Fourier Series and Partial Differential Equations Lecture Notes

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Hilary Term

Contents

Preface

These lecture notes are designed to accompany the first year course "Fourier Series and Partial Differential Equations" and are taken largely from notes originally written by Dr Yves Capdeboscq, Dr Alan Day and Dr Janet Dyson.

The first part of this course of lectures introduces Fourier series, concentrating on their practical application rather than proofs of convergence. We will then discuss how the heat equation, wave equation and Laplace's equation arise in physical models. In each case we will explore basic techniques for solving the equations in several independent variables, and elementary uniqueness theorems.

Reading material

Fourier series.

- D. W. Jordan and P. Smith, Mathematical Techniques (Oxford University Press, 3rd Edition, 2003), Chapter 26.
- E. Kreyszig, Advanced Engineering Mathematics (Wiley, 8th Edition, 1999), Chapter 10.
- W. A. Strauss, Partial Differential Equations: An Introduction (Wiley, 1st Edition, 1992), Chapter 5.

Heat equation.

- G. F. Carrier and C. E. Pearson, Partial Differential Equations: Theory and Technique (Academic Press, 2nd Edition, 1998), Chapter 1.
- W. A. Strauss, Partial Differential Equations: An Introduction (Wiley, 1st Edition, 1992), Chapters 1–4.
- E. Kreyszig, Advanced Engineering Mathematics (Wiley, 8th Edition, 1999), Chapter 12.

Wave equation.

- G. F. Carrier and C. E. Pearson, Partial Differential Equations: Theory and Technique (Academic Press, 2nd Edition, 1998), Chapter 3.
- W. A. Strauss, Partial Differential Equations: An Introduction (Wiley, 1st Edition, 1992), Chapters 1–4.

- E. Kreyszig, Advanced Engineering Mathematics (Wiley, 8th Edition, 1999), Chapter 12.

Laplace's equation.

- G. F. Carrier and C. E. Pearson, Partial Differential Equations: Theory and Technique (Academic Press, 2nd Edition, 1998), Chapter 4.
- W. A. Strauss, Partial Differential Equations: An Introduction (Wiley, 1st Edition, 1992), Chapter 6.

Chapter 1

Introduction

In this chapter we introduce the concept of initial and boundary value problems, and the equations that we shall study throughout this course.

1.1 Initial and boundary value problems

Consider a second-order ordinary differential equation (ODE)

$$y'' = f(x, y, y'), \tag{1.1}$$

where $y' = \mathrm{d}y/\mathrm{d}x$ and $y'' = \mathrm{d}^2y/\mathrm{d}x^2$. The problem is to find $y(x)$, subject to appropriate additional information.

1.1.1 Initial-value problem (IVP)

Suppose that $y(a) = p$ and $y'(a) = q$ are prescribed.

Figure 1

$$y = p + q(x - a).$$

1.1.2 Boundary-value problem (BVP)

Suppose that $y(x)$ is defined on an interval $[a, b]$ and $y(a) = A$ and $y(b) = B$ are prescribed.

Figure 2

1.1.3 Existence and uniqueness

Recall that solutions may not exist, or if they exist they may not be unique.

IVP: $y'' = 6y^{\frac{1}{3}}$, $y(0) = 0$, $y'(0) = 0$ has solutions $y(x) = 0$, $y(x) = x^3$ (non-uniqueness);

BVP1: $y'' + y = 0$, $y(0) = 1$, $y(2\pi) = 0$ has no solution (non-existence);

BVP2: $y'' + y = 0$, $y(0) = 0$, $y(2\pi) = 0$ has infinitely many solutions, $y(x) = c \sin x$, where c is an arbitrary constant (non-uniqueness).

1.2 Some preliminaries

We state, but do not prove, two preliminary results.

Theorem 1.1 (Leibniz's Integral Rule) Let $F(x,t)$ and $\partial F/\partial t$ be continuous in both x and t in some region of the (x,t) plane including $(t,x) \in [t_0, t_1] \times [a(t), b(t)]$, and the functions $a(t)$ and $b(t)$ and their derivatives be continuous for $t \in [t_0, t_1]$. Then

$$G(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)}^{b(t)} F(x,t)\,\mathrm{d}x = b'(t)F(b(t),t) - a'(t)F(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial F(x,t)}{\partial t}\,\mathrm{d}x. \quad (1.2)$$

As a result, if $a(t)$ and $b(t)$ are constants with

$$G(t) = \int_a^b F(x,t)\,\mathrm{d}x, \quad (1.3)$$

then

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \int_a^b \frac{\partial F(x,t)}{\partial t}\,\mathrm{d}x. \quad (1.4)$$

Lemma 1.2 If $f(x)$ is continuous then

$$\frac{1}{h} \int_a^{a+h} f(x)\,\mathrm{d}x \to f(a) \text{ as } h \to 0.$$

Note that

$$\frac{G(t+h) - G(t)}{h} = \int_a^b \frac{F(x,t+h) - F(x,t)}{h}\,\mathrm{d}x, \quad (1.5)$$

and the integrand tends to $\partial F(x,t)/\partial t$ as $h \to 0$.

1.3 The equations we shall study

It is proposed to study three linear second-order partial differential equations (PDEs) that have applications throughout the physical sciences.

1.3.1 The heat equation

Also known as the diffusion equation, we will find $T(x,t)$ such that

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad (1.6)$$

where, for example, $T(x,t)$ is a temperature at position x and time t, and κ is a positive constant—the *thermal diffusivity*.

1.3.2 The wave equation

Here, we will look at finding $y(x,t)$ such that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{1.7}$$

where, for example, $y(x,t)$ is the transverse displacement of a stretched string at position x and time t, and c is a positive constant—the *wave speed*.

1.3.3 Laplace's equation

In this case the problem is to find $T(x,y)$ such that

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{1.8}$$

where, for example, $T(x,y)$ may be a temperature and x and y are Cartesian coordinates in the plane. In this case, Laplace's equation models a two-dimensional system at steady state in time: in three space-dimensions the temperature $T(x,y,z,t)$ satisfies the heat equation

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right). \tag{1.9}$$

Note that equation (1.9) reduces to (3.8) if T is independent of y and z. If the temperature field is *static*, T is independent of time, t, and is a solution of *Laplace's equation in* $\mathbb{R}^3$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0, \tag{1.10}$$

and, in the special case in which T is also independent of z, of *Laplace's equation in* $\mathbb{R}^2$,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{1.11}$$

Chapter 2

Fourier series

In the following chapters, we will look at methods for solving the PDEs described in Chapter 1. In order to incorporate general initial or boundary conditions into our solutions, it will be necessary to have some understanding of Fourier series.

For example, we can see that the series

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \tag{2.1}$$

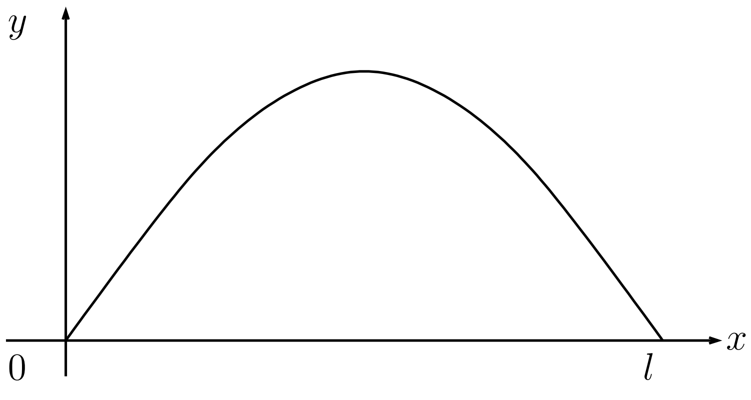
is a solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad x \in [0, L], \quad t \geq 0, \tag{2.2}$$

which satisfies the boundary conditions

$$y(0,t) = 0 = y(L,t). \tag{2.3}$$

We may view $y(x,t)$ as the solution of the problem which models a vibrating string of length L pinned at both ends, *e.g.* a guitar string.



We would like to find a solution with initial conditions

$$y(x,0) = \alpha \sin\left(\frac{\pi x}{L}\right), \qquad \frac{\partial y}{\partial t}(x,0) = 0, \tag{2.4}$$

and we do this by calculating A_n and B_n as follows: from equation (2.1) we have

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.5}$$

and

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{2.6}$$

Hence, we want A_n, B_n such that

$$\alpha \sin\left(\frac{\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \qquad 0 = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{2.7}$$

By inspection we see that $A_1 = \alpha$, $A_n = 0$ for $n \neq 1$ and $B_n = 0\ \forall n$. Thus, for these initial conditions, the solution is

$$y(x,t) = \alpha \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi ct}{L}\right). \tag{2.8}$$

If we would like to take more general initial conditions

$$y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = g(x), \tag{2.9}$$

we need to find $\{A_n, B_n\}$ such that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \qquad g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{2.10}$$

These are known the Fourier sine series of the functions f and g.

2.1 Periodic, even and odd functions

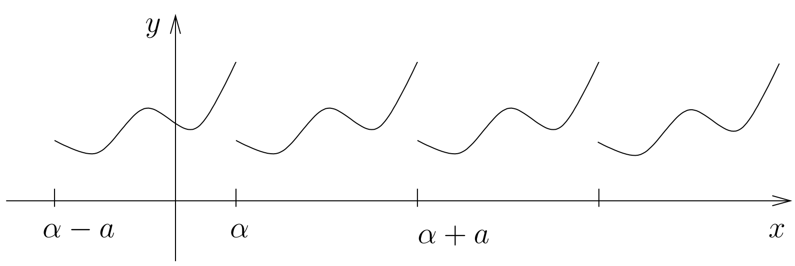
Definition f is a *periodic function* if there is an $a > 0$ such that

$$f(x+a) = f(x), \qquad \forall x \in \mathbb{R}. \tag{2.11}$$

If this is the case a is called a *period* for f. Note that the period is not unique, but if there is a smallest such a, it is called the *prime period* of f.

Notes.

1. Observe that this means that $f(x) = c$ for c constant does not have a prime period.
2. Examples of periodic functions are $\sin x$ with prime period 2π and $\cos(2\pi x/a)$ with prime period a. Examples of non-periodic functions are x and x^2.
3. Observe that if f is defined on the half-open interval $(\alpha, \alpha + a]$ we can extend it to be a periodic function by demanding it is periodic with period a. This is called a periodic extension.



Definition Formally, we define the *periodic extension*, F, of f as follows: given $x \in \mathbb{R}$ there exists a unique integer m such that $x - ma \in (\alpha, \alpha + a]$. If we then set $F(x) = f(x - ma)$, we can see that F is periodic with period a.

