Numerical Linear Algebra QS 5 *(MT 2017)*

Optional questions: Qns 2, 7

1. If $A \in \mathbb{R}^{n\times n}$ is such that $\operatorname{diag}(A) = I$, what polynomial p_k satisfies

$$x - x^{(k)} = p_k(A)(x - x^{(0)})$$

where x is the exact solution of $Ax = b$ and $\{x^{(j)}, j = 0, 1, \ldots\}$ are the iterate vectors produced by Jacobi iteration?

If $x^{(0)} = 0$ is chosen, for what polynomial is $x^{(k)} = q_k(A)b$?

2. (Section C exam questions 2001) If $A = M - N$ with A, M, $N \in \mathbb{R}^{n\times n}$, A, M nonsingular and $M^{-1}N$ diagonalisable, prove that the iteration

$$Mx^{(k)} = Nx^{(k-1)} + b, \qquad k = 1, 2, \ldots$$

will generate a sequence $\{x^{(k)}\}$ which converges to the solution x of $Ax = b$ for any starting guess $x^{(0)}$ if and only if the eigenvalues λ of $M^{-1}N$ satisfy $|\lambda| < 1$. Further, if $M^{-1}N$ is symmetric so that there is a basis $\{v_i, i = 1, \ldots, n\}$ for $\mathbb{R}^n$ of orthonormal eigenvectors of $M^{-1}N$, show that

$$v_i^T(x - x^{(k)}) = \lambda_i^k\, v_i^T(x - x^{(0)})$$

where λ_i is the eigenvalue corresponding to the eigenvector v_i.

For the remainder of this question you should assume that $|\lambda| < 1$ is a necessary and sufficient condition for convergence regardless of whether $M^{-1}N$ is or is not diagonalisable.

The Successive Overelaxation Method (SOR) is: $x^{(0)}$ arbitrary,

for $k = 1, 2, \ldots$

  for $i = 1, \ldots, n$

$$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \omega\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right)\Big/ a_{ii}$$

  end

end

where $A = \{a_{ij}, i, j = 1, \ldots, n\}$, $b = \{b_i, i = 1, \ldots, n\}$ and $\omega \in \mathbb{R}$. In terms of the diagonal matrix D, the strictly lower triangular matrix L and the strictly upper triangular matrix U such that $A = D + L + U$, write the SOR iteration in matrix form.

Suppose that $D = I$. By considering the determinant of the SOR iteration matrix or otherwise show that if $\omega \notin (0, 2)$ then SOR iteration can not be convergent. If further, $L^2 = 0$, show that the SOR iteration is convergent if and only if the eigenvalues of $(I - \omega L)A$ lie in $B(1/\omega, 1/\omega)$ where $B(a, b)$ is the open disc with centre a and radius b.

3. Suppose that $A = M - N \in \mathbb{R}^{m\times m}$ and it is desired to solve the linear system $Ax = b$. State a theorem which gives necessary and sufficient conditions for convergence of the vector sequence $\{x^{(i)}\}$ generated by the iteration: for initial guess $x^{(0)}$ compute

$$Mx^{(i+1)} = Nx^{(i)} + b, \qquad i = 0, 1, \ldots .$$

For Jacobi iteration what is M? Prove that if $A = \{a_{i,j}\}$ and

$$|a_{i,i}| > \sum_{j\neq i} |a_{i,j}|, \qquad i = 1, 2, \ldots, m \quad (\star)$$

then Jacobi iteration converges. If $m = n^2$ and $A = D + L + U$ where $D = \mathrm{diag}(D_1, D_2, \ldots, D_n)$ and $D_i \in \mathbb{R}^{n\times n}, i = 1, 2, \ldots, n$ are the diagonal blocks of A, L is the lower triangular part of $A - D$ and U is the upper triangular part of $A - D$, prove that the iteration based on the splitting $M = D$ will converge provided that the condition $(\star)$ holds. If for each $i = 1, 2, \ldots, n$, D_i is the tridiagonal matrix

$$\begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}$$

and

$$A = \begin{pmatrix} D_1 & -I & & & \\ -I & D_2 & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & D_{n-1} & -I \\ & & & -I & D_n \end{pmatrix}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, find the eigenvalues of the iteration matrix based on the splitting $M = D$ and show that the spectral radius of the iteration matrix is

$$1 - \pi^2/(n+1)^2 + O(n^{-4})$$

for large n.

