# Practical Example

$$4u_{j,k} - u_{j+1,k} - u_{j-1,k} - u_{j,k+1} - u_{j,k-1} = h^2 f_{j,k}$$

for $j, k = 1, \ldots, n$ with $u_{0,k}, u_{n+1,k}, u_{j,0}, u_{j,n+1}$ given.

$$4u_{j,k} - u_{j+1,k} - u_{j-1,k} - u_{j,k+1} - u_{j,k-1} = h^2 f_{j,k}$$

for $j, k = 1, \ldots, n$ with $u_{0,k}, u_{n+1,k}, u_{j,0}, u_{j,n+1}$ given.

eg.Jacobi iteration for this problem is: guess $u^{(0)}$

for iterates $i = 1, 2, \ldots$
    for $j = 1, \ldots, n$
        for $k = 1, \ldots, n$

$$u^{(i)}_{j,k} = \frac{1}{4}\left[u^{(i-1)}_{j+1,k} + u^{(i-1)}_{j-1,k} + u^{(i-1)}_{j,k+1} + u^{(i-1)}_{j,k-1} + h^2 f_{j,k}\right]$$

        endo
    endo
endo

Proposition for $r, s = 1, \ldots, n$

$$\lambda^{r,s} = \frac{1}{2}(\cos r\pi h + \cos s\pi h)$$

is an eigenvalue of the Jacobi iteration matrix for $A$ with eigenvector $\underline{v}^{r,s}$ having entries

$$v_{jk}^{r,s} = \sin rj\pi h \ \sin sk\pi h$$

Proof direct calculation: see exercises

Remark shows Jacobi iteration converges as $-1 < \lambda^{r,s} < 1$ for each $r, s$ but

$$\rho(\text{Jacobi}) = \frac{1}{2}(\cos \pi h + \cos \pi h) = \cos \pi h = 1 - \frac{\pi^2 h^2}{2} + O(h^4)$$

so very close to 1 for small $h \Rightarrow$ slow convergence.

But recalling (in this notation)

$$\underline{u} - \underline{u}^{(i)} = (M^{-1}N)^i(\underline{u} - \underline{u}^{(0)}) = \sum_{r,s=1}^{n} \alpha_{r,s}(\lambda^{r,s})^i \underline{v}^{r,s}$$

we see that $\lambda^{r,s}$ small $\Rightarrow$ error component in $\underline{v}^{r,s}$ reduces very quickly to zero, so $\underline{u} - \underline{u}^{(i)}$ is quickly dominated by components $\underline{v}^{r,s}$ for which $|\lambda^{r,s}| \simeq 1$.

More interesting for our purpose: relaxed Jacobi: for $\theta A x = \theta b$, $\theta \in \mathbb{R}^+,\ M = D,\ N = (1-\theta)D - \theta(L+U)$

$$\Rightarrow M^{-1}N = (1-\theta)I - \theta D^{-1}(L+U)$$

has eigenvalues $1 - \theta + \frac{\theta}{2}(\cos r\pi h + \cos s\pi h)$

so e.g. for $\theta = \frac{1}{2}$, eigenvalues $\frac{1}{2} + \frac{1}{4}(\cos r\pi h + \cos s\pi h) \in (0,1)$

AND high frequency eigenvectors ($r, s$ large) correspond to small eigenvalues ($\lambda^{r,s} \ll 1$) $\Rightarrow$ a 'smoother'