B.6 Renewal theory, insurance ruin and queueing theory (Optional)

(a) Clearly, the first customer arrives at an \( \text{Exp}(\lambda) \) distributed arrival time \( A_1 \), say, and leaves after further time \( G_1 \), where \( G_1 \) is independent of \( A_1 \) and has density \( g \). The departure time \( T_1 = A_1 + G_1 \) has as its density the convolution of \( g \) with the exponential density, i.e.

\[
\int_0^t g(s)\lambda e^{-\lambda(t-s)}ds.
\]

Given the \( n \)th customer has just been served, the lack of memory property of the current inter-arrival time shows that the post-\( T_n \) arrival process is again a Poisson process independent from the past, and the next customer arrives after a further \( \text{Exp}(\lambda) \) distributed time \( A_{n+1} \). Together with his independent service time \( G_{n+1} \), we conclude that \( T_{n+1} = T_n + A_{n+1} + G_{n+1} \). Therefore, the times between two consecutive departures after completed service are independent identically distributed, hence the associated counting process a renewal process.

(b) The Strong Law of Renewal Theory gives

\[
\frac{X_t}{t} \to \frac{1}{\mathbb{E}(A_1 + G_1)} = \frac{1}{\nu + 1/\lambda} \quad \text{almost surely as } t \to \infty.
\]

(c) Let \( N \) be the original arrival process of potential customers, then

\[
\frac{X_t}{N_t} = \frac{X_t/t}{N_t/t} \to \frac{1/(\nu + 1/\lambda)}{\lambda} = \frac{1}{\lambda\nu + 1} \quad \text{almost surely as } t \to \infty.
\]

on \( \{X_t/t \to 1/(\nu + 1/\lambda)\} \cap \{N_t/t \to \lambda\} \), but now

\[
\mathbb{P}(X_t/N_t \to 1/(\lambda\nu + 1)) \geq \mathbb{P}(X_t/t \to 1/(\nu + 1/\lambda), N_t/t \to \lambda) = 1,
\]

as required.

(d) Denote by \( S_n = G_1 + \cdots + G_n \) the total time before \( T_n \) that the server is busy. Then clearly for \( t = T_n \), the Strong Law of Large Numbers gives us for the proportion of time that the server is busy

\[
\frac{S_n}{T_n} = \frac{S_n/n}{T_n/n} \to \frac{\nu}{\nu + 1/\lambda} \quad \text{a.s. as } n \to \infty.
\]

The total time \( B_t \) that the server was busy before time \( t \) satisfies \( S_{X_t} \leq B_t < S_{X_t+1} \).

Remember that \( T_{X_t} \leq t < T_{X_t+1} \), and hence

\[
\frac{S_{X_t}}{T_{X_t}} \times \frac{T_{X_t}}{T_{X_t+1}} = \frac{S_{X_t}}{T_{X_t+1}} \leq \frac{B_t}{t} \leq \frac{S_{X_t+1}}{T_{X_t}} = \frac{T_{X_t+1}}{T_{X_t}} \times \frac{S_{X_t+1}}{T_{X_t+1}}
\]

from which we see

\[
\frac{\nu}{\nu + 1/\lambda} \leq \lim_{t \to \infty} \frac{B_t}{t} \leq \frac{\nu}{\nu + 1/\lambda} \quad \text{almost surely, using } X_t \to \infty \text{ and } T_n/T_{n+1} \to 1 \text{ almost surely. The latter can be obtained from the Strong Law of Large Numbers again, applied twice to } (T_n/n) \times ((n + 1)/T_{n+1}) \times (n/(n + 1)).
\]

2. \( N_i(T_n) \) is the number of visits to \( i \) by the jump chain up to time \( n \), and \( N(T_n) = n \), so \( N_i(T_n)/N(T_n) \to \eta_i \) almost surely as \( n \to \infty \), by the Ergodic Theorem for the jump chain.

With \( T_{N(t)} \leq t < T_{N(t)+1} \), we obtain

\[
N_i(T_{N(t)})/N(T_{N(t)} + 1) \leq N_i(t)/N(t) \leq N_i(T_{N(t)+1})/N(T_{N(t)}),
\]
and both bounds are easily seen to converge to \( \eta_i \) almost surely.