2.1.1 Properties of periodic functions

If f, g are periodic functions with period a, then:

1. f, g are also periodic functions with period na for any $n \in \mathbb{N}$;

2. for any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is periodic with period a;

3. fg is periodic with period a;

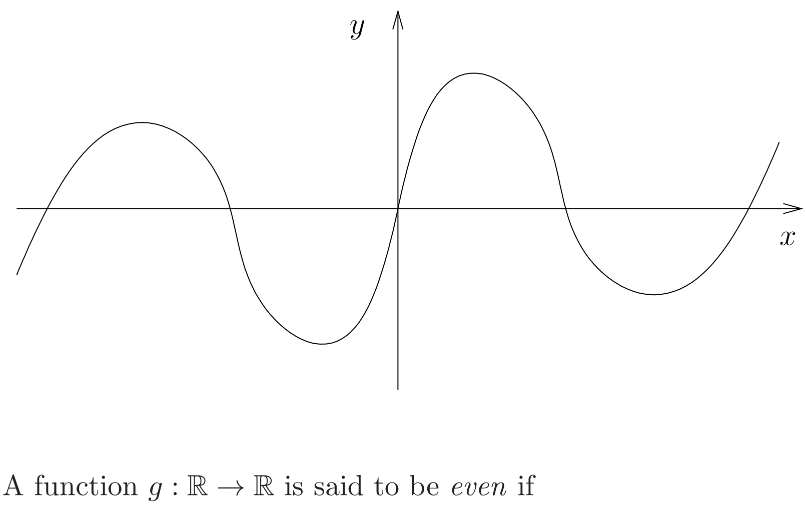
4. for any $\lambda > 0$, $f(\lambda x)$ is periodic with period a/λ,
$$f(\lambda(x + a/\lambda)) = f(\lambda x + a) = f(\lambda x); \tag{2.12}$$

5. for any $\alpha \in \mathbb{R}$,
$$\int_0^a f(x)\,\mathrm{d}x = \int_\alpha^{\alpha+a} f(x)\,\mathrm{d}x, \tag{2.13}$$
since
$$\int_\alpha^{\alpha+a} f(x)\,\mathrm{d}x = \int_\alpha^a f(x)\,\mathrm{d}x + \int_a^{\alpha+a} f(x)\,\mathrm{d}x. \tag{2.14}$$

2.1.2 Odd and even functions

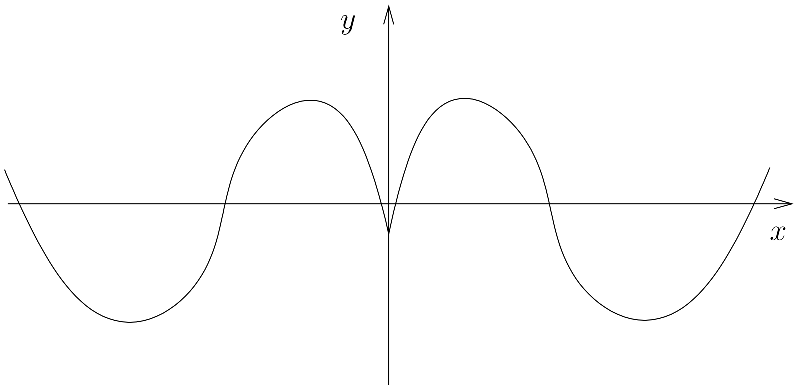
Definition A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *odd* if
$$f(x) = -f(-x), \qquad \forall x \in \mathbb{R}. \tag{2.15}$$

For example, $\sin(\lambda x)$ for $\lambda \in \mathbb{R}$ and x^{2n+1} for $n \in \mathbb{N}$ are both odd functions.



Definition A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *even* if
$$g(x) = g(-x), \qquad \forall x \in \mathbb{R}. \tag{2.16}$$

Examples of even functions are $\cos(\lambda x)$ for $\lambda \in \mathbb{R}$ and x^{2n} for $n \in \mathbb{N}$.



Notes. If f, f_1 are odd functions and g, g_1 are even functions then:

1. $f(0) = 0$ because $f(0) = -f(-0) = -f(0)$;
2. for any $\alpha \in \mathbb{R}$,
$$\int_{-\alpha}^{\alpha} f(x)\,\mathrm{d}x = 0; \tag{2.17}$$
3. for any $\alpha \in \mathbb{R}$,
$$\int_{-\alpha}^{\alpha} g(x)\,\mathrm{d}x = 2\int_{0}^{\alpha} g(x)\,\mathrm{d}x; \tag{2.18}$$
4. the functions $h(x) = f(x)g(x)$, $h_1(x) = f(x)f_1(x)$ and $h_2(x) = g(x)g_1(x)$ are odd, even and even, respectively.

2.2 Fourier series for functions of period 2π

Let f be a function of period 2π. We would like to get an expansion for f of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right], \tag{2.19}$$

where the a_n and b_n are constants. Remember that $\sin(nx)$ and $\cos(nx)$ are periodic with period 2π. We have two questions to answer.

Question 1: if equation (2.19) is true, can we find the a_n and b_n in terms of f?

Question 2: with these a_n, b_n, when, if ever, is equation (2.19) true?

2.2.1 Question 1

Suppose equation (2.19) is true and that we can integrate it term by term, *i.e.*

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0\int_{-\pi}^{\pi}\mathrm{d}x + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(nx)\,\mathrm{d}x + b_n\int_{-\pi}^{\pi}\sin(nx)\,\mathrm{d}x\right]. \tag{2.20}$$

Since

$$\int_{-\pi}^{\pi}\mathrm{d}x = 2\pi, \quad \int_{-\pi}^{\pi}\cos(nx)\,\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi}\sin(nx)\,\mathrm{d}x = 0, \tag{2.21}$$

we must have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x. \tag{2.22}$$

Thus if equation (2.19) is true, then we know a_0.

Note that due to the properties of periodic functions, we could use $\int_0^{2\pi} f(x)\,\mathrm{d}x$ instead of $\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$ in the preceding. More generally, we could integrate over any interval of length 2π.

Lemma 2.1 Let $n, m \in \mathbb{N} \setminus 0$. We then have the following equalities:

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0, \tag{2.23}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{nm}, \tag{2.24}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{nm}, \tag{2.25}$$

where δ_{nm} is the Kronecker delta defined by

$$\delta_{nm} = \begin{cases} 0 & n \neq m, \\ 1 & n = m. \end{cases} \tag{2.26}$$

Proof. Equation (2.23) is trivial as $\sin(mx)\cos(nx)$ is odd. For equation (2.24) we compute, for $n \neq m$,

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} \left[-\cos\{(m+n)x\} + \cos\{(m-n)x\}\right]\,\mathrm{d}x, \\ &= \frac{1}{2}\left[\frac{-\sin\{(m+n)x\}}{m+n} + \frac{\sin\{(m-n)x\}}{m-n}\right]_{-\pi}^{\pi}, \\ &= 0. \end{aligned} \tag{2.27}$$

If $n = m$ we have

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \left[\frac{1-\cos(2nx)}{2}\right]\,\mathrm{d}x = \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} = \pi. \tag{2.28}$$

Similar computations yield equation (2.25). □

Thus, to find a_n and b_n, we assume equation (2.19) is true. Multiplying both sides by $\cos(mx)$ and integrating term-wise gives

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x &= \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos(mx)\,\mathrm{d}x \\ &\quad + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,\mathrm{d}x \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,\mathrm{d}x. \end{aligned} \tag{2.29}$$

The first term on the right-hand side is trivially zero for $m \neq 0$. Using Lemma 2.1 for the remaining terms gives

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \sum_{n=1}^{\infty} a_n \pi \delta_{nm} = \pi a_m, \tag{2.30}$$

and hence

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x. \tag{2.31}$$

Note that this also holds for $m = 0$ (which is the reason for the factor of 1/2).

Multiplying equation (2.19) by $\sin(mx)$ and integrating term-wise, we similarly obtain

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x. \tag{2.32}$$

Definition Suppose f is such that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x, \tag{2.33}$$

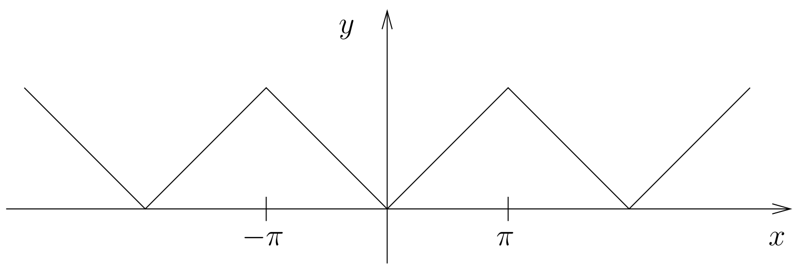
exist. Then we shall write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right], \tag{2.34}$$

and call the series on the right-hand side the *Fourier series* for f, whether or not it converges to f. The constants a_n and b_n are called the *Fourier coefficients* of f.

Example 2.1 Find the Fourier series of the function f which is periodic with period 2π and such that

$$f(x) = |x|, \quad x \in (-\pi, \pi]. \tag{2.35}$$



To find a_n, b_n first notice that f is even, so $f(x)\sin(nx)$ is odd and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = 0, \tag{2.36}$$

for every n. Also, $f(x)\cos(nx)$ is even, so

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x, \\ &= \frac{2}{\pi}\left(\left[\frac{x\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,\mathrm{d}x\right), \\ &= -\frac{2}{\pi}\left(\left[\frac{-\cos(nx)}{n^2}\right]_0^{\pi}\right), \\ &= \frac{2}{\pi}\left(\frac{\cos(n\pi)-\cos(0)}{n^2}\right), \\ &= \frac{2}{\pi}\frac{[(-1)^n-1]}{n^2}. \end{aligned} \tag{2.37}$$

Note that this is not valid for $n = 0$. In fact, $a_0 = \pi$. If n is even, $n = 2m$ say, we have

$$a_{2m} = \frac{2((-1)^{2m}-1)}{\pi(2m)^2} = 0. \tag{2.38}$$

If n is odd, $n = 2m + 1$ say, we obtain

$$a_{2m+1} = \frac{2(-1-1)}{\pi(2m+1)^2} = \frac{-4}{\pi(2m+1)^2}. \tag{2.39}$$

2.2.2 Sine and cosine series

Let f be 2π-periodic. If f is odd then

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \tag{2.40}$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin(ns)\,\mathrm{d}s = \frac{2}{\pi}\int_0^{\pi} f(s)\sin(ns)\,\mathrm{d}s, \tag{2.41}$$

i.e. f has a Fourier sine series. In this case $a_n = 0$ because $f(x)\cos(nx)$ is odd. This is also true if $f(x) = -f(-x)$ for $x \neq n\pi$, $n \in \mathbb{Z}$, i.e. f is odd apart from the end points and zero.

