Conjugate Gradient Algorithm choose

x_0 , $r_0 = b - Ax_0 = p_0$ and for $k = 0, 1, 2, \ldots$

$$\begin{aligned}
\alpha_k &= p_k^T r_k / p_k^T A p_k \\
x_{k+1} &= x_k + \alpha_k p_k \\
r_{k+1} &= b - A x_{k+1} (= r_k - \alpha_k A p_k) \\
\beta_k &= -p_k^T A r_{k+1} / p_k^T A p_k (= r_{k+1}^T r_{k+1} / r_k^T r_k) \\
p_{k+1} &= r_{k+1} + \beta_k p_k
\end{aligned}$$

Preconditioned Conjugate Gradient Algorithm choose x_0,

$r_0 = b - Ax_0$, solve $Pz_0 = r_0$, $p_0 = z_0$, $k = 0, 1, \ldots$

$$\begin{aligned}
\alpha_k &= z_k^T r_k / p_k^T A p_k \\
x_{k+1} &= x_k + \alpha_k p_k \\
r_{k+1} &= r_k - \alpha_k A p_k \\
\text{Solve } P z_{k+1} &= r_{k+1} \\
\beta_k &= z_{k+1}^T r_{k+1} / z_k^T r_k \\
p_{k+1} &= z_{k+1} + \beta_k p_k
\end{aligned}$$

Convergence:

$$
\begin{aligned}
\frac{\|x - x_k\|_A}{\|x - x_0\|_A} &\le \min_{p \in \Pi_k, p(0)=1} \max_j |p(\lambda_j)| \\
&\le \min_{p \in \Pi_k, p(0)=1} \max_{t \in [a,b]} |p(t)| \\
&\le 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k
\end{aligned}
$$

So fast convergence if

(i) A has few distant eigenvalues (or few clusters)

(ii) κ is small, e.g. if $\kappa = 9$

$$\frac{\|x - x_k\|_A}{\|x - x_0\|_A} \leq 2\left(\frac{3-1}{3+1}\right)^k = \frac{2}{2^k}$$

error halving at each iteration.

$$\kappa = \frac{\lambda_{\max}(P^{-1}A)}{\lambda_{\min}(P^{-1}A)}$$

Example: 5-point Finite Difference approx of Laplacian

$$-\nabla^2 u = f \quad \text{in } \Omega, \quad u = g \text{ on } \partial\Omega$$

Finite Differences:

$$A \sim h^{-2} \quad \begin{array}{ccccc} & & -1 & & \\ & & | & & \\ -1 & — & 4 & — & -1 \\ & & | & & \\ & & -1 & & \end{array}$$

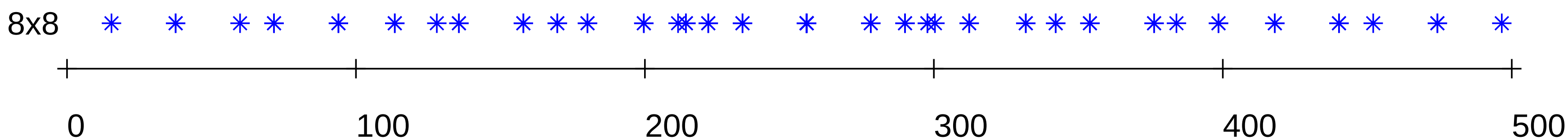
eg. on unit square A is block tridiagonal:

$$\underbrace{h^{-2}\begin{bmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & B & -I \\ & & 0 & -I & B \end{bmatrix}}_{A \in \mathbb{R}^{n^2 \times n^2}}, \quad \underbrace{\begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}}_{B \in \mathbb{R}^{n \times n}}$$

5-point finite difference approx of Laplacian
Using discrete Fourier analysis eigenvalues known:

$$\lambda = h^{-2}\,[4 - 2\cos(r\pi h) - 2\cos(s\pi h)]\,, \quad r, s = 1, \ldots, n$$

$$\Rightarrow \qquad \kappa = \frac{4}{\pi^2}\, h^{-2}, \qquad \lambda_{\min} \approx 2\pi^2,\ \lambda_{\max} \approx 8h^{-2}$$



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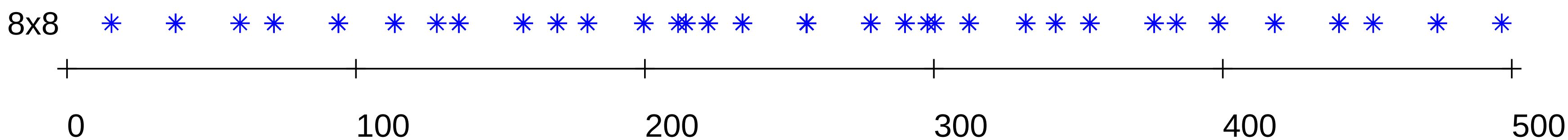

