Polynomial Iterative Methods

Simple iteration

$$
\begin{aligned}
x^{(k)} &= (M^{-1}N)x^{(k-1)} + \hat{b} \ , \ \ \hat{b} = M^{-1}b \\
\text{and} \qquad x &= (M^{-1}N)x + \hat{b} \\
\Rightarrow x - x^{(k)} &= (M^{-1}N)(x - x^{(k-1)}) \\
\Rightarrow x - x^{(k)} &= (M^{-1}N)^k(x - x^{(0)}) = S^k(x - x^{(0)})
\end{aligned}
$$

if $S = M^{-1}N$. ie.

$$
x - x^{(k)} = p_k(S)(x - x^{(0)}) = S^k(x - x^{(0)}) \ , \ \ p_k(z) = z^k
$$

Now if

$$\boldsymbol{x} - \boldsymbol{x}^{(0)} = \sum_{i=1}^{n} \boldsymbol{\alpha}_i \boldsymbol{v}_i, \quad \boldsymbol{S}\boldsymbol{v}_i = \boldsymbol{\lambda}_i \boldsymbol{v}_i$$

$$\boldsymbol{x} - \boldsymbol{y}^{(k)} = \sum_{i=1}^{n} \boldsymbol{\alpha}_i \boldsymbol{p}_k(\boldsymbol{S}) \boldsymbol{v}_i = \sum_{i=1}^{n} \boldsymbol{\alpha}_i \boldsymbol{p}_k(\boldsymbol{\lambda}_i) \boldsymbol{v}_i$$

Idea: $\boldsymbol{x} - \boldsymbol{y}^{(k)}$ should be small if $\boldsymbol{p}_k(\boldsymbol{\lambda}_i)$ is small

If S is symmetric we can say more
(since S orthogonaly diagonalisable)

$$\Rightarrow S \;=\; U\Lambda U^T\;,\;\; UU^T = I$$

$\uparrow$ diag Matrix of eigenvalues

$$\Rightarrow S^2 = U\Lambda U^T U\Lambda U^T = U\Lambda^2 U^T, \ldots, S^k = U\Lambda^k U^T$$

$$\Rightarrow p(S) \;=\; Up(\Lambda)U^T \qquad \text{any polynomial } p$$

$$\Rightarrow \|p(S)\|_2 \;=\; \|Up(\Lambda)U^T\|_2 = \|p(\Lambda)\|_2$$

$$= \left\| \begin{bmatrix} p(\lambda_1) & & & O \\ & p(\lambda_2) & & \\ & & \ddots & \\ O & & & p(\lambda_n) \end{bmatrix} \right\|_2$$

$$= \max_i \; |p(\lambda_i)|$$

So for polynomial iteration $x - y^{(k)} = p_k(S)(x - x^0)$

$$\|x-y^{(k)}\|_2 \leq \|p_k(S)\|_2 \|x-x^{(0)}\|_2 = \max_i |p_k(\lambda_i)| \; \|x-x^{(0)}\|_2$$

So desire $p_k \in \Pi_k$ is small at eigenvalues with $p_k(1) = 1$.

Notes:

- if $\lambda_i = 1$ for some i
  $\Rightarrow Sv_i = v_i \Leftrightarrow Mv_i = Nv_i \Leftrightarrow Av_i = 0$ ie. A singular.
- If only k distinct eigenvalues of S: choose p_k to have these as roots.

  For such a p_k, $\|x - y^{(k)}\|_2 = 0$ i.e. termination after k steps!

Candidates for $\{p_k\}$? : Chebyshev polynomials

Suppose $\lambda_i(S) \in [a,b]$, $1 \notin [a,b]$ then

$$\|x - y^{(k)}\|_2 \le \max_{i} |p_k(\lambda_i)| \|x - x^{(0)}\|_2$$
$$\le \max_{t \in [a,b]} |p_k(t)| \|x - x^{(0)}\|_2$$

and the polynomials which

$$\operatorname*{minimise}_{p \in \Pi_k,\ p(1)=1} \max_{t \in [a,b]} |p(t)|$$

are shifted and scaled Chebyshev polynomials

Chebyshev polynomials are defined on $[-1, 1]$ by $T_0(t) = 1$ and for $m = 1, 2, \ldots$ by

$$T_m(t) \;=\; \begin{cases} \frac{1}{2^{m-1}} \cos m\theta & (0 \le \theta \le \pi) \\ \qquad \text{where } t = \cos\theta & -1 \le t \le 1 \\ & \\ \frac{1}{2^{m-1}} \cosh m\theta & \\ \qquad \text{where } t = \cosh\theta & t \ge 1 \\ & \\ (-1)^m T_m(-t) & t \le -1 \end{cases}$$

$$T_0 = 1, \;\; T_1 = t,$$

$$T_2 = \frac{1}{2}\cos\, 2\theta = \frac{1}{2}(2\cos^2\theta - 1) = t^2 - \frac{1}{2}, \;\; \ldots$$

In general since

$$\begin{aligned}\cos(m+1)\,\theta &= \cos m\theta \cos\theta - \sin m\theta \sin\theta \\ \cos(m-1)\,\theta &= \cos m\theta \cos\theta + \sin m\theta \sin\theta\end{aligned}$$

we have

$$\cos(m+1)\,\theta + \cos(m-1)\,\theta = 2\cos m\theta \cos\theta$$

i.e. $2^m T_{m+1}(t) + 2^{m-2} T_{m-1}(t) = 2 \cdot 2^{m-1} T_m(t) t$

or

$$T_{m+1}(t) = t\,T_m(t) - \frac{1}{4} T_{m-1}(t)\ ,\ m = 2, 3, \ldots \qquad (\star)$$

so

$$T_3(t) = t(t^2 - \frac{1}{2}) - \frac{1}{4}t = t^3 - \frac{3}{4}t\ , \quad \text{etc.}$$

If $\lambda_i(S) \in [a,b], 1 \notin [a,b]$, need to shift using linear map

$$\begin{array}{ccccc} [a,b] & \mapsto & [-1,1] & : & t = \frac{2r-a-b}{b-a} \\ \in & & \in & & \\ r & & t & & \end{array}$$

$$\Rightarrow \hat{T}_k(r) = T_k\left(\frac{2r-a-b}{b-a}\right) \Big/ T_k\left(\frac{2-a-b}{b-a}\right)$$

satisfies $\hat{T}_k(1) = 1$ and minimises

$$\max_{r \in [a,b]} |p(r)|$$

over all polynomials of degree $\leq k$.

Using $p_k = \hat{T}_k$ is called the
Chebyshev semi-iterative method.

Remarks:

1. need estimates $a \lesssim \lambda_{\text{min}}(S)$ and $b \gtrsim \lambda_{\text{max}}(S)$ in order to construct $\hat{T}_k$'s on a reasonable interval
2. can use the 3-term recurrance $(\star)$ to make the algorithm more efficient than using the coefficients $\beta_e^{(k)}$ of the polynomials $T_k(z)$ explicitly (see exercise)
3. Convergence: we have

$$\|x - y^{(k)}\|_2 \leq \max_{r \in [a,b]} |\hat{T}_k(r)| \|x - x^{(0)}\|$$

if $\lambda_i \in [a, b]$, $1 \notin [a, b]$

$$\max_{r \in [a,b]} |\hat{T}_k(r)| = |\hat{T}_k(a)| = |\hat{T}_k(b)|$$

$$= \frac{|T_k(1)|}{|T_k(\frac{2-a-b}{b-a})|}$$

as max error always attained at end points.