If f is even then

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{2.42}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(s)\cos(ns)\,\mathrm{d}s, \tag{2.43}$$

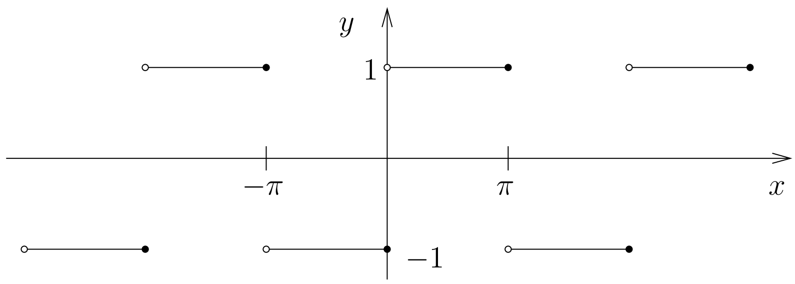
i.e. f has a Fourier cosine series.

2.2.3 Question 2

Recall Question 2: with these a_n, b_n, when, if ever, is equation (2.19) true? Consider what happens in the following example.

Example 2.2 Consider the Fourier series of the function f which is periodic with period 2π and such that

$$f(x) = \begin{cases} 1 & 0 < x \leq \pi, \\ -1 & -\pi < x \leq 0. \end{cases} \tag{2.44}$$



Note that f is odd, so we can conclude that $f(x)\cos(nx)$ is odd, giving $a_n = 0$ without computation. On the other hand, $f(x)\sin(nx)$ is even, so

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\,\mathrm{d}x = -\frac{2\left[(-1)^n - 1\right]}{n\pi}, \tag{2.45}$$

i.e.

$$b_{2m} = 0, \qquad b_{2m+1} = \frac{4}{(2m+1)\pi}, \tag{2.46}$$

and hence

$$f(x) \sim \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1}\sin\left[(2m+1)x\right]. \tag{2.47}$$

Consider Question 2 for this case: when is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]? \tag{2.48}$$

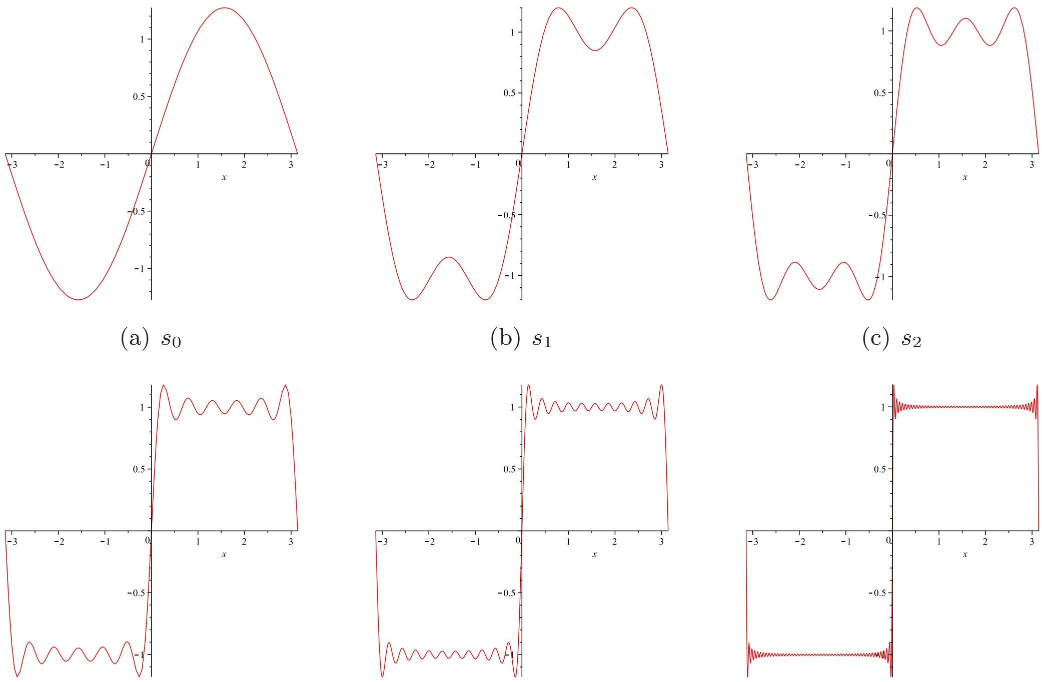
Recall that

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)x]}{2m+1}, \tag{2.49}$$

means $\lim_{n\to\infty} s_n(x)$ where

$$s_n(x) = \frac{4}{\pi}\sum_{m=0}^{n}\frac{\sin[(2m+1)x]}{2m+1}. \tag{2.50}$$

The question is therefore, does $s_n(x)$ converge for each x? If it does, is the limit $f(x)$? Some partial sums, s_n, are plotted in Figure 2.1.



(a) s_0 (b) s_1 (c) s_2

(d) s_5 (e) s_{10} (f) s_{50}

Figure 2.1: Convergence of the Fourier series for the function of Section 2.2.3.

2.3 Convergence of Fourier series

For the previous example it does appear that except at points of discontinuity the partial sums do converge to $f(x)$. At points of discontinuity they converge to zero. A similar result is true also for most functions which appear in applications. To present this result we first need to discuss one-sided limits.

Definition We say that the *right-hand* limit of f at c is

$$f(c_+) = \lim_{\substack{h \to 0 \\ h > 0}} f(c+h), \tag{2.51}$$

if this exists. Similarly, the *left-hand limit* of f at c is

$$f(c_-) = \lim_{\substack{h \to 0 \\ h > 0}} f(c-h), \tag{2.52}$$

if this exists.

The existence part is important since, for example, $f(x) = \sin(1/x)$ does not have these limits at zero.

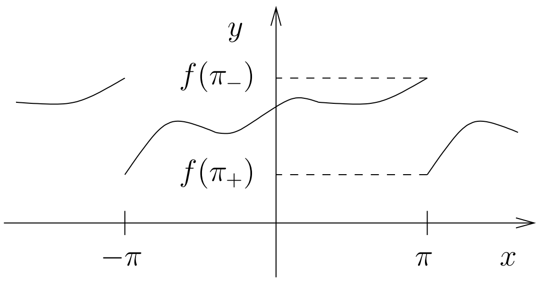
Definition The function f is *piecewise continuous* on an interval (a, b) if we can divide (a, b) into a finite number of sub-intervals, on each of which f is defined and continuous, and the left- and right-hand limits at the endpoints of each sub-interval exist.

Theorem 2.2 (Convergence theorem) Let f be a periodic function with period 2π, with f and f' piecewise continuous on $(-\pi, \pi)$. Then the Fourier series of f at x converges to the value $\frac{1}{2}[f(x_+) + f(x_-)]$, *i.e.*

$$\frac{1}{2}\left[f(x_+) + f(x_-))\right] = \frac{1}{2}a_0 + \sum_{n=0}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]. \tag{2.53}$$

Note that if f is continuous at x, then $f(x_+) = f(x_-) = f(x)$ so the Fourier series converges to $f(x)$.

Note that if a function is defined on an interval of length 2π, we can find the Fourier series of its periodic extension and equation (2.53) will then hold on the original interval. But we have to be careful at the end points of the interval: *e.g.* if f is defined on $(-\pi, \pi]$ then at $\pm\pi$ the Fourier Series of f converges to $\frac{1}{2}\left[f(\pi_-) + f((-\pi)_+)\right]$.



For Example 2.2 we have, by Theorem 2.2, that

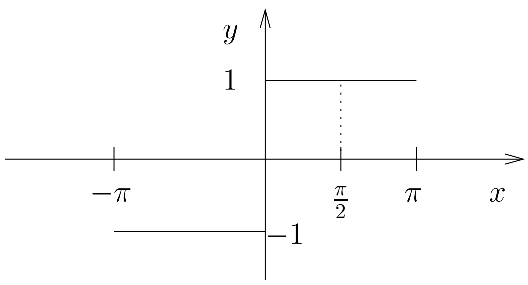
$$\frac{1}{2}\left[f(x_+) + f(x_-)\right] = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin[(2m+1)x], \tag{2.54}$$

where both sides reduce to zero at $x = 0, \pm\pi$. At $x = \pi/2$ we obtain

$$1 = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sin\left[\frac{(2m+1)\pi}{2}\right] = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}, \tag{2.55}$$

and hence

$$\frac{\pi}{4} = \sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}. \tag{2.56}$$



2.3.1 Rate of convergence

When using Fourier series in practical situations, we often need to truncate the series at some finite value of n. In this case, we would be interested in questions such as how good is the convergence? Also, what about the speed of the convergence? In general, the more derivatives f has, the faster the convergence. We can roughly say that if the discontinuity is in the p^{th} derivative, then a_n, b_n decay like n^{-p-1}.

Lemma 2.3 Assume that F is continuous on $[-\pi, \pi]$, and $F' = f$ is piecewise continuous on $(-\pi, \pi)$. Let a_n, b_n be the Fourier coefficients of f and A_n, B_n be the Fourier coefficients of F. Then, $A_n = -b_n/n$ and $B_n = a_n/n$.

Proof. The proof is an integration by parts, and is not shown here. □

In fact, this is best seen using *complex* Fourier coefficients, $c_n := a_n + ib_n$. Then the lemma says that

$$c_n(f') = -inc_n(f). \tag{2.57}$$

This can be iterated and used to solve ODEs. For example, if f is a 2π periodic function, with Fourier coefficients $c_n(f)$, and y is the solution of the differential equation

$$y^{(5)}(x) + a_4 y^{(4)}(x) + a_3 y^{(3)}(x) + a_2 y^{(2)}(x) + a_1 y^{(1)}(x) + a_0 y(x) = f(x), \tag{2.58}$$

then the Fourier coefficients of y, $c_n(y)$, $n \geq 0$, are given by

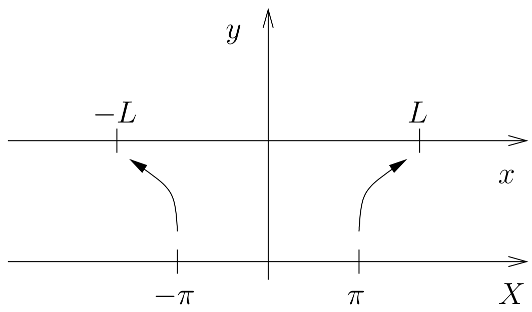
$$\left[(-in)^5 + a_4(-in)^4 + a_3(-in)^3 + a_2(-in)^2 + a_1(-in) + a_0\right] c_n(y) = c_n(f). \tag{2.59}$$

2.3.2 Gibbs Phenomenon

As can be seen in Figure 2.1, at a point of discontinuity, the partial sums always overshoot the limiting values. This overshoot does not tend to zero as more terms are taken, but the width of the overshooting region does tend to zero. This is known as the Gibbs Phenomenon.

2.4 Functions of any period

Consider a function f of period $2L$ ($L > 0$). We want a series in $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$. To do this we make the transformation $X = \pi x/L$.