4. Show that the (iteration) matrix

$$T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

has eigenvalues $\frac{2}{3}$, $-\frac{1}{3}$ and $-\frac{1}{3}$ and that $\left(T - \frac{2}{3}I\right)\left(T + \frac{1}{3}I\right) = 0$. Convince yourself (by using matlab, maple or otherwise) that no power of T is the zero matrix, but $T^k \to 0$ as $k \to \infty$. Define, however, a polynomial iterative method which will terminate after 2 iterations.

5. Verify that for arguements $|t| \geq 1$ the definition

$$T_k(t) = \frac{1}{2^{k-1}} \cosh k \left(\cosh^{-1} t\right)$$

defines the same (Chebyshev) polynomial as

$$T_k(x) = \frac{1}{2^{k-1}} \cos k \left(\cos^{-1} t\right)$$

for $|t| \leq 1$. Verify that (at least) for $t > \cosh(ln2)$, $T_k(t) \to \infty$ as $k \to \infty$.

6. If the eigenvalues λ of an iteration matrix S satisfy $|\lambda| \leq \rho < 1$ and $y^{(k)}$, $k = 2, 3, \cdots$ are the iterates obtained by using Chebyshev polynomials

$$\hat{T}_k(x) = \frac{T_k\left(\frac{x}{\rho}\right)}{T_k\left(\frac{1}{\rho}\right)} \quad \text{(i.e. shifted onto } [-\rho, \rho] \text{ and scaled so that } \hat{T}_k(1) = 1)$$

in a polynomial iteration based on S, show that

$$T_{k+1}(\frac{1}{\rho})\, e^{(k+1)} = \frac{1}{\rho}\, T_k(\frac{1}{\rho})\, S\, e^{(k)} - \frac{1}{4}\, T_{k-1}(\frac{1}{\rho})\, e^{(k-1)}$$

(at least for $k \geq 3$) where $e^{(k)} = y^{(k)} - x$ and x is the exact solution satisfying $x = S\,x + g$. (You may want first to obtain a 3–term recurrence for the $\hat{T}_k$). From this show that

$$y^{(k+1)} = w_{k+1} \left(S\, y^{(k)} + g - y^{(k-1)}\right) + y^{(k-1)}$$

where

$$w_{k+1} = \frac{1}{\rho} \frac{T_k\left(\frac{1}{\rho}\right)}{T_{k+1}\left(\frac{1}{\rho}\right)} = 1 + \frac{1}{4} \frac{T_{k-1}\left(\frac{1}{\rho}\right)}{T_{k+1}\left(\frac{1}{\rho}\right)}.$$

This shows that it is unnecessary to compute the iterates $x^{(k)}$ of the simple iteration involving S since the w_k can be computed easily using the 3 term recurrence for Chebyshev polynomials.

7. Let $A \in \mathbb{R}^{n^2 \times n^2}$ be the matrix which arises from 5–point finite difference replacement of the Laplacian with Dirichlet boundary conditions on a regular grid on the unit square. (Assume that A is scaled so that it has 4's on the diagonal). It is desired to solve $(\sigma\, I + A)\, x = b$ where $0 < \sigma \in \mathbb{R}$ using polynomial iteration based on the simple Jacobi iteration $(\sigma\, I + D)\, x^{(k+1)} = -(L+U)\, x^{(k)} + b$. By using Geshgorins Theorem (or otherwise) show that the eigenvalues of the iteration matrix satisfy $|\lambda| \leq \frac{4}{4+\sigma}$. Hence estimate convergence of Chebyshev polynomial iteration (as in previous question) if $\sigma = \frac{1}{2}$.