\( t \mapsto N_i(t) \) is a renewal process with mean inter-renewal time \( m_i = \mathbb{E}(H_i^{(1)}) \), so the Strong Law of Renewal Theory yields \( N_i(t)/t \rightarrow 1/m_i \) almost surely as \( t \rightarrow \infty \). Hence

\[
\frac{N(t)}{t} = \frac{N_i(t)/t}{N(t)/N_i(t)} \rightarrow \frac{1/m_i}{\eta_i} \quad \text{almost surely as } t \rightarrow \infty.
\]

Since \( N(t)/t \) does not depend on \( i \), neither does \( 1/(m_i \eta_i) \).

3. (a) We have \( q_{0,0} = \lambda, q_{n,n+1}, q_{0,0} = \mu, q_{1,0} = \mu, q_{n+2,n}, n \in \mathbb{N} \).

The detailed balance equations have no (non-zero) solution, so we have to solve \( \xi Q = 0 \):

\[
-\xi_0 \lambda + \xi_0 \mu = 0, \quad \xi_0 \lambda - \xi_0 (\lambda + \mu) + \xi_1 \mu + \xi_2 \mu = 0 \\
\xi_n \lambda - \xi_{n+1} (\lambda + \mu) + \xi_{n+3} \mu = 0, \quad n \in \mathbb{N}.
\]

For a solution of the form \( \xi_n = \alpha^n \xi_0 \) we get \( \lambda - \alpha (\lambda + \mu) + \alpha^2 \mu = 0 \), hence \( \alpha = 1 \) or \( \mu \alpha^2 + \mu \alpha - \lambda = 0 \), and the only solution that can lead to a probability distribution \( \xi \) is for \( \alpha = \sqrt{1/4 + \lambda/\mu} - 1/2 \in (0,1) \) if \( 0 < \lambda < 2\mu \).

We also need \( \xi_0 = (\mu/\lambda) \xi_0 \) and

\[
1 = \xi_0 + \sum_{n \in \mathbb{N}} \xi_n = \xi_0 \left( \frac{\mu}{\lambda + \mu} + \frac{1}{1 - \alpha} \right)
\]

so that we get \( \xi_n = \alpha^n (1 - \alpha) \frac{\lambda}{\lambda + \mu(1 - \alpha)} \) and \( \xi_0 = \frac{\mu(1 - \alpha)}{\lambda + \mu(1 - \alpha)} \). It is easily checked that this solution also satisfies

\[
\xi_0 \lambda - \xi_0 (\lambda + \mu) + \xi_1 \mu + \xi_2 \mu = 0,
\]

as required.

(b) We are interested in numbers of customers. So denote the counting process of customers entering the system by \( (N_t)_{t \geq 0} \) and the counting process of customers entering single service by \( (S_t)_{t \geq 0} \).

Single service is not represented by a (set of) state(s). A customer is served alone if he arrives when the system is empty or if he is alone in the queue when a customer leaves, so we are counting transitions from 0 to 0 and from 1 to 0. Let us treat these two types of customers separately, counted by \( S_t^{(1)} \) and \( S_t^{(2)} \), respectively.

\( S_t^{(1)} \) and \( S_t^{(2)} \) are delayed renewal processes, by the strong Markov property at successive times of the respective type of transition. We require the mean of the inter-renewal distribution. For \( S_t^{(1)} \), this is \( m_\emptyset = \mathbb{E}_\emptyset (H_\emptyset) = 1/(\lambda \xi_\emptyset) \), since renewals exactly correspond to the transitions after return times to \( \emptyset \) by the queue. For \( S_t^{(2)} \), not every return time to 1 leads to a renewal, each such return time leads to a renewal with probability \( q = \mu/(\lambda + \mu) \), independently, so that a geom(\( q \)) number \( N \) of such inter-return times \( R_j \) are needed, and

\[
\mathbb{E} \left( \sum_{j=1}^{N} R_j \right) = \mathbb{E} \left( \sum_{j=1}^{\infty} 1_{\{N \geq j\}} R_j \right) = \sum_{j=1}^{\infty} \mathbb{P}(N \geq j) \mathbb{E}(R_j) = \mathbb{E}(N) \mathbb{E}(R_1) = m_1/q = 1/(\mu \xi_1).
\]