Formally, we define $g(X) = f(x) = f(LX/\pi)$ so that

$$g(X+2\pi) = f\left(\frac{L(X+2\pi)}{\pi}\right) = f\left(\frac{LX}{\pi}+2L\right) = f\left(\frac{LX}{\pi}\right) = g(X), \tag{2.60}$$

and g is 2π-periodic. Hence the previous theory holds for g, *i.e.* if we can write

$$g(X) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nX) + b_n\sin(nX)\right], \tag{2.61}$$

then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\cos(nX)\,\mathrm{d}X, \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \end{aligned} \tag{2.62}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\sin(nX)\,\mathrm{d}X, \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \end{aligned} \tag{2.63}$$

So if we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right], \tag{2.64}$$

then (2.61) holds, so

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{2.65}$$

The series in equation (2.64) is called the Fourier series for f and a_n and b_n are the Fourier coefficients of f. Again, we use $\sim$ if we do not know whether or not it converges. By Theorem 2.2, under suitable conditions the series in equation (2.61) converges to

$$\frac{g(X_+) + g(X_-)}{2}, \tag{2.66}$$

so we obtain

Theorem 2.4 Let f be a periodic function of period $2L$ which is sufficiently well-behaved. Then the Fourier series of f at x converges to

$$\frac{f(x_+) + f(x_-)}{2}, \tag{2.67}$$

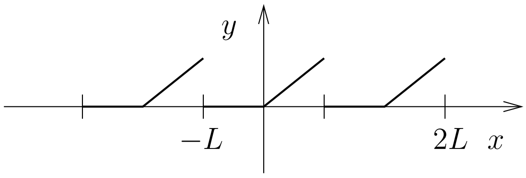
so equation (2.64) holds at any point where f is continuous.

Example 2.3 Find the Fourier series of the $2L$-periodic extension of

$$f(x) = \begin{cases} x & x \in (0, L], \\ 0 & x \in (-L, 0]. \end{cases} \tag{2.68}$$

Hence show that

$$\frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}. \tag{2.69}$$



We have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{L[(-1)^n - 1]}{n^2\pi^2}, \quad n \neq 0, \tag{2.70}$$

as in Example 2.1. So we have $a_{2m} = 0$ for $m > 0$ and

$$a_{2m+1} = \frac{-2L}{(2m+1)^2\pi^2}. \tag{2.71}$$

For a_0 we calculate

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x = \frac{1}{L}\int_{0}^{L} x\,\mathrm{d}x = \frac{L}{2}, \tag{2.72}$$

and for b_n

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \\ &= \frac{1}{L}\left(\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x\right), \\ &= \frac{1}{L}\left(-\frac{L^2(-1)^n}{n\pi} + \left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L\right), \\ &= (-1)^{n+1}\frac{L}{n\pi}. \end{aligned}$$

So

$$f(x) \sim \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2}\cos\left[\frac{(2m+1)\pi x}{L}\right] + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right). \tag{2.73}$$

By Theorem 2.4, if $x \in [0, L)$ we obtain

$$x = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2}\cos\left[\frac{(2m+1)\pi x}{L}\right] + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{L}{m\pi}\sin\left(\frac{m\pi x}{L}\right). \tag{2.74}$$

because f is continuous on $[0, L)$. If we put $x = 0$ we calculate

$$0 = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2}, \tag{2.75}$$

which proves (2.69). If we set $x = L$ in equation (2.73) we obtain

$$\frac{f(L_+) + f(L_-)}{2} = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2} \cos[(2m+1)\pi], \tag{2.76}$$

giving

$$\frac{0 + L}{2} = \frac{L}{4} + \sum_{m=0}^{\infty} \frac{-2L}{(2m+1)^2\pi^2}, \tag{2.77}$$

which gives equation (2.69) again.

2.4.1 Sine and cosine series

Given a function f defined on $[0, L]$ we require an expansion with only cosine terms or only sine terms. This will be done by extending f to be even (for only cosine terms) or odd (for only sine terms) on $(-L, L]$ and then extending to a $2L$-period function. The series obtained will then be valid on $(0, L)$.

Definition If f is defined on $[0, L]$, the *even extension* for f, denoted by f_e, is the periodic extension of

$$f_e(x) = \begin{cases} f(x) & x \in [0, L], \\ f(-x) & x \in (-L, 0), \end{cases} \tag{2.78}$$

so that we have $f_e(x) = f_e(-x)$ for all x. Thus:

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{2.79}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \tag{2.80}$$

is called the *Fourier cosine series* of f.

Definition The *odd extension* for f, denoted by f_o, is the periodic extension of

$$f_o(x) = \begin{cases} f(x) & x \in [0, L], \\ -f(-x) & x \in (-L, 0), \end{cases} \tag{2.81}$$

so that $f_o(x) = -f_o(-x)$ for all $x \neq nL$. Similarly,

$$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.82}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

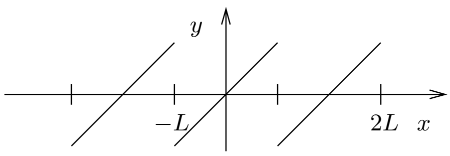
is called the *Fourier sine series* for f.

Note. For f_o to really be odd we must have $f_o(0) = 0$ and also $f_o(L) = -f_o(-L) = -f_o(L)$ (the last equality follows from periodicity) giving $f_o(L) = 0$ and therefore $f_o(nL) = 0$ for all $n \in \mathbb{Z}$. However, the value at of f at these isolated points does not affect the Fourier series.

Example 2.4 Find the Fourier sine and cosine expansions of $f(x) = x$ for $x \in [0, L]$.

Sine expansion The odd extension is defined by

$$f_o(x) = \begin{cases} x & x \in [0, L], \\ -(-x) & x \in (-L, 0). \end{cases} \tag{2.83}$$



In this case

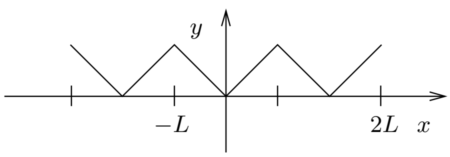
$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = (-1)^{n+1}\frac{2L}{n\pi}, \tag{2.84}$$

as in Example 2.3. For $x \in [0, L)$ we therefore obtain

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right). \tag{2.85}$$

Cosine expansion The even extension is given by

$$f_e(x) = \begin{cases} x & x \in [0, L], \\ -x & x \in [-L, 0). \end{cases} \tag{2.86}$$



Now,

$$a_0 = \frac{2}{L}\int_0^L x\,\mathrm{d}x = L, \tag{2.87}$$

and

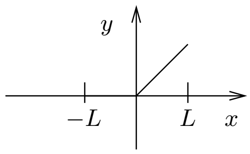
$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \begin{cases} 0 & n = 2m \text{ even}, \\ \frac{-4L}{(2m+1)^2\pi^2} & n = 2m+1 \text{ odd}. \end{cases} \tag{2.88}$$

Thus for $x \in [0, L]$ we get

$$x = \frac{L}{2} + \sum_{m=0}^{\infty} -\frac{4L}{(2m+1)^2\pi^2}\cos\left[\frac{(2m+1)\pi x}{L}\right]. \tag{2.89}$$

Recall that (2.74) is the Fourier series of

$$h(x) = \begin{cases} x & x \in [0, L], \\ 0 & x \in (-L, 0). \end{cases} \tag{2.90}$$



Looking at the results from the previous example this indicates that the Fourier series of $f(x) + g(x)$ equals the Fourier series of $f(x)$ plus the Fourier series of $g(x)$.

Chapter 3

The heat equation

In this chapter we shall look at the heat equation in one space dimension, learning a method for its derivation, and some techniques for solving.

3.1 Derivation in one space dimension

A straight rigid metal rod lies along the x-axis. The lateral surface is insulated to prevent heat loss.

Figure 3

Let ρ be the mass density per unit length, c be the specific heat, $T(x,t)$ be the temperature and:

- $+q(x,t)$ be the heat flux from $-$ to $+$;
- $-q(x,t)$ be the heat flux from $+$ to $-$.

Consider any interval $[a, a+h]$:

$$\begin{aligned} \text{internal energy} &= \int_a^{a+h} \rho c T(x,t)\,\mathrm{d}x; & (3.1) \\ \text{net heat flux out of } [a,a+h] &= q(a+h,t) - q(a,t). & (3.2) \end{aligned}$$

By conservation of energy, for every interval $[a, a+h]$,

$$\text{rate of change of internal energy} + \text{net heat flux out} = 0. \tag{3.3}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^{a+h} \rho c T(x,t)\,\mathrm{d}x + [q(a+h,t) - q(a,t)] = 0. \tag{3.4}$$

Hence, by Leibniz,

$$\frac{1}{h}\int_a^{a+h} \rho c \frac{\partial T}{\partial t}(x,t)\,\mathrm{d}x + \left[\frac{q(a+h,t) - q(a,t)}{h}\right] = 0, \tag{3.5}$$

and on letting $h \to 0$ we get the equation

$$\rho c \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{3.6}$$

In order to close the system, we need to describe how the heat flux varies as a function of x, t and T.

3.1.1 Fourier's law

In one space dimension, the law of heat conduction, also known as Fourier's law, states that the time rate of heat transfer through a material is proportional to the negative gradient in the temperature:

$$q(x,t) = -k \frac{\partial T}{\partial x}, \tag{3.7}$$

where k is the *thermal conductivity.* The negative sign reflects the fact that heat flows from high temperatures to low temperatures.

On substituting from equation (3.7) into (3.6) we arrive at the heat equation in one space dimension:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{3.8}$$

where $\kappa = k/\rho c$ is the *thermal diffusivity.*

3.2 Units and nondimensionalisation

Consider the units of the variables (x, t and T) and parameter (κ) associated with the heat equation. We will use the following notation to denote the dimensions of a variable or parameter:

$$[p] = \text{dimensions of } p. \tag{3.9}$$

In SI units we have

$$[x] = \text{m (metres)}, \quad [t] = \text{s (seconds)}, \quad [T] = \text{K (Kelvin)}. \tag{3.10}$$

For the heat equation,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{3.11}$$

we see that the left-hand side has units Ks^{-1}. The term $\partial^2 T/\partial x^2$ has units Km^{-2}. Hence for the units of the right-hand side to balance those of the left-hand side, the units of κ must be $[\kappa] = \text{m}^2\text{s}^{-1}$.

