O1 History of Mathematics
Lecture XI
19th-century rigour in real analysis, part 1

Monday 14th November 2016
(Week 6)

Summary

- New difficulties emerge
- Continuity and convergence
- Integration
- The Fundamental Theorem of Calculus
- New ideas about integration

Recall from lecture VIII: Fourier series, 1822

Joseph Fourier, *Théorie analytique de la chaleur* [*Analytic theory of heat*] (1822):

Suppose that $\phi(x) = a \sin x + b \sin 2x + c \sin 3x + \cdots$

and also that $\phi(x) = x\phi'(0) + \frac{1}{6}x^3\phi'''(0) + \cdots$

After many pages of calculations, multiplying and comparing power series, Fourier found that the coefficient of $\sin nx$ must be

$$\frac{2}{\pi}\int_0^\pi \phi(x) \sin nx \, dx$$

Fourier's derivation was based on 'naive' manipulations of infinite series. It was ingenious but non-rigorous, shaky.

BUT it led to profound results

New doubts in the early 19th century

Fourier's work converged with more philosophical investigation to stimulate questions concerning:

- functions — what exactly should they be?
- convergence — what exactly should it be?
- convergence of functions — what properties are preserved?
- integration — what exactly should it be?
- existence of limits — what are the essential properties of real numbers? [Lecture XII]

Recall from Lecture VIII: Cauchy sequences, 1821

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0,\ u_1,\ u_2, \ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit* s*: in other words, it is necessary and sufficient that for infinitely large values of the number* n*, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit* s*, and consequently from each other, by infinitely small quantities.*

A theorem of Cauchy (1821)

Cauchy, *Cours d'analyse*, pp. 131–132:

> *When the various terms of a series are functions of a variable x, continuous with respect to this variable in the neighbourhood of a particular value for which the series is convergent, the sum s of the series is also, in the neighbourhood of this value, a continuous function of x.*

Abel's counterexample, a footnote in his article on the binomial series, Crelle's Journal, 1826: the series

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

is discontinuous for every value $(2m+1)\pi$ of x, for integer m.

Which type of continuity? Which type of convergence?

For continuity of s, we need:
$\forall \varepsilon > 0 : \exists \delta : \forall a : |a| < \delta \Rightarrow |s(x+a) - s(x)| < \varepsilon$

Certainly, each partial sum s_n is continuous:
$\forall \varepsilon > 0 : \exists \delta : \forall a : |a| < \delta \Rightarrow |s_n(x+a) - s_n(x)| < \varepsilon$

The series converges, so $r_n = s - s_n$ becomes arbitrarily small and remains so for small increments of x:
$\forall \varepsilon > 0 : \exists N : \forall n : n > N \Rightarrow |r_n(x)| < \varepsilon \wedge |r_n(x+a)| < \varepsilon$

We have $s(x+a) = s_n(x+a) + r_n(x+a)$, hence:

$$\begin{aligned} |s(x+a) - s(x)| &= |s_n(x+a) + r_n(x+a) - s_n(x) - r_n(x)| \\ &\leq |s_n(x+a) - s_n(x)| + |r_n(x+a)| + |r_n(x)| \\ &\leq 3\varepsilon \end{aligned}$$

But the N above depends on ε, x and a — we need this to be independent of x, i.e., we need uniform convergence...

Conditions for the theorem to work

The need for uniform convergence was gradually recognised:

- Karl Weierstrass (lectures in Berlin), 1841;
- Emmanuel Björling (Uppsala), 1846 (some doubt and debate);
- Gabriel Stokes (Cambridge), 1847;
- Phillip Seidel (Berlin), 1848.

See: G. H. Hardy, ‘Sir George Stokes and the concept of uniform convergence’, *Proc. Camb. Phil. Soc.* **19** (1918), 148–156 (also: *Collected Papers of G. H. Hardy*, vol. VII, 505–513)

G. H. Hardy (1918)

The discovery of the notion of uniform convergence is generally and rightly attributed to Weierstrass, Stokes, and Seidel. The idea is present implicitly in Abel's proof of his celebrated theorem on the continuity of power series; but the three mathematicians mentioned were the first to recognise it explicitly and formulate it in general terms. * *Their work was quite independent. ... Each, as it happens has some special claim to recognition. Weierstrass's discovery was the earliest, and he alone fully realised its far-reaching importance ... Stokes has the actual priority of publication; and Seidel's work is but a year later and, while narrower in its scope than that of Stokes, is even sharper and clearer.*

** The idea was rediscovered by Cauchy, five or six years after the publication of the work of Stokes and Seidel.*

Integration

- Recall that in the 17th century, ‘integration’ was designed for ‘quadrature’, for measuring space or calculating area
- in the 18th century, ‘integration’ was essentially regarded as the inverse of differentiation

Integration in the 18th century (1)

Leonhard Euler, *Institutiones calculi integralis* [*Foundations of integral calculus*] (1768):

Definition 1: Integral calculus is the method of finding, from a given relationship between differentials, a relationship between the quantities themselves: and the operation by which this is carried out is usually called integration.

Corollary 1: Therefore where differential calculus teaches us to investigate the relationship between differentials from a given relationship between variable quantities, integral calculus supplies the inverse method.