Now we obtain from the strong law of renewal theory

\[
\frac{S_t}{N_t} = \frac{S_t^{(1)} + S_t^{(2)}}{N_t} \rightarrow \frac{\lambda \xi_\emptyset + \mu \xi_1}{\lambda}.
\]
4. (a) Since $R$ has independent increments, the $Z_i$ are independent, $i \geq 1$. Since $R$ has stationary increments, the $Z_i$ are identically distributed. We calculate by conditioning on $N_\epsilon$

\[ \mathbb{E}(Z_1) = \mathbb{E}(R_\varepsilon - u) = \mathbb{E} \left( \varepsilon - \sum_{k=1}^{N_\varepsilon} A_k \right) = \varepsilon - \sum_{n=0}^{\infty} \mathbb{P}(N_t = n) n \mathbb{E}(A_1) = \varepsilon - \frac{\lambda \varepsilon}{\mu}. \]

The Strong Law of Large Numbers now yields

\[ \frac{R_{\varepsilon n}}{\varepsilon} = \frac{1}{n} \left( Z_1 + \ldots + Z_n \right) \rightarrow \frac{\mathbb{E}(Z_1)}{\varepsilon} = c - \frac{\lambda}{\mu}, \]

a.s., as $n \to \infty$, as required. Now, clearly $c = \lambda / \mu$ and $n \varepsilon \to \infty$ implies that also $R_{\varepsilon n} \to \infty$, a.s.

(b) First note that $Z_n = S_n - S_{n-1} = cY_n - A_n$, $n \geq 1$, are identically distributed and independent, since $Y_n$, $n \geq 1$, and $A_n$, $n \geq 1$, are all independent. Similarly, $\tilde{Z}_n = S_{n+1} + A_{n+1} - (S_n + A_n) = cY_{n+1} - A_n$, $n \geq 1$, are independent and identically distributed. Note also that $Z_n$ and $\tilde{Z}_n$ have the same distribution with

\[ \mathbb{E}(\tilde{Z}_1) = \mathbb{E}(Z_1) = \mathbb{E}(cY_1 - A_1) = \frac{c}{\lambda} - \frac{1}{\lambda}. \]

Now, the Strong Law of Large Numbers yield

\[ \frac{R_{T_n}}{n} = \frac{u}{n} + \frac{Z_1 + \ldots + Z_n}{n} \rightarrow \frac{c}{\lambda} - \frac{1}{\mu}, \]

and

\[ \frac{R_{T_n}}{n} = \frac{u + cY_1}{n} + \frac{\tilde{Z}_1 + \ldots + \tilde{Z}_n}{n} \rightarrow \frac{c}{\lambda} - \frac{1}{\mu}, \]

a.s. as $n \to \infty$.

(c) Now use $R_{T_{N_t}} \leq R_t \leq R_{T_{N_t+1}-}$ and show that the two bounds tend to the same limit. First, clearly

\[ \frac{N_t}{t} \to \lambda \quad \text{a.s. as } t \to \infty, \]

by the Strong Law of Renewal Theory applied to the renewal process $N$. Now

\[ \frac{R_{T_{N_t}}}{t} = \frac{S_{N_t}}{N_t} \to \left( \frac{c}{\lambda} - \frac{1}{\mu} \right) \lambda \quad \text{a.s. as } t \to \infty \]

and similarly

\[ \frac{R_{T_{N_t+1}-}}{t} = \frac{S_{N_t+1} + A_{N_t+1}}{N_t + 1} \to \left( \frac{c}{\lambda} - \frac{1}{\mu} \right) \lambda \quad \text{a.s. as } t \to \infty. \]

5. (a) Just express in terms of the jump chain

\[ \mathbb{P}(A_1 > T_k | X_0 = m) = \mathbb{P}(M_1 = m - 1, \ldots, M_k = m - k | M_0 = m) = \left( \frac{\mu}{\lambda + \mu} \right)^k \]

for $m \geq k$. 
(b) Now condition on the initial state to get
\[ \mathbb{P}(A_1 > T_k) = \sum_{i=k}^{\infty} \rho^i (1 - \rho) \left( \frac{\mu}{\lambda + \mu} \right)^k = \left( \frac{\lambda}{\lambda + \mu} \right)^k. \]