We can *non-dimensionalise* the heat equation by scaling our variables and parameters. For example, let

$$x = l\hat{x}, \quad t = \tau\hat{t}, \quad T = T_0\hat{T}, \tag{3.12}$$

where l, τ and T_0 are a typical lengthscale, timescale and temperature, respectively, for the problem under consideration. Then

$$\frac{\partial}{\partial t} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\frac{\partial}{\partial \hat{t}} = \frac{1}{\tau}\frac{\partial}{\partial \hat{t}}, \tag{3.13}$$

$$\frac{\partial}{\partial x} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}} = \frac{1}{l}\frac{\partial}{\partial \hat{x}}, \tag{3.14}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}}\left(\frac{1}{l}\frac{\partial}{\partial \hat{x}}\right) = \frac{1}{l^2}\frac{\partial^2}{\partial \hat{x}^2}, \tag{3.15}$$

and substituting into the heat equation we have

$$\frac{T_0}{\tau}\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\kappa T_0}{l^2}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}. \tag{3.16}$$

Rearranging gives

$$\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\kappa \tau}{l^2}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}. \tag{3.17}$$

Considering the problem on a timescale where $\tau = l^2/\kappa$ gives

$$\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial^2 \hat{T}}{\partial \hat{x}^2}. \tag{3.18}$$

Notice that now

$$[\hat{x}] = 1, \quad [\hat{t}] = 1, \quad [\hat{T}] = 1, \tag{3.19}$$

since

$$[l] = \mathrm{m}, \quad [\tau] = \left[\frac{l^2}{\kappa}\right] = \mathrm{s}, \quad [T_0] = \mathrm{K}. \tag{3.20}$$

This means that we can compare heat problems on different scales: for example, two systems with different l and κ will exhibit comparable behaviour on the same time scales if l^2/κ is the same in each problem.

3.3 Heat conduction in a finite rod

Let the rod occupy the interval $[0, L]$. If we look for solutions of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{3.21}$$

which are separable, $T(x,t) = F(x)G(t)$, we find that

$$\underbrace{\frac{\kappa}{F(x)}F''(x)}_{\text{independent of } t} = \underbrace{\frac{1}{G(t)}G'(t)}_{\text{independent of } x}, \tag{3.22}$$

and hence both sides are constant (independent of both x and t). If the constant is $-\kappa\lambda^2$, $F(x)$ satisfies the ODE

$$F''(x) = -\lambda^2 F(x), \tag{3.23}$$

the solution of which is

$$F(x) = A\sin(\lambda x) + B\cos(\lambda x). \tag{3.24}$$

If the ends are held at zero temperature then $F(0) = F(L) = 0$. The boundary condition at $x = 0$ gives $B = 0$ and the boundary condition at $x = L$ gives

$$A \sin(\lambda L) = 0. \tag{3.25}$$

Since we want $A \neq 0$ to avoid a non-trivial solution, it must be that $\sin(\lambda L) = 0$ *i.e.* λ must be such that $\lambda L = n\pi$ where n is a positive integer. Hence λ must be one of the numbers

$$\left\{ \frac{n\pi}{L} : n = 1, 2, 3, \ldots \right\}. \tag{3.26}$$

Moreover $G(t)$ satisfies the ODE

$$G'(t) = -\kappa\lambda^2 G(t) = -\frac{\kappa n^2\pi^2}{L^2} G(t), \tag{3.27}$$

and, therefore, $G(t) \propto e^{-n^2\pi^2\kappa t/L^2}$. Hence we have the separable solution

$$T_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}, \tag{3.28}$$

which satisfies the heat equation (3.8) and the boundary conditions

$$T(0,t) = 0 \text{ and } T(L,t) = 0 \text{ for } t > 0. \tag{3.29}$$

The general solution can therefore be written as a linear combination of the T_n so that

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.30}$$

3.4 Initial and boundary value problem (IBVP)

We now consider finding the solution of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.31}$$

subject to the initial condition

$$T(x,0) = f(x), \quad 0 \leq x \leq L, \tag{3.32}$$

and the boundary conditions

$$T(0,t) = 0 \text{ and } T(L,t) = 0 \text{ for } t > 0. \tag{3.33}$$

In view of our preceding discussion we look for a solution as an infinite sum

$$T(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.34}$$

Example 3.1 Solve the IBVP for the case

$$T(x,0) = \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) = f(x), \quad 0 \leq x \leq L. \tag{3.35}$$

From equation (3.34) we see that

$$T(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right). \tag{3.36}$$

Comparing terms we see that $a_1 = 1$, $a_2 = 1/2$ and $a_n = 0$ $(n \geq 3)$ so that the solution is

$$T(x,t) = \sin\left(\frac{\pi x}{L}\right) e^{-\pi^2\kappa t/L^2} + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) e^{-4\pi^2\kappa t/L^2}. \tag{3.37}$$

3.4.1 Application of Fourier series

To solve for more general initial conditions, we can use Fourier series to determine the constants a_n:

$$T(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L. \tag{3.38}$$

The question is now, given $f(x)$, can it be expanded as a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L? \tag{3.39}$$

From the lectures on Fourier series, we know that such an expansion exists if *e.g.* f is piecewise continuously differentiable on $[0, L]$. The coefficients a_n are determined by the orthogonality relations:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x = \begin{cases} 0, & m \ne n, \\ \frac{1}{2}L, & m = n. \end{cases} \tag{3.40}$$

Thus

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x. \tag{3.41}$$

Example 3.2 Find the solution of the IBVP when

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le L_1 \text{ and } L_2 \le x \le L, \\ 1 & \text{for } L_1 < x < L_2. \end{cases} \tag{3.42}$$

Here $f(x)$ has the Fourier sine expansion

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right), \tag{3.43}$$

and the solution of IBVP is

$$T(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.44}$$

3.5 Uniqueness

We have constructed a solution of our IBVP, and found a formula for it as the sum of an infinite series, but is it the only solution?

Theorem 3.1 The IBVP has only one solution.

Proof. Let U be a solution of the same IBVP, *i.e.*

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \, t > 0, \tag{3.45}$$

subject to the initial condition

$$U(x,0) = f(x), \quad 0 \le x \le L, \tag{3.46}$$

and the boundary conditions

$$U(0,t) = 0 \text{ and } U(L,t) = 0 \text{ for } t > 0. \tag{3.47}$$

Now consider the difference $W := U - T$. Then W satisfies the IBVP

$$\frac{\partial W}{\partial t} = \kappa \frac{\partial^2 W}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.48}$$

$$W(x,0) = 0, \quad 0 \le x \le L, \tag{3.49}$$

and the boundary conditions

$$W(0,t) = 0 \text{ and } W(L,t) = 0 \text{ for } t > 0. \tag{3.50}$$

Let

$$I(t) := \frac{1}{2}\int_0^L [W(x,t)]^2\, \mathrm{d}x. \tag{3.51}$$

Evidently $I(t) \ge 0$ and $I(0) = 0$. By Leibniz's rule,

$$\begin{aligned} I'(t) &= \int_0^L W \frac{\partial W}{\partial t}\, \mathrm{d}x, &(3.52)\\ &= \kappa \int_0^L W \frac{\partial^2 W}{\partial x^2}\, \mathrm{d}x, &(3.53)\\ &= \kappa \int_0^L \left[\frac{\partial}{\partial x}\left(W \frac{\partial W}{\partial x}\right) - \left(\frac{\partial W}{\partial x}\right)^2 \right] \mathrm{d}x. &(3.54) \end{aligned}$$

On carrying out the integration and using the boundary conditions at $x = 0$ and $x = L$ we see that

$$I'(t) = -\kappa \int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x \le 0, \tag{3.55}$$

and, therefore, I cannot increase. Hence

$$0 \le I(t) \le I(0) = 0, \tag{3.56}$$

and $I(t) = 0$ for every $t \ge 0$. Thus

$$\int_0^L [W(x,t)]^2 \mathrm{d}x = 0, \tag{3.57}$$

for every $t \ge 0$ and so $W = 0$ and $U = T$, which proves the theorem. □

3.6 Non-zero steady state

It may be that the temperatures of the ends $x = 0$ and $x = L$ are prescribed and constant but not equal to zero.

Example 3.3 Solve the IBVP

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,\, t > 0, \tag{3.58}$$

subject to the initial condition

$$T(x,0) = 0, \quad 0 \le x \le L, \tag{3.59}$$

and the boundary conditions

$$T(0,t) = T_0 \text{ and } T(L,t) = T_1 \text{ for } t > 0. \tag{3.60}$$

We cannot use separation of variables and Fourier series right at the outset. However, we conjecture that, as $t \to \infty$, $T(x,t) \to U(x)$, where

$$\kappa \frac{\mathrm{d}^2 U}{\mathrm{d}x^2} = 0, \quad U(0) = T_0 \text{ and } U(L) = T_1, \tag{3.61}$$

i.e.

$$U(x) = T_0 \left(1 - \frac{x}{L}\right) + T_1 \left(\frac{x}{L}\right). \tag{3.62}$$

If we now put $S(x,t) := T(x,t) - U(x)$, we find that S is a solution of the IBVP

$$\frac{\partial S}{\partial t} = \kappa \frac{\partial^2 S}{\partial x^2}, \quad 0 < x < L,\ t > 0, \tag{3.63}$$

with

$$S(0,t) = 0 \text{ and } S(L,t) = 0 \text{ for } t > 0, \tag{3.64}$$

and

$$S(x,0) = -T_0 \left(1 - \frac{x}{L}\right) - T_1 \left(\frac{x}{L}\right). \tag{3.65}$$

In view of the form of the boundary conditions, this IBVP can be solved by our previous methods. The solution is

$$S(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[-T_0 + (-1)^n T_1\right] \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}, \tag{3.66}$$

and so

$$T(x,t) = T_0 \left(1 - \frac{x}{L}\right) + T_1 \left(\frac{x}{L}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[-T_0 + (-1)^n T_1\right] \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}. \tag{3.67}$$

3.7 Other boundary conditions

Other boundary conditions are possible, *e.g.* at an end which is *thermally insulated* the heat flux is zero. Thus $-kT_x = 0$ there and, therefore, $T_x = 0$. If both ends are thermally insulated we look for separable solutions of the heat equation of the form

$$T(x,t) = F(x)G(t), \tag{3.68}$$

where $F'(0) = F'(L) = 0$. We find that $F'' = -\lambda^2 F$, $G' = -\lambda^2 \kappa G$, and $F = a\cos(\lambda x)$, where $\sin(\lambda L) = 0$ and so L is one of the numbers $\{n\pi/L : n = 0, 1, 2, 3, \ldots\}$. The separable solutions in these circumstances are

$$a_0 \text{ and } a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2} \quad (n = 1, 2, 3, \ldots). \tag{3.69}$$

Thus if we consider the IBVP

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,\ t > 0, \tag{3.70}$$

with boundary conditions

$$\frac{\partial T}{\partial x}(0,t) = 0 \text{ and } \frac{\partial T}{\partial x}(L,t) = 0 \text{ for } t > 0, \tag{3.71}$$

and initial condition

$$T(x,0) = f(x) \text{ for } 0 \leq x \leq L, \tag{3.72}$$

we look for a solution

$$T(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2\kappa t/L^2}, \tag{3.73}$$

where the prescribed $f(x)$ has the Fourier cosine expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L. \tag{3.74}$$

The required coefficients are

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad (n = 0, 1, 2, 3, \ldots). \tag{3.75}$$

Note that, as $t \to \infty$,

$$T(x,t) \to \frac{1}{2}a_0 = \frac{1}{L}\int_0^L f(s)\,\mathrm{d}s, \tag{3.76}$$

the extreme right-hand side being the mean initial temperature. The uniqueness of $T(x,t)$, for a given $f(x)$, can be established much as before.