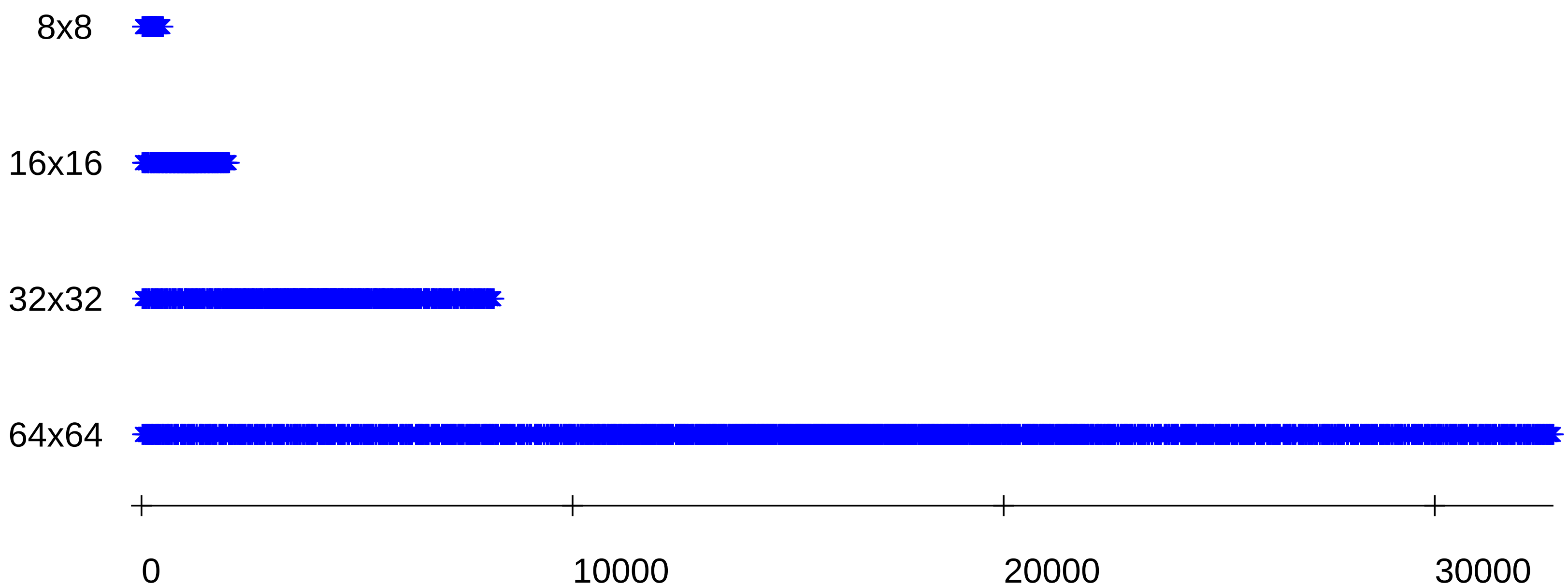

Multigrid/Multilevel preconditioning

appropriate methods (smoothing and grid transfers) converge in a number of iterations independent of h
$\Rightarrow$ optimal solvers for Laplacian problems

Number of PCG iterations, Preconditioner is 1 V-cycles (contraction factor: η)

$\|\mathbf{r}^{(k)}\|/\|\mathbf{r}^{(0)}\| \leq 10^{-4}$

1 relaxed Jacobi iteration for pre- and post-smoothing.

grid	PCG iterations (MG contraction)	n
8×8	4 (0.10)	49
16×16	4 (0.11)	225
32×32	4 (0.12)	961
64×64	4 (0.14)	3969
128×128	5 (0.16)	16129

Multigrid:

Convergence bound: $\|\mathbf{u} - \mathbf{u}^{(k)}\|_A \leq \eta \|\mathbf{u} - \mathbf{u}^{(k-1)}\|_A$
η typically 0.1

$\Rightarrow$ multigrid is a great preconditioner for Laplacian because

$$\|\mathbf{u} - \mathbf{u}^{(k)}\|_A \leq \eta \|\mathbf{u} - \mathbf{u}^{(k-1)}\|_A$$

$$\Rightarrow 1 - \eta \leq \lambda_{\min}(P^{-1}A), \lambda_{\max}(P^{-1}A) \leq 1 + \eta$$

when P^{-1} is the action of a single multigrid cycle.

Hence $\kappa \leq (1+\eta)/(1-\eta)$ which is typically $1.1/0.9 \approx 1.22$

Many other uses of these methods in diverse application areas