The Singular Value Decomposition (SVD)

Theorem: Given $A \in \mathbb{R}^{m \times n}$ there exist orthogonal matrices

$$U = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^{m \times m} \quad (u_i \in \mathbb{R}^m \text{ each } i)$$

$$V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n \times n} \quad (v_i \in \mathbb{R}^n \text{ each } i)$$

such that

$$U^T A V = \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p), p = \min\{m, n\}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$ (the singular values).

$\{u_i\}$ are the left singular vectors,
$\{v_i\}$ are the right singular vectors

Properties of SVD:
if $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0, \sigma_{r+1} = \ldots = \sigma_p = 0$, then

- r= rank(A) = dim$\{Ay : y \in \mathbb{R}^n\}$ = dim(range(A))
- $v_{r+1}, \ldots, v_n \in \mathbb{R}^n$ are an orthonormal basis for ker(A) (=Null(A))
- $u_1, \ldots, u_r \in \mathbb{R}^m$ are an orthonormal basis for range(A)
- $\|A\|_2 = \sigma_1$
- $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2$

Proof: Exercise except all basically follow from any $x \in \mathbb{R}^n$ can be written as $\sum_{i=1}^{n} \alpha_i v_i$ so

$$\begin{aligned}
Ax &= U\Sigma V^T x = U\Sigma\left(\sum_{i=1}^{n} \alpha_i V^T v_i\right) \\
&= U\Sigma \sum_{i=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = U \sum_{i=1}^{n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sigma_i \alpha_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
&= \sum_{i=1}^{n} \alpha_i \sigma_i u_i
\end{aligned}$$

Also there are ‘nearness’ results like

$$\|A - \sum_{i=1}^{s} \sigma_i u_i v_i^T\|_2 \leq \|A - B\|_2$$

for all B of rank s
lead to data compression.

The Golub-Reinsch bidiagonalisation algorithm is used to compute the SVD: available in matlab (svd), but not covered here

A final comment:
For a symmetric matrix which is positive (semi-)definite, the SVD is simply diagonalisation:

$U = V$ and Σ is the diagonal matrix of the (non-negative) eigenvalues:

$$A = U\Sigma U^T \quad \text{or} \quad AU = U\Sigma$$

That is, the columns, u_i of U are the eigenvectors of A and $Au_i = \sigma_i u_i = \lambda_i u_i$

For a symmetrix and indefinite matrix, the singular values are the absolute values of the eigenvalues.