(c) Now
\[ \mathbb{P}(M_k = i | A_1 > T_k) = \frac{\mathbb{P}(M_k = i, \ldots, M_0 = i + k)}{\mathbb{P}(A_1 > T_k)} = \frac{\rho^{i+k} (1 - \rho) \left( \frac{\mu}{\lambda + \mu} \right)^k}{\left( \frac{\lambda}{\lambda + \mu} \right)^k} = \rho^i (1 - \rho). \]

(d) Now calculate for \( i \geq 2 \)
\[ \mathbb{P}(X_{A_1} = i) = \sum_{n=1}^{\infty} \mathbb{P}(A_1 = T_n, M_n = i) \]
\[ = \sum_{n=1}^{\infty} \rho^{i+n-2} (1 - \rho) \left( \frac{\mu}{\lambda + \mu} \right)^{n-1} \frac{\lambda}{\lambda + \mu} \]
\[ = \rho^{i-1} (1 - \rho) \frac{\lambda}{\lambda + \mu} \frac{1}{1 - \frac{\lambda}{\lambda + \mu}} = \rho^i (1 - \rho). \]

\( \xi \) cannot be the stationary distribution for \( (X_{A_m})_{m \geq 0} \), since \( \xi_0 > 0 \) but \( \mathbb{P}(X_{A_m} = 0) = 0 \) for all \( m \geq 1 \).

6. (a) By one-step analysis
\[ \tilde{m}(t) = \mathbb{E}(\tilde{X}_t) = \int_0^t (1 + m(t - x)) g(x) dx = G(t) + \int_0^t m(t - x) g(x) dx. \]

Writing \( \tilde{m}'(t) = g(t) + \sum_{k=1}^{\infty} g * f^{*k}(t) \), we check that
\[ \int_0^t \int_0^x g(y) dy f(t - x) dx = \int_0^t g(y) \int_t^{t-y} f(t - y - z) dz dy = \int_0^t g(y) \int_0^{t-y} f(x) dx dy \]
and similarly
\[ \int_0^t \int_0^x g * f^{*k}(y) dy f(t - x) dx = \int_0^t g * f^{*k}(y) \int_0^{t-y} f(x) dx dy \]
\[ - \int_0^t \int_0^y g(z) f^{*k}(y - z) dz \int_0^{t-y} f(x) dx dy \]
\[ = \int_0^t g(z) \int_0^{t-z} \int_0^{t-w} f^{*k}(s) f(w - s) ds dw dz \]
\[ = \int_0^t g(z) \int_0^{t-z} f^{*(k+1)}(w) dw dz. \]

Then, summing over \( k \geq 0 \) (\( k = 0 \) from previous calculation), we see that
\[ \int_0^t \tilde{m}(x) f(t - x) dx = \int_0^t g(z) m(t - z) dy \]
as required.
(b) We condition on the last arrival before time $t$ and that it is the $k$th arrival

$$
\mathbb{P}(\tilde{E}_t > y) = \sum_{k \geq 0} \mathbb{P}(\tilde{E}_t > y, X_t = k)
$$

$$
= G(t + y) + \sum_{k \geq 1} \int_0^t \mathbb{P}(\tilde{E}_t > y, X_t = k| T_k = x) g \ast f^{*(k-1)}(x) \, dx
$$

$$
= G(t + y) + \sum_{k \geq 1} \int_0^t \mathbb{P}(Y_k > t - x + y) g \ast f^{*(k-1)}(x) \, dx
$$

$$
= G(t + y) + \int_0^t F(t - x + y) \tilde{m}'(x) \, dx
$$

as required. We used the convention that $g \ast f^{*(0)} = g$.

(c) In this case $\tilde{m}(t) = t/\mu$ solves the second formula in (a) since by integration by parts

$$
G(t) + \frac{1}{\mu} \int_0^t (t-x)f(x) \, dx
$$

$$
= \frac{1}{\mu} \int_0^t \overline{F}(y) \, dy - \frac{1}{\mu} \int_0^t \overline{F}(x) \, dx - \frac{1}{\mu} \left[ (t-x)\overline{F}(x) \right]_0^t = \frac{t}{\mu}.
$$

Therefore $\tilde{m}'(x) = 1/\mu$ for all $x \geq 0$ and hence from (b)

$$
\mathbb{P}(\tilde{E}_t > y) = G(t + y) + \frac{1}{\mu} \int_0^t F(t + y - x) \, dx
$$

$$
= G(t + y) + \int_y^{t+y} g(z) \, dz = G(y)
$$

as required.