The logistic map
Sir Robert May (Baron May of Oxford)

\[ x_{n+1} = g(x_n) = \lambda x_n (1 - x_n). \]

1976
$x_{n+1} = f(x_n)$

$f(x) = \mu x (1 - x)$

$0 \leq x \leq 1$
Evolution of a map:
1) Choose initial conditions $\mu = 0.8, \ x_0 = 0.7$
2) Proceed vertically until you hit $f(x)$
3) Proceed horizontally until you hit $y=x$
4) Repeat 2)
5) Repeat 3)
   
Evolution of the logistic map

Fixed point?
Phenomenology of the logistic map

\[ f(x) = \mu x(1 - x) \quad 0 \leq x \leq 1 \]

What's going on? Analyze first a) → b) → c) , ...
\[ \delta = \lim_{k \to \infty} \left( \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \right) = 4.669201609 \ldots \]
I called my parents that evening and told them that I had discovered something truly remarkable, that, when I had understood it, would make me a famous man.

$$\delta = \lim_{k \to \infty} \left( \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \right) = 4.669201609 \ldots$$
Magnifying the bifurcation diagram reveals it is a fractal.
- Geometric representation

\[ f(x) = \mu x(1 - x) \quad 0 \leq x \leq 1 \]

- Evolution of the logistic map

How do we analyze the existence/stability of a fixed point?
• Fixed points

- Condition for existence: \[ x_f = f(x_f) \]
- Logistic map: \[ x_f (1 - \mu + \mu x_f) = 0 \]
- Notice: since \( 0 \leq x \leq 1 \) the second fixed point exists only for \( \mu \geq 1 \)

• Stability

- Define the distance of \( x_n \) from the fixed point \( x_f \)
  \[ \delta_n = x_n - x_f \]
- Consider a neighborhood of \( x_f \)
  \[ |\delta_{n+1}| = |x_{n+1} - x_f| = |f(x_f + \delta_n) - x_f| \approx |f'(x_f)| |\delta_n| \]
- The requirement \( |\delta_{n+1}| < |\delta_n| \) implies
  \[ \left| \frac{df}{dx} \right|_{x=x_f} < 1 \]

Logistic map?
Stability and the Logistic Map

- Stability condition: \[ \frac{df}{dx} = \mu(1 - 2x) < 1 \]

- First fixed point: stable (attractor) for \( \mu < 1 \)

- Second fixed point: stable (attractor) for \( 1 < \mu < 3 \)

- No coexistence of 2 stable fixed points for these parameters (transcritical bifurcation)

What about \( 3 \leq \mu \leq 4 \)?
- **Period doubling**

\[ f(x) \]

- **Evolution of the logistic map**

Observations:

1) The map oscillates between two values of \( x \)

2) Period doubling:

\[ x_{n+2} = x_n \]

What is it happening?
Period doubling

- At $\mu = 3$, the fixed point $x^*$ becomes unstable, since
  \[
  \left. \frac{df}{dx} \right|_{x=x_f} > 1
  \]

- Observation: an attracting 2-cycle starts \(\text{(flip)-bifurcation}\)

  The points are found solving the equations
  \[
  x_2 = \mu x_1 (1 - x_1)
  
  x_1 = \mu x_2 (1 - x_2)
  \]
  and thus:
  \[
  x_{1,2} = (1 + \mu \pm \sqrt{\mu^2 - 2\mu - 3}) / 2\mu
  \]

Why do these points appear?

These points form a 2-cycle for $f(x)$

However, the relation $x_{n+2} = f(x_n)$ suggests they are fixed points for the iterated map $f^2(x)$

- Stability analysis for $f^2(x) - 1 < \mu^2 (1 - 2x_1)(1 - 2x_2) < 1$
  and thus: $3 < \mu < 1 + \sqrt{6} = 3.44949$

- For $\mu = 1 + \sqrt{6}$, loss of stability and bifurcation to a 4-cycle

Now, graphically..
• Bifurcation diagram  Plot of fixed points vs $\mu$

$\mu = 3.4, \ x_0 = 0.2$

$\mu = 3.5, \ x_0 = 0.2$

$\mu = 2.8, \ x_0 = 0.2$

$\mu_\infty = 3.5699$
• Bifurcation diagram  Plot of fixed points vs  $\mu$

Observations:
1) Infinite series of period doublings at *pitchfork-like (flip) bifurcations*
2) After a point
   \[ \mu_\infty = 3.5699 \]
   chaos seems to appear
3) Regions where stable periodic cycles exist occur for
   \[ \mu \geq \mu_\infty \]

What is general?
General points:

1) Period doubling is a quite general route to chaos (other possibilities, e.g. intermittency)

2) Period doublings exhibit universal properties, e.g. they are characterized by certain numbers that do not depend on the nature of the map. For example, the ratio of the spacings between consecutive values of at the bifurcation points approaches the universal “Feigenbaum” constant. The latter occurs for all maps that have a quadratic maximum

\[ \delta = \lim_{k \to \infty} \left( \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \right) \approx 4.669201609 \ldots \]

3) Thus, we can predict where the cascade of period doublings ends, and something else starts

4) The something else looks chaotic, however, can we quantify how chaotic really is?

How do we characterize/quantify chaos?

Chaos: rapid divergence of nearby points in phase space

Measure of divergence: Lyapunov exponent \( \lambda \)
Lyapunov exponent

• One-dimensional system with initial conditions \( x \) and \( x + \epsilon \) with \( \epsilon \ll 1 \)

• After \( n \) iterations, their divergency is approximately

\[ \epsilon(n) \approx \epsilon e^{n\lambda} \]

- If \( \lambda < 0 \) there is convergence \( \Rightarrow \) no chaos
- If \( \lambda > 0 \) there is divergence \( \Rightarrow \) chaos

• One dimensional systems

\[ x_{n+1} = f(x_n) = f^n(x_0) \]

After \( n \) steps

\[ f^n(x_0 + \epsilon) - f^n(x_0) \approx \epsilon e^{n\lambda} \]

Thus:

\[ \lambda \approx \frac{1}{n} \ln \left[ \frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon} \right] \approx \frac{1}{n} \ln \left[ \frac{df^n(x)}{dx} \bigg|_{x=x_0} \right] \]

\[ = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \]

Logistic map

![Logistic map graph]