Chapter 4

The wave equation

In this chapter we look at the wave equation, concentrating on applications to waves on strings. We discuss methods for solution and also uniqueness of solutions.

4.1 Derivation in one space dimension

Consider a flexible string stretched to a tension T, with mass density ρ, undergoing small transverse vibrations. First suppose the string to be at rest along the x-axis in the (x, y)-plane. A point initially at $x\boldsymbol{i}$ is displaced to $\boldsymbol{r}(x,t) = x\boldsymbol{i} + y(x,t)\boldsymbol{j}$, where $y(x,t)$ is the transverse displacement and $\boldsymbol{i}$ and $\boldsymbol{j}$ are the usual unit vectors along the coordinate axes. We will assume that $|\partial y/\partial x| \ll 1$ and ignore gravity and air-resistance.

Figure 4

The vector

$$\boldsymbol{\tau} := \frac{\partial \boldsymbol{r}}{\partial x} = \boldsymbol{i} + \frac{\partial y}{\partial x}\boldsymbol{j}, \tag{4.1}$$

is a *tangent* vector to the string and, since

$$|\boldsymbol{\tau}| = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} = 1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{1}{8}\left(\frac{\partial y}{\partial x}\right)^4 + \cdots, \tag{4.2}$$

it is approximately a unit tangent. Thus, in the figure:

- $+T\boldsymbol{\tau}$ = force exerted by $+$ on $-$;
- $-T\boldsymbol{\tau}$ = force exerted by $-$ on $+$.

The *velocity* and *acceleration* vectors are

$$\boldsymbol{v} = \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial y}{\partial t}\boldsymbol{j}, \qquad \boldsymbol{a} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \frac{\partial^2 y}{\partial t^2}\boldsymbol{j}, \tag{4.3}$$

respectively.

Consider the piece of string which occupies the interval $[a, a+h]$, where, at a later stage in the argument, $h \to 0$:

Figure 5

$$\text{net force} = T\boldsymbol{\tau}(a+h,t) - T\boldsymbol{\tau}(a,t); \tag{4.4}$$

$$\text{momentum} = \int_a^{a+h} \rho \boldsymbol{v}(x,t)\,\mathrm{d}x. \tag{4.5}$$

By *Newton's Second Law*, for every interval $[a, a+h]$,

$$\text{net force} = \text{rate of change of momentum}, \tag{4.6}$$

$$\Longrightarrow T\boldsymbol{\tau}(a+h,t) - T\boldsymbol{\tau}(a,t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^{a+h} \rho \boldsymbol{v}(x,t)\,\mathrm{d}x. \tag{4.7}$$

On using Leibniz's rule, and dividing through by h, we see that

$$T\left(\frac{\boldsymbol{\tau}(a+h,t) - \boldsymbol{\tau}(a,t)}{h}\right) = \frac{1}{h}\int_a^{a+h} \rho \frac{\partial \boldsymbol{v}}{\partial t}(x,t)\,\mathrm{d}x, \tag{4.8}$$

and, on letting $h \to 0$, that

$$T\frac{\partial \boldsymbol{\tau}}{\partial x}(a,t) = \rho \frac{\partial \boldsymbol{v}}{\partial t}(a,t), \tag{4.9}$$

for every a. Thus, if we substitute for $\boldsymbol{\tau}$ and $\boldsymbol{v}$ in terms of the displacement y, we have

$$\rho \frac{\partial^2 y}{\partial t^2}\boldsymbol{j} = T\frac{\partial^2 y}{\partial x^2}\boldsymbol{j}, \tag{4.10}$$

and, hence, the wave equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \tag{4.11}$$

or

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.12}$$

where $c = \sqrt{T/\rho}$ is the *wave speed*.

4.2 Units and nondimensionalisation

Consider the units of the variables (x, t and y) and parameter (c) associated with the wave equation. For the wave equation,

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.13}$$

we see that the left-hand side has units ms^{-2}. The term $\partial^2 y/\partial x^2$ has units m^{-1}. Hence for the units of the right-hand side to balance those of the left-hand side, the units of c must be $[c] = \mathrm{ms}^{-1}$, as we expect.

As before, can *non-dimensionalise* the wave equation by scaling our variables and parameters. For example, let

$$x = l\hat{x}, \quad t = \tau\hat{t}, \quad y = l\hat{y}, \tag{4.14}$$

where l, and τ are a typical lengthscale and timescale, respectively, for the problem under consideration. Then

$$\frac{\partial}{\partial t} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\frac{\partial}{\partial \hat{t}} = \frac{1}{\tau}\frac{\partial}{\partial \hat{t}}, \tag{4.15}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\mathrm{d}\hat{t}}{\mathrm{d}t}\frac{\partial}{\partial \hat{t}}\left(\frac{1}{\tau}\frac{\partial}{\partial \hat{t}}\right) = \frac{1}{\tau^2}\frac{\partial^2}{\partial \hat{t}^2}, \tag{4.16}$$

$$\frac{\partial}{\partial x} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}} = \frac{1}{l}\frac{\partial}{\partial \hat{x}}, \tag{4.17}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\mathrm{d}\hat{x}}{\mathrm{d}x}\frac{\partial}{\partial \hat{x}}\left(\frac{1}{l}\frac{\partial}{\partial \hat{x}}\right) = \frac{1}{l^2}\frac{\partial^2}{\partial \hat{x}^2}, \tag{4.18}$$

and substituting into the wave equation we have

$$\frac{l}{\tau^2}\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{c^2 l}{l^2}\frac{\partial^2 \hat{y}}{\partial \hat{x}^2}. \tag{4.19}$$

Rearranging gives

$$\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{c^2\tau^2}{l^2}\frac{\partial^2 \hat{y}}{\partial \hat{x}^2}. \tag{4.20}$$

Considering the problem on a timescale where $\tau = l/c$ gives

$$\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{y}}{\partial \hat{x}^2}. \tag{4.21}$$

Notice that now

$$[\hat{x}] = 1, \quad [\hat{t}] = 1, \quad [\hat{y}] = 1, \tag{4.22}$$

since

$$[l] = \mathrm{m}, \quad [\tau] = \left[\frac{l}{c}\right] = \mathrm{s}. \tag{4.23}$$

This gives a relationship between problems with different lengthscales and wave speeds.

4.3 Normal modes of vibration for a finite string

A string is stretched between $x = 0$ and $x = L$ and the ends are held fixed. If the string is plucked, what notes do we hear? The question suggests that we want a solution which is periodic in time, with a period to be determined.

The displacement $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.24}$$

with boundary conditions

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t > 0. \tag{4.25}$$

We use the separation of variables technique *i.e.* we attempt to find some (not all) solutions of equations (4.24) and (4.25) in the separable form

$$y(x,t) = F(x)G(t). \tag{4.26}$$

Substituting from (4.26) into (4.24) gives

$$\underbrace{c^2\frac{F''(x)}{F(x)}}_{\text{independent of } t} = \underbrace{\frac{1}{G(t)}G''(t)}_{\text{independent of } x} . \tag{4.27}$$

Hence both sides are constant, independent of both x and t, and we take this constant to be equal to $-\omega^2$ to get

$$F''(x) = -\frac{\omega^2}{c^2}F(x), \quad G''(t) = -\omega^2 G. \tag{4.28}$$

The ODE for $F(x)$ is to be solved subject to the boundary conditions

$$F(0) = F(L) = 0. \tag{4.29}$$

We have

$$F(x) = A\sin\left(\frac{\omega x}{c}\right) + B\cos\left(\frac{\omega x}{c}\right), \tag{4.30}$$

where the boundary condition at $x = 0$ gives $B = 0$, and the boundary condition at $x = L$ gives

$$A\sin\left(\frac{\omega L}{c}\right) = 0. \tag{4.31}$$

Since we want $A \neq 0$, otherwise $F = 0$ and $y = 0$, it must be that $\sin(\omega L/c) = 0$, *i.e.* ω must be such that $\omega L/c = n\pi$, where n is a positive integer, and ω must be one of the numbers

$$\left\{\frac{n\pi c}{L} : n = 1, 2, 3, \ldots\right\}. \tag{4.32}$$

The ODE for $G(t)$ has the solution

$$G(t) = a\cos(\omega t) + b\sin(\omega t), \tag{4.33}$$

and so equations (4.24) and (4.25) have solutions of the form

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right], \tag{4.34}$$

where $n = 1, 2, 3, \ldots$, and a_n and b_n are arbitrary constants. Such a solution is known as a *normal mode*.

A normal mode is periodic in t,

$$y(x, t+p) = y(x,t), \tag{4.35}$$

with *period*

$$p = \frac{2\pi}{\omega} = \frac{2L}{nc}, \tag{4.36}$$

and *frequency* (pitch)

$$\frac{1}{p} = \frac{\omega}{2\pi} = \frac{nc}{2L}. \tag{4.37}$$

The case $n = 1$ corresponds to the *fundamental frequency* $c/(2L)$, and all other normal frequencies are integer multiples of the fundamental frequency.

Note the graphs of the functions $\sin(n\pi x/L)$:

Figure 6

Figure 7

Figure 8

The general solution can be written as a super-position of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right], \tag{4.38}$$

and satisfies (4.24) and (4.25).

4.4 Initial-and-boundary value problems for finite strings

Consider the following IBVP: find $y(x,t)$ such that

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.39}$$

and y satisfies the initial conditions

$$y(x,0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L, \tag{4.40}$$

and the boundary conditions

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t > 0, \tag{4.41}$$

where $f(x)$ and $g(x)$ are known functions. According to (4.40), the initial transverse displacement and the initial transverse velocity are prescribed. If *e.g.*

$$f(x) = \begin{cases} 2hx/L & 0 \leq x \leq L/2, \\ 2h(L-x)/L & L/2 \leq x \leq L, \end{cases} \tag{4.42}$$

and $g(x) = 0$, $0 \leq x \leq L$, the mid-point of the string is pulled aside a distance h and the string is released from rest.

