times \frac{1}{2}\frac{t_{\mathrm{m}}}{\mathrm{St}}, \tag{25}$$

and so

$$\frac{t_{\mathrm{m}}}{\mathrm{St}} \sim \frac{1}{q} \qquad \text{as} \quad t_{\mathrm{m}}/\mathrm{St} \to 0. \tag{26}$$

(f) In the mushy region, $\partial\theta/\partial t = q$ so θ takes a time $1/q$ to go from $\theta = 0$ to $\theta = 1$ when a purely liquid region exists.

(g) When all melting is done the mushy layer disappears and we are left with just solid and liquid and an interface $x = s$. In the solid we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= -q, && \text{in} \quad s(t) \le x \le 1, \\ u &= -1, && \text{on} \quad x = 1, \\ u &= 0, && \text{on} \quad x = s(t), \\ \frac{\partial u}{\partial x} &= 0, && \text{on} \quad x = s(t), \end{aligned}$$

which gives $u = -q(x-s)^2/2$ and $s = 1 - \sqrt{2/q}$ as required.

(h) When all melting is done the mushy layer disappears and we are left with just solid and liquid and we have reached the previous state we are reduced to solving a regular

Stefan problem again:

$$\frac{\partial^2 u}{\partial x^2} = -q \qquad 0 \le x \le s(t), \tag{27}$$
$$\frac{\partial^2 u}{\partial x^2} = -q \qquad s(t) \le x \le 1, \tag{28}$$
$$\frac{\partial u}{\partial x} = 0 \qquad x = 0, \tag{29}$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x}, \qquad x = s(t) \tag{30}$$
$$u^+ = u^- = 0 \qquad x = s(t), \tag{31}$$
$$u = -1, \qquad x = 1. \tag{32}$$

This gives

$$u = \frac{q}{2}(s^2 - x^2) \qquad 0 \le x \le s(t), \tag{33}$$
$$u = (s - x)\left[\frac{1}{1-s} + \frac{q}{2}(x-1)\right], \qquad s(t) \le x \le 1 \tag{34}$$

and so

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{q}{2}(1+s) - \frac{1}{1-s}, \tag{35}$$

which finally gives

$$s = 1 - \sqrt{\frac{2}{q}} \qquad \text{at} \quad t = 0 \tag{36}$$

as required.

(i) The system is superheated if $\partial u^+/\partial x > 0$ at $x = s^+$. Now

$$\left.\frac{\partial u^+}{\partial x}\right|_{x=s^+} = -\frac{1}{1-s} + \frac{1}{2}q(1-s) \tag{37}$$
$$= (1-s)\left[\frac{q}{2} - \frac{1}{(1-s)^2}\right]. \tag{38}$$

Now $s > 1 - \sqrt{2/q}$ for all time, so $q/2 - 1/(1-s)^2 < 0$ and $1 - s^2 > 0$ and therefore $\partial u^+/\partial x < 0$ and the system is not superheated.

As $t \to \infty$, $\mathrm{d}s/\mathrm{d}t \to 0$ so

$$\frac{q}{2}(1+s) = \frac{1}{1-s} \qquad \Rightarrow \qquad s = \sqrt{1 - \frac{2}{q}} \tag{39}$$

as required.