Iterative solution methods for $\boldsymbol{Ax = b}$, $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{n \times n}}$

idea: split $\boldsymbol{A = M - N}$, so easy to solve systems with $\boldsymbol{M}$, then iterate:

Guess $\boldsymbol{x^{(0)}}$

solve $\boldsymbol{Mx^{(k)} = Nx^{(k-1)} + b}$ for $\boldsymbol{k = 1, 2, \ldots}$

basic point: if $\{\boldsymbol{x^{(k)}}\}$ converges (to $\boldsymbol{x}$, say) then

$$\boldsymbol{Mx = Nx + b}, \quad \text{ie. } \boldsymbol{Ax = b}$$

ie. it converges to the solution.

Jacobi's method: $M = \text{diag}(A), \quad (N = A - M)$

In practice: componentwise

for iterates $k = 1, 2, \ldots$
    for rows (equations) $i = 1, \ldots, n$

$$x_i^{(k)} = \frac{1}{a_{i,i}} \left(- \sum_{j=1, j \neq i}^{n} a_{i,j} x_i^{(k-1)} + b_i \right)$$

    endo
endo

better ? use most recently updated value of x_i

for iterates $k = 1, 2, \ldots$

for rows $i = 1, \ldots, n$

$$x_i^{(k)} = \frac{1}{a_{i,i}} \left(-\sum_{j=1}^{i-1} a_{i,j} x_j^{(k)} - \sum_{j=i+1}^{n} a_{i,j} x_j^{(k-1)} + b_i \right)$$

endo

endo

This is Gauss-Seidel iteration: rearranging

$$\sum_{j=1}^{i} a_{ij} x_j^{(k)} = -\sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i$$

which is $\qquad (L + D)x^{(k)} = -Ux^{(k-1)} + b$

when $D =$ diag(A), $L =$ strict lower triangular of A, $U =$ strict upper triangular of A

i.e. Solve $Mx^{(k)} = Nx^{(k-1)} + b, \quad M = L + D$ is achieved by forwards substitution.

better still ? take Gauss-Seidel $x^{(k)}$ iterate and average with $x^{(k-1)}$

$$x_i^{(k)} = \omega \underbrace{\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \frac{1}{a_{ii}}}_{\text{exactly as in Gauss-Seidel}} + (1-\omega) x_i^{(k-1)}$$

$\omega \in \mathbb{R}$ relaxation parameter

$\omega < 1$	gives underelaxation	cautious & slow
$\omega = 1$	Gauss-Seidel	
$\omega > 1$	overelaxation	$\Rightarrow$ Successive Overelaxation Method (SOR)

By rearranging as for Gauss-Seidel: matrix form

$$(D + \omega L) x^{(k)} = \omega b + \left[(1 - \omega) D - \omega U \right] x^{(k-1)}$$

Symmetry sometimes useful to preserve, so if A symmetric ($\Leftrightarrow U = L^T$):
Symmetric SOR (SSOR)

$$\begin{aligned}(D+\omega L)x^{(k-\frac{1}{2})} &= \omega b + \left[(1-\omega)D - \omega U\right] x^{(k-1)} \\ (D+\omega U)x^{(k)} &= \omega b + \left[(1-\omega)D - \omega L\right] x^{(k-\frac{1}{2})}\end{aligned}$$

corresponds to $M = (D+\omega L)D^{-1}(D+\omega U)$ which is symmetric.

