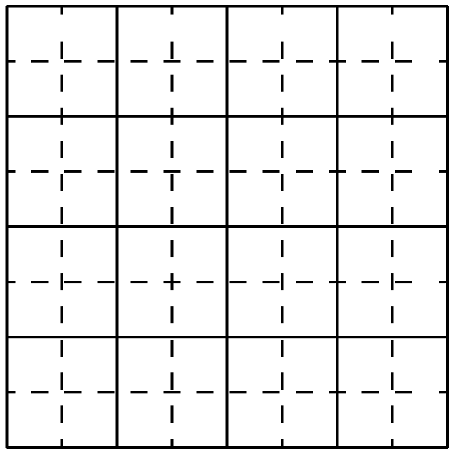
Idea of a 'smoother' leads to Multigrid



guess $\boldsymbol{u}^{\mathbf{0}}$

on fine grid smooth (i.e. for 3 relaxed Jacobi iteration)

$\boldsymbol{u}^{\mathbf{0}} \longrightarrow \boldsymbol{u}^{s}$

$\boldsymbol{u} - \boldsymbol{u}^{s} = \boldsymbol{e}^{s}$ smoother than $\boldsymbol{u} - \boldsymbol{u}^{\mathbf{0}} = \boldsymbol{e}^{\mathbf{0}}$

$$\boldsymbol{A}\boldsymbol{e}^{s} = \boldsymbol{A}\boldsymbol{u} - \boldsymbol{A}\boldsymbol{u}^{s} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{u}^{s} = \boldsymbol{r}^{s} \quad \text{(residual)}$$

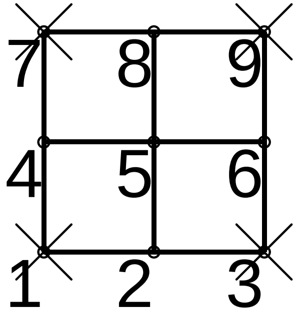
same as original problem, but $\boldsymbol{e}^{s}$ smoother $\Rightarrow$ solve on coarser grid i.e. use a coarse grid representation $\overline{\boldsymbol{A}}$ of $\boldsymbol{A}$ and solve $\overline{\boldsymbol{A}}\overline{\boldsymbol{e}}^{s} = \overline{\boldsymbol{r}}^{s}$ where $\overline{\boldsymbol{e}}^{s}, \overline{\boldsymbol{r}}^{s}$ are coarse grid restrictions of $\boldsymbol{e}^{s}, \boldsymbol{r}^{s}$ respectively.

So need grid transfer operators:

Restriction: fine $\longrightarrow$ coarse

Prolongation: coarse $\longrightarrow$ fine

Prolongation:



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_7 \\ x_9 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}$$

$\uparrow P$

Restriction:

normally $\boldsymbol{R} = \boldsymbol{\alpha P^T}$ where $\boldsymbol{\alpha} \in \mathbb{R}$ is such that

$$\boldsymbol{\alpha P^T P}\mathbf{e} = \mathbf{RPe} = \mathbf{e}$$

$\mathbf{e}$ being the vector of all ones

eg. $\boldsymbol{\alpha} = \frac{\mathbf{4}}{\mathbf{9}}$ for the $\boldsymbol{P}$ above.

Basic point:

$$\begin{array}{rrcll} \text{if} & \boldsymbol{x} & = & \boldsymbol{P}\overline{\boldsymbol{x}} & \qquad \overline{\boldsymbol{x}} \text{ `short'} \\ \text{or} & \overline{\boldsymbol{x}} & = & \boldsymbol{R}\boldsymbol{x} & \qquad \boldsymbol{x} \text{ `long'} \end{array}$$

then little loss of accuracy if and only if $\boldsymbol{x}$ is a 'smooth vector' i.e. a vector of coefficients representing a non-oscillatory function.