Figure 9

We know that the general solution can be written as in equation (4.38), and hence the boundary conditions must satisfy

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad \frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) b_n \sin\left(\frac{n\pi x}{L}\right). \tag{4.43}$$

Example 4.1 Solve the IBVP for the case

$$f(x) = A\sin\left(\frac{\pi x}{L}\right) + B\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right), \quad g(x) = 0. \tag{4.44}$$

Since

$$f(x) = A\sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}B\sin\left(\frac{2\pi x}{L}\right), \tag{4.45}$$

the solution is obtained by taking $a_1 = A$, $a_2 = B/2$, $a_n = 0$ for $n \geq 3$ and $b_n = 0\ \forall n$ to give

$$y(x,t) = A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) + \frac{1}{2}B\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right). \tag{4.46}$$

Example 4.2 Solve the IBVP for the case

$$f(x) = 0, \quad g(x) = \sin^3\left(\frac{\pi x}{L}\right). \tag{4.47}$$

Since

$$g(x) = \frac{3}{4}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{L}\right), \tag{4.48}$$

we take $a_n = 0\ \forall n$ and identify

$$\left(\frac{\pi c}{L}\right)b_1 = \frac{3}{4}, \quad \left(\frac{2\pi c}{L}\right)b_2 = 0, \quad \left(\frac{3\pi c}{L}\right)b_3 = -\frac{1}{4}, \quad b_n = 0 \text{ for } n \geq 4, \tag{4.49}$$

to arrive at

$$y(x,t) = \frac{3L}{4\pi c}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi ct}{L}\right) - \frac{L}{12\pi c}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi ct}{L}\right). \tag{4.50}$$

4.4.1 Application of Fourier series

To solve for more general initial conditions, we again look for a solution as a superposition of normal modes:

$$y(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right], \tag{4.51}$$

so that we arrive at the problem: given $f(x)$ and $g(x)$ can they be expanded as Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L, \tag{4.52}$$

$$g(x) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)b_n\sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L? \tag{4.53}$$

From the lectures on Fourier Series we know that such an expansion as (4.52) exists if *e.g.* f and g are piecewise continuously differentiable on $[0, L]$. The coefficients are determined by the orthogonality relations:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \begin{cases} 0 & m \neq n, \\ \frac{1}{2}L & m = n. \end{cases} \tag{4.54}$$

Thus

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \tag{4.55}$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{4.56}$$

Example 4.3 (Guitar or lute) For the special case (4.42),

$$a_n = \frac{2}{L}.\frac{2h}{L}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x + \frac{2}{L}.\frac{2h}{L}\int_{\frac{L}{2}}^L (L-x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \tag{4.57}$$

$$= \frac{8h}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right), \tag{4.58}$$

and

$$b_n = \frac{2}{n\pi c}\int_0^L 0.\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = 0. \tag{4.59}$$

Hence the solution is

$$y(x,t) = \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \tag{4.60}$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi ct}{L}\right) - \ldots\right]. \tag{4.61}$$

Example 4.4 (Piano) The initial transverse displacement is zero and the section $[l_1, l_2]$ is given an initial transverse velocity v. Here $f(x) = 0$ for $0 \leq x \leq L$, and

$$g(x) = \begin{cases} 0 & \text{for } 0 \leq x < L_1 \text{ and } L_2 < x \leq L, \\ v & \text{for } L_1 \leq x \leq L_2. \end{cases} \tag{4.62}$$

Thus $a_n = 0$ and

$$b_n = \frac{2}{n\pi c}\int_{L_1}^{L_2} v\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{2vL}{n^2\pi^2 c}\left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right]. \tag{4.63}$$

The transverse displacement is

$$y(x,t) = \frac{2vL}{\pi^2 c}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right). \tag{4.64}$$

4.5 Normal modes for a weighted string

A string of length $2L$ has fixed ends and a mass m is attached to the mid-point; find the normal frequencies of vibration.

Figure 10

Let the string occupy the interval $[-L, L]$, the mass being attached at $x = 0$. Let $y^-(x,t)$ and $y^+(x,t)$ be the transverse displacements for $-L \le x < 0$ and $0 < x \le L$, respectively. Then y^- and y^+ must satisfy the wave equations

$$\frac{\partial^2 y^-}{\partial t^2} = c^2 \frac{\partial^2 y^-}{\partial x^2}, \quad \frac{\partial^2 y^+}{\partial t^2} = c^2 \frac{\partial^2 y^+}{\partial x^2}, \tag{4.65}$$

and the boundary conditions

$$y^-(-L,t) = 0 \text{ and } y^+(L,t) = 0 \text{ for } t > 0. \tag{4.66}$$

What conditions hold at the mass m? There are two: firstly,

$$y^-(0,t) = y^+(0,t) \text{ for } t > 0; \tag{4.67}$$

and, secondly, the mass m is subject to Newton's Second Law,

Figure 11

$$m\frac{\partial^2 y}{\partial t^2}(0,t)\boldsymbol{j} = T\left(\boldsymbol{i} + \frac{\partial y^+}{\partial x}\boldsymbol{j}\right) - T\left(\boldsymbol{i} + \frac{\partial y^-}{\partial x}\boldsymbol{j}\right), \tag{4.68}$$

i.e.

$$m\frac{\partial^2 y}{\partial t^2}(0,t) = T\left[\frac{\partial y^+}{\partial x}(0,t) - \frac{\partial y^-}{\partial x}(0,t)\right] \text{ for } t > 0. \tag{4.69}$$

If we apply separation of variables arguments to (4.65) and (4.66) we see that y^-, y^+ must be of the form

$$y^-(x,t) = A\sin\left(\frac{\omega}{c}(L+x)\right)\cos(\omega t + \epsilon), \tag{4.70}$$

$$y^+(x,t) = B\sin\left(\frac{\omega}{c}(L-x)\right)\cos(\omega t + \epsilon), \tag{4.71}$$

where A, B, ϵ are constants and $\omega/(2\pi)$ is the, as yet unknown, normal frequency. Substitution into the boundary conditions (4.67) and (4.69) gives

$$A\sin\left(\frac{\omega L}{c}\right) = B\sin\left(\frac{\omega L}{c}\right), \tag{4.72}$$

and

$$-m\omega^2 A\sin\left(\frac{\omega L}{c}\right) = T\left[-B\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega L}{c}\right) - A\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega L}{c}\right)\right], \tag{4.73}$$

i.e.

$$A\left[\left(\frac{m\omega c}{T}\right)\sin\left(\frac{\omega L}{c}\right) - \cos\left(\frac{\omega L}{c}\right)\right] = B\cos\left(\frac{\omega L}{c}\right). \tag{4.74}$$

If the linear homogeneous equations (4.72) and (4.74) are to have non-trivial solutions (A and B not both zero) then the determinant must equal zero:

$$\sin\left(\frac{\omega L}{c}\right)\cos\left(\frac{\omega L}{c}\right) = \sin\left(\frac{\omega L}{c}\right)\left\{\left(\frac{m\omega c}{T}\right)\sin\left(\frac{\omega L}{c}\right) - \cos\left(\frac{\omega L}{c}\right)\right\}. \tag{4.75}$$

Thus either

$$\sin\left(\frac{\omega L}{c}\right) = 0 \quad \text{or} \quad \cot\left(\frac{\omega L}{c}\right) = \frac{m\omega c}{2T}. \tag{4.76}$$

The first equality in equation (4.76) implies

$$\frac{\omega L}{c} = n\pi \quad \text{i.e.} \quad \frac{\omega}{2\pi} = \frac{nc}{2L} = 2n.\frac{c}{2.2L}, \tag{4.77}$$

These correspond to the normal frequencies of a string of length $2L$ for which $x = 0$ is a node. There is no simple formula for the solutions of the other equality. If we put $\theta = \omega L/c$ then $\omega = c\theta/L$, where

$$\cot\theta = \frac{mc^2}{2TL}\theta = \frac{m}{2\rho L}\theta, \tag{4.78}$$

and we see there are infinitely many roots $\theta_1, \theta_2, \theta_3, \ldots$ and these determine infinitely many normal frequencies in addition to those given by (4.77).

Figure 12

4.6 Uniqueness of an IBVP for a finite string

We consider the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.79}$$

and prove a uniqueness theorem based on energy considerations. The *kinetic energy* of the string is

$$\frac{1}{2}\int_0^L \rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x. \tag{4.80}$$

The *stress energy* is the product of the tension and the extension, where the extension is

$$\int_0^L \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\, \mathrm{d}x - L = \int_0^L \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \ldots\right] \mathrm{d}x - L, \tag{4.81}$$

$$\approx \frac{1}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. \tag{4.82}$$

Thus

$$E(t) := \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right] \mathrm{d}x, \tag{4.83}$$

is the *energy* of the string. The energy appears to depend upon the time but in important cases it is actually constant.

Lemma 4.1 If $y(x,t)$ is a solution of the wave equation (4.79) and satisfies the boundary conditions

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t \geq 0, \tag{4.84}$$

then $E(t)$ is constant for $t \geq 0$.

Proof. Leibniz's rule applied to equation (4.83) gives

$$E'(t) = \int_0^L \left[\rho \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] \mathrm{d}x, \tag{4.85}$$

and, on substituting for $\rho\, \partial^2 y/\partial t^2$ from the wave equation (4.79), we find that

$$\begin{aligned} E'(t) &= T\int_0^L \left[\frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] \mathrm{d}x, & (4.86)\\ &= T\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right) \mathrm{d}x, & (4.87)\\ &= T\left[\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right]_0^L, & (4.88)\end{aligned}$$

hence $E'(t) = 0$ since the boundary conditions (4.84) tell us that $\partial y/\partial t(0,t) = 0$ and $\partial y/\partial t(L,t) = 0$ for $t \geq 0$. Thus $E(t)$ is constant. □

Theorem 4.2 (Uniqueness) For each pair of functions f and g, the IBVP

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.89}$$

$$y(x,0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = g(x) \text{ for } 0 \leq x \leq L, \tag{4.90}$$

$$y(0,t) = 0 \text{ and } y(L,t) = 0 \text{ for } t > 0, \tag{4.91}$$

has at most one solution. [That a solution exists we know since we have constructed a solution with the aid of separation of variables and Fourier series.]

Proof. Let $y(x,t)$ and $u(x,t)$ both be solutions of the IBVP. Consider the difference $w := y - u$ and the associated energy

$$E(t) := \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + T\left(\frac{\partial w}{\partial x}\right)^2\right] \mathrm{d}x. \tag{4.92}$$

Note that $E \geq 0$. Now $w(x,t)$ is a solution of the IBVP

$$\rho\frac{\partial^2 w}{\partial t^2} = T\frac{\partial^2 w}{\partial x^2}, \quad \text{for } 0 < x < L \text{ and } t > 0, \tag{4.93}$$

$$w(x,0) = 0, \quad \frac{\partial w}{\partial t}(x,0) = 0 \quad (0 \leq x \leq L) \tag{4.94}$$

$$w(0,t) = 0 \text{ and } w(L,t) = 0 \quad (t > 0). \tag{4.95}$$

By Lemma 4.1 and the boundary conditions (4.95), $E(t) \equiv$ constant, and, in view of the initial conditions (4.94), $E(0) = 0$. Hence $E(t) = 0$ for every $t > 0$. Thus $\partial w/\partial x = 0$, $\partial w/\partial t = 0$ and $w(x,t)$ is independent of both x and t. Using (4.95) again tells us that $w(x,t) \equiv 0$. Thus $y = u$ and the solution is unique. □

4.7 The general solution of the wave equation

The wave equation is untypical among PDEs in that it is possible to write down all the solutions. Note that if $F : \mathbb{R} \to \mathbb{R}$ is any (twice differentiable) function, and

$$y(x,t) := F(x - ct), \tag{4.96}$$

then

$$\frac{\partial y}{\partial x} = F'(x - ct), \quad \frac{\partial^2 y}{\partial x^2} = F''(x - ct), \tag{4.97}$$

and

$$\frac{\partial y}{\partial t} = -cF'(x - ct), \quad \frac{\partial^2 y}{\partial t^2} = c^2 F''(x - ct), \tag{4.98}$$

and so (4.96) is a solution of the wave equation. Equation (4.96) represents a wave of constant shape propagating in the positive x-direction with speed c.