Corollary 2: Clearly just as in Analysis two operations are always contrary to each other, as subtraction to addition, division to multiplication, extraction of roots to raising of powers, so also by similar reasoning integral calculus is contrary to differential calculus.

(See: *Mathematics emerging*, §14.2.1.)

Integration in the 18th century (2)

Definition 2: Since the differentiation of any function of x has a form of this kind: $X\partial x$, when such a differential form $X\partial x$ is proposed, in which X is any function of x, that function whose differential $= X\partial x$ is called its integral, and is usually indicated by the prefix $\int$, so that $\int X\partial x$ denotes that variable quantity whose differential $= X\partial x$.

Corollary 2: Therefore just as the letter ∂ is the sign of differentiation, so we use the letter $\int$ as the sign of integration, and thus these two signs are mutually contrary to each other, as though they destroy each other: certainly $\int \partial X = X$, because by the former is denoted the quantity whose differential is ∂X, which in both cases is X.

Integration in the 18th century (3)

Corollary 3: Therefore since the differentials of these functions of x

$$x^2, \qquad x^n, \qquad \sqrt{(aa - xx)}$$

are

$$2x\partial x, \qquad nx^{n-1}\partial x, \qquad \frac{-x\partial x}{\sqrt{(aa - xx)}}$$

then adjoining the sign of integration $\int$, they are seen to become:

$$\int 2x\partial x = xx; \quad \int nx^{n-1}\partial x = x^n; \quad \int \frac{-x\partial x}{\sqrt{(aa - xx)}} = \sqrt{(aa - xx)}$$

whence the use of this sign is more clearly seen.

Some 19th-century ideas

Recall that Fourier coefficients are given by $\dfrac{2}{\pi}\displaystyle\int_0^\pi \phi(x)\sin nx\,dx$.

It is not always possible to solve such an integral algebraically.

Fourier (1822): but we can draw the curve of $\phi(x)$, and hence that of $\phi(x)\sin nx$, under which there is clearly an area.

Fourier thus returned to the idea of integral as area and influenced Cauchy almost immediately...

A theory of definite integrals (1823)

Augustin-Louis Cauchy, *Résumé des leçons ... sur le calcul infinitésimal* [*Summary of lessons ... on the infinitesimal calculus*], 1823, Lesson 21:

> *Suppose* $f(x)$ *continuous between* $x = x_0$ *and* $x = X$. *Choose* $x_1, x_2, \ldots, x_{n-1}$ *between these limits. Define*
>
> $$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

[much discussion of dependence on partition followed by:]

> *If the numerical values of the elements are made to decrease indefinitely by increasing their number, the value of* S *will become essentially constant, or in other words, it will finish by attaining a certain limit which will depend only on the form of the function* $f(x)$ *and the boundary values* $x = x_0$, $x = X$ *given to the variable* x. *This limit is what one calls a definite integral.*

Cauchy and the Fundamental Theorem of Calculus

Cauchy's 26th lesson:

> *If in the definite integral* $\int_{x_0}^{X} f(x)\,dx$ *one makes one of the two limits vary, for example the quantity* X*, the integral itself will vary with this quantity; and if one replaces the variable limit* X *by* x*, there results a new function of* x*, ... Let*
>
> $$\mathcal{F}(x) = \int_{x_0}^{x} f(x)\,dx$$
>
> *be this new function.*

Proves that $\mathcal{F}'(x) = f(x)$, and also

- that if $\varpi'(x) = 0$, then $\varpi(x) = \text{const.}$, from which is derived
- that if $F'(x) = f(x)$, then $\int_{x_0}^{X} f(x)\,dx = F(X) - F(x_0)$.

The Fundamental Theorem of Calculus

What is the Fundamental Theorem of Calculus?

- integration is the inverse of differentiation?
- integration 'as a sum' is the same as integration 'by rule'?
- Newton's integration is the same as Leibniz's integration?
- Cauchy's integration is the same as Euler's integration?
- 18th-century integration is the same as 17th-century integration?
- 19th-century integration is the same as 18th-century integration?

Riemann's integral (1853)

Here function $f(x)$ is no longer required to be continuous on $[a, b]$. As Cauchy, take $x_1 < x_2 < \cdots < x_{n-1}$. Define $\delta_1 := x_1 - a$, $\delta_2 := x_2 - x_1$, ..., $\delta_n := b - x_{n-1}$. Choose numbers ε_i between 0 and 1. Then define

$$S := \delta_1 f(a + \varepsilon_1\delta_1) + \delta_2 f(x_1 + \varepsilon_2\delta_2) \\ + \delta_3 f(x_2 + \varepsilon_3\delta_3) + \cdots + \delta_n f(a_{n-1} + \varepsilon_n\delta_n)$$

If this has the property that it comes infinitely close to a fixed value A when all the δ_i become infinitely small, then this is the value of $\int_a^b f(x)\,dx$.

Note: many variants over the years, all called Riemann integral

Lebesgue's integral (1901)

Uses measure theory (Borel, 1894).

Can integrate highly discontinuous functions, such as f where

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Has highly effective theorems for integrating infinite series.

Morphed at the hands of mathematicians and teachers — e.g., Frederic and Marcel Riesz (Budapest), Aubrey Ingleton (Oxford) — into the integral described by using limits of step functions.