Coarse grid operator: $\overline{\boldsymbol{A}}$: 2 possibilities:

(i) 5 point formula on $\boldsymbol{2h}$ mesh

(ii) $\overline{\boldsymbol{A}} = \boldsymbol{RAP} = \boldsymbol{\alpha P^T AP}$
(Galerkin coarse grid operator)

2-grid algorithm:

Choose u_0

$\texttt{for two} - \texttt{grid iterations}\ i = 0\ \texttt{until}$ convergence $\texttt{do}$

$\quad (\texttt{pre}-)\texttt{smooth}:\ u_i \longrightarrow u^s$

$\quad \texttt{calculate residual}:\ r^s = b - Au^s$

$\quad \texttt{restrict residual}:\ r^s \longrightarrow \overline{r}^s \qquad (\overline{r}^s = Rr^s)$

$\quad \texttt{solve}\ \overline{A}\,\overline{e}^s = \overline{r}^s\ \texttt{to get coarse grid correction}$

$\quad \texttt{prolong}:\ \overline{e}^s \longrightarrow e^s \qquad (e^s = P\overline{e}^s)$

$\quad \texttt{update}:\ u_{i+1} \leftarrow u^s + e^s$

$\quad (\texttt{sometimes})\ \texttt{post} - \texttt{smooth}:\ u_{i+1} \longrightarrow u_{i+1}$

$\texttt{enddo}$

Note:

$$u_{i+1} \leftarrow u^s + P\overline{A}^{-1}R(b - Au^s)$$

If smoother is based on a splitting $A = M - N$ then the iteration matrix is $M^{-1}N$ and we have eg. for 2 smoothing steps (so $u^s = u^{(2)}$)

$$\begin{aligned} u^{(1)} &= (M^{-1}N)u^{(0)} + M^{-1}b \\ u^{(2)} &= (M^{-1}N)u^{(1)} + M^{-1}b \\ &= (M^{-1}N)^2u^{(0)} + (I + M^{-1}N)M^{-1}b \qquad (\star) \end{aligned}$$

but also the exact solution satisfies

$$u = (M^{-1}N)u + M^{-1}b$$

$$\Rightarrow u = (M^{-1}N)^2u + (I + M^{-1}N)M^{-1}b \qquad (+)$$

so $(+) - (\star)$ gives

$$u - u^{(2)} = (M^{-1}N)^2(u - u^{(0)}).$$

Note also for the residual using $(\star)$ and $(+)$ we have

$$b - Au^{(2)} = A(u - u^{(2)}) = A(M^{-1}N)^2(u - u^{(0)}).$$

So 2-grid iterate is

$$\begin{aligned} u^{(2)} \quad &+ \quad P\overline{A}^{-1}R(b - Au^{(2)}) \\ &= \quad u^{(2)} + P\overline{A}^{-1}R \;\; A\;(M^{-1}N)^2(u - u^{(0)}) \end{aligned}$$

so that the error after a single 2-grid iteration is

$$u - u^{(2)} - P\overline{A}^{-1}R \;\; A\;(M^{-1}N)^2(u - u^{(0)})$$

$$= (A^{-1} - P\overline{A}^{-1}R) \;\; A\;(M^{-1}N)^2(u - u^{(0)})$$

In general the j^{th} 2-grid iteration (for iterate u_j with $u^{(0)} = u_0$) is

$$u_j = \left[(M^{-1}N)^2 u_{j-1} + (I + M^{-1}N)M^{-1}b\right] + P\overline{A}^{-1}R\left(b - A\left[(M^{-1}N)^2 u_{j-1} + (I + M^{-1}N)M^{-1}b\right]\right)$$

and the error $e_j = u - u_j$ therefore satisfies

$$e_j = (A^{-1} - P\overline{A}^{-1}R)\ \ A\,(M^{-1}N)^2 e_{j-1}$$

or in general if ν pre-smoothing steps and μ post-smooting steps are used

$$e_j = (M^{-1}N)^{\mu}(A^{-1} - P\overline{A}^{-1}R)\ \ A\,(M^{-1}N)^{\nu} e_{j-1}$$