Important point: if A sparse then these methods only need use the non-zero entries of A e.g.

$$\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} \text{ becomes } \sum_{\{j<i:a_{ij}\neq 0\}} a_{ij} x_j^{(k)}$$

Convergence of simple iterations:

$$\begin{aligned} Mx^{(k)} &= Nx^{(k-1)} + b \\ \text{and } Ax = b \Rightarrow \quad Mx &= Nx + b \qquad\qquad (A = M - N) \\ \text{so } M(x - x^{(k)}) &= N(x - x^{(k-1)}) \\ x - x^{(k)} &= M^{-1}N(x - x^{(k-1)}) \\ &= (M^{-1}N)^k (x - x^{(0)}) \end{aligned}$$

$M^{-1}N$ is called the iteration matrix

So $\|x - x^{(k)}\| \to 0$ at least if

$$\|(M^{-1}N)^k\| \|x - x^{(0)}\| \leq \underbrace{\|M^{-1}N\|^k}_{\uparrow} \quad \underbrace{\|x - x^{(0)}\|}_{\text{unknown error in initial guess}}$$

$\to 0$ if $\|M^{-1}N\| < 1$

this is a sufficient condition for convergence.

Notation

$\rho(A) =$ max $\{|\lambda| : \lambda$ an eigenvalue of $A\}$
the spectral radius

Theorem

If $M^{-1}N$ is diagonalisable, then $\|x - x^{(k)}\| \to 0$ as $k \to \infty$ for any initial guess $x^{(0)}$ if and only if $\rho(M^{-1}N) < 1$

But diagonalisation is not necessary since even if $\boldsymbol{M^{-1}N}$ is not diagonalisable, it is triangularisable

i.e. $\exists$ triangular matrix $\boldsymbol{T}$ with $\boldsymbol{M^{-1}N = QTQ^T}$ for some orthogonal matrix $\boldsymbol{Q}$ (Schur decomposition).

More useful for our purpose here (only!) is the existence of a *Jordan canonical form*:

$$\boldsymbol{M^{-1}N = XJX^{-1}, J =} \begin{bmatrix} \boldsymbol{J_1} & & \boldsymbol{O} \\ & \ddots & \\ \boldsymbol{O} & & \boldsymbol{J_p} \end{bmatrix}$$

where $\boldsymbol{J_i} \in \mathbb{R}^{\boldsymbol{n_i \times n_i}}$ and $\sum_{\boldsymbol{i=1,\ldots,p}} \boldsymbol{n_i = n}$ with

$$\boldsymbol{J_i =} \begin{bmatrix} \boldsymbol{\lambda_i} & \boldsymbol{1} & & \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{1} \\ & & & \boldsymbol{\lambda_i} \end{bmatrix}$$

This is *a Jordan block*

Thus $(M^{-1}N)^k = (XJX^{-1})^k = XJ^kX^{-1}$ and, as $k \to \infty$

$$(M^{-1}N)^k \to 0 \Leftrightarrow J^k \to 0 \Leftrightarrow J_i^k \to 0 \text{ all } i.$$

That is, we obtain convergence if and only if for every Jordan block, its powers tend to zero as $k \to \infty$.

First consider if $\lambda_i = 0$, then write $J_i = \widehat{J} \in \mathbb{R}^{\widehat{n} \times \widehat{n}}$ and

$$\widehat{J} = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}, \widehat{J^2} = \begin{bmatrix} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

and generally the diagonal of 1's moves up toward the top right for each succesive power. Thus $\widehat{J^{\widehat{n}}} = 0$

Now consider when $\lambda_i \neq 0$: we have

$$\begin{aligned} J_i^k &= \left(\lambda_i I + \widehat{J}\right)^k \\ &= \sum_{r=0}^{k} \binom{k}{r} \widehat{J}^r \, \lambda_i^{k-r} \quad \text{since } I, \widehat{J} \text{ commute} \\ &= \sum_{r=0}^{n_i} \binom{k}{r} \widehat{J}^r \, \lambda_i^{k-r} \\ &\to 0 \quad \text{as } k \to \infty \text{ since } \lambda_i^{k-r} \to 0 \\ &\qquad \text{if and only if } |\lambda_i| < 1 \text{ each } i. \end{aligned}$$

Thus $(M^{-1}N)^k \to 0$ as $k \to \infty \Leftrightarrow |\lambda_i| < 1$ each i, hence convergence since $x - x^{(k)} = (M^{-1}N)^k (x - x^{(0)})$.

Note the powers of J can grow considerably before eventual convergence.