Figure 13

Similarly, if $G : \mathbb{R} \to \mathbb{R}$ is any (twice differentiable) function and

$$y(x,t) := G(x + ct), \tag{4.99}$$

then $y(x,t)$ is a solution of the wave equation. Equation (4.99) represents a wave of constant shape propagating in the negative x-direction with speed c.

Figure 14

Again, the sum

$$y(x,t) := F(x - ct) + G(x + ct), \tag{4.100}$$

is a solution of the wave equation. It will now be shown that *every solution of the wave equation must be of the form* (4.100).

To verify this introduce new independent variables

$$\xi := x - ct, \quad \eta := x + ct, \tag{4.101}$$

and seek a solution $y(x,t) = Y(\xi, \eta)$. Then

$$\frac{\partial y}{\partial x} = \frac{\partial Y}{\partial \xi} + \frac{\partial Y}{\partial \eta}, \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 Y}{\partial \xi^2} + 2\frac{\partial^2 Y}{\partial \xi \partial \eta} + \frac{\partial^2 Y}{\partial \eta^2}, \tag{4.102}$$

$$\frac{\partial y}{\partial t} = -c\frac{\partial Y}{\partial \xi} + c\frac{\partial Y}{\partial \eta}, \quad \frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 Y}{\partial \xi^2} - 2c^2\frac{\partial^2 Y}{\partial \xi \partial \eta} + c^2\frac{\partial^2 Y}{\partial \eta^2}, \tag{4.103}$$

and substitution into the wave equation gives

$$\frac{\partial^2 Y}{\partial \xi^2} + 2\frac{\partial^2 Y}{\partial \xi \partial \eta} + \frac{\partial^2 Y}{\partial \eta^2} = \frac{\partial^2 Y}{\partial \xi^2} - 2\frac{\partial^2 Y}{\partial \xi \partial \eta} + \frac{\partial^2 Y}{\partial \eta^2}. \tag{4.104}$$

Hence in the new variables the wave equation transforms to the equation

$$\frac{\partial^2 Y}{\partial \xi \partial \eta} = 0, \tag{4.105}$$

i.e.

$$\frac{\partial}{\partial \xi}\left(\frac{\partial Y}{\partial \eta}\right) = 0. \tag{4.106}$$

Thus $\partial Y/\partial \eta$ is independent of ξ and is a function of η only, say $G'(\eta)$, *i.e.*

$$\frac{\partial Y}{\partial \eta} = G'(\eta), \tag{4.107}$$

and so

$$\frac{\partial}{\partial \eta}\left[Y - G(\eta)\right] = 0. \tag{4.108}$$

Thus $Y - G(\eta)$ is a function of ξ only, say $F(\xi)$, and therefore

$$Y - G(\eta) = F(\xi), \tag{4.109}$$

and

$$Y(\xi, \eta) = F(\xi) + G(\eta) \quad \Longrightarrow \quad y(x,t) = F(x - ct) + G(x + ct). \tag{4.110}$$

Further use of this conclusion will be made later.

Example 4.5 A string occupies $-\infty < x \leq 0$ and is fixed at $x = 0$. A wave $y(x,t) = F(x - ct)$ is incident from $x < 0$. Find the reflected wave.

Figure 15

The solution of the wave equation is

$$y = \underbrace{F(x - ct)}_{\text{incident}} + \underbrace{G(x + ct)}_{\text{reflected}}, \tag{4.111}$$

where G is to be found. The condition $y(0,t) = 0$ is to be satisfied for all t. Hence $F(-ct) + G(ct) = 0$, for all t, and so $G(\theta) = -F(-\theta)$ for all θ. Thus

$$y(x,t) = \underbrace{F(x - ct)}_{\text{incident}} - \underbrace{F(-x - ct)}_{\text{reflected}}. \tag{4.112}$$

4.8 Waves on infinite strings: D'Alembert's formula

When the string occupies the interval $(-\infty, \infty)$ the fact that the solution has the form (4.100),

$$y(x,t) = F(x - ct) + G(x + ct), \tag{4.113}$$

is especially useful since we cannot now use separation of variables and Fourier series.

Consider the following IVP: find $y(x,t)$ if

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad \text{for } -\infty < x < \infty \text{ and } t > 0, \tag{4.114}$$

$$y(x,0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x,0) = g(x) \text{ for } -\infty < x < \infty, \tag{4.115}$$

where f and g are prescribed functions.

To solve this we attempt to choose F and G in (4.113) so as to satisfy the initial conditions (4.115):

$$F(x) + G(x) = f(x), \quad -cF'(x) + cG'(x) = g(x). \tag{4.116}$$

The second integrates to give

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a, \tag{4.117}$$

where a is a constant. Hence

$$F(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s - a\right], \tag{4.118}$$

$$G(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a\right], \tag{4.119}$$

and so

$$y(x,t) = \frac{1}{2}\left[f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(s)\,\mathrm{d}s - a\right] + \frac{1}{2}\left[f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,\mathrm{d}s + a\right]. \tag{4.120}$$

Thus we arrive at *D'Alembert's Formula*

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{4.121}$$

The argument shows that, for given f and g, the IVP has one and only one solution (*i.e.* existence and uniqueness).

Next let us ask how the solution at a point $(x_0, t_0) = P$, in the upper half of the (x,t)-plane, depends upon the data f and g. We have

$$y(x_0,t_0) = \frac{1}{2}[f(x_0-ct_0) + f(x_0+ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(x)\,\mathrm{d}x, \tag{4.122}$$

and this equation is usefully interpreted in terms of the following diagram.

Figure 16

4.8.1 Characteristic diagram

Through P draw the characteristic lines $x - ct = x_0 - ct_0$, $x + ct = x_0 + ct_0$ to intersect the x-axis at Q and R, as shown. Then

$$y(P) = \frac{1}{2}[f(Q) + f(R)] + \frac{1}{2c}\int_Q^R g(x)\,\mathrm{d}x. \tag{4.123}$$

Hence:

- $y(P)$ depends on f only through the values taken by f at Q and R;
- $y(P)$ depends on g only through the values taken by g on the interval QR.

The interval QR is the *domain of dependence* of P. This means that arbitrary alterations to f and g outside the interval QR have no effect on $y(P)$. The D'Alembert formula provides an explicit formula for y but care is required if f and g have different analytic behaviours in different stretches.

Example 4.6 Find the solution of the wave equation for which

$$y(x,0) = 0, \quad -\infty < x < \infty, \tag{4.124}$$

$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} 0 & x < -l, \\ vx/l & -l \le x \le l, \\ 0 & x > l. \end{cases} \tag{4.125}$$

In this case $g(x) = \partial y/\partial t(x,0)$ changes its analytic behaviour at the points $(-l,0)$ and $(l,0)$. We construct the characteristics through these points and thus divide up the upper-half of the (x,t)-plane into six regions $\mathbb{R}_1, \ldots, \mathbb{R}_6$, as shown ($\mathbb{R}_1$ is below $x + ct = -l$, $\mathbb{R}_2$ is above $x + ct = -l$ and above $x - ct = -l$, $\mathbb{R}_3$ is below $x - ct = -l$ and below $x + ct = l$, $\mathbb{R}_4$ is above $x + ct = l$ and above $x - ct = -l$, $\mathbb{R}_5$ is above $x + ct = l$ and above $x - ct = l$ and $\mathbb{R}_6$ is below $x - ct = l$).

Figure 17

In $\mathbb{R}_1$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0. \tag{4.126}$$

In $\mathbb{R}_2$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{-l} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-l}^{x+ct} \frac{vs}{l}\,\mathrm{d}s = \frac{v}{4lc}\left[(x+ct)^2 - l^2\right]. \tag{4.127}$$

In $\mathbb{R}_3$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{vs}{l}\,\mathrm{d}s = \frac{v}{4lc}\left[(x+ct)^2 - (x-ct)^2)\right] = \frac{vxt}{l}. \tag{4.128}$$

In $\mathbb{R}_4$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{-l} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-l}^{l} \frac{vs}{l}\,\mathrm{d}s + \frac{1}{2c}\int_{l}^{x+ct} 0\,\mathrm{d}s = 0. \tag{4.129}$$

In $\mathbb{R}_5$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{l} \frac{vs}{l}\,\mathrm{d}s + \frac{1}{2c}\int_{l}^{x+ct} 0\,\mathrm{d}s = \frac{v}{4lc}\left[l^2 - (x-ct)^2\right]. \tag{4.130}$$

In $\mathbb{R}_6$,

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0. \tag{4.131}$$

From this information we can find the shape of the string. We illustrate this for a time $t > l/c$:

- for $x < -l - ct$ the point $(x,t) \in \mathbb{R}_1$ and $y(x,t) = 0$;
- for $-l - ct < x < l - ct$, $(x,t) \in \mathbb{R}_2$ and $y(x,t) = \frac{v}{4lc}\left[(x+ct)^2 - l^2\right]$;
- for $l - ct < x < -l + ct$, $(x,t) \in \mathbb{R}_4$ and $y(x,t) = 0$;
- for $-l + ct < x < l + ct$, $(x,t) \in \mathbb{R}_5$ and $y(x,t) = \frac{v}{4lc}\left[l^2 - (x-ct)^2\right]$;
- for $x > l + ct$, $(x,t) \in \mathbb{R}_6$ and $y(x,t) = 0$.

Figure 18

As t increases we see two packets of displacement, one moving to the left with speed c and the other to the right with speed c. Between them the displacement is zero.

Chapter 5

Laplace's equation in the plane

Steady two-dimensional heat flow is governed by Laplace's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{5.1}$$

If r and θ are the usual plane polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{5.2}$$

Laplace's equation becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{5.3}$$

For rectangles, discs, the exteriors of discs, and annuli, we can use separation of variables and Fourier series to construct solutions of (5.1) and (5.3).

5.1 BVP in cartesian coordinates

Consider the following BVP: find the solution of Laplace's equation (5.1) which is defined on the rectangle $\{(x, y) : 0 \le x \le a,\ 0 \le y \le b\}$ and satisfies the boundary conditions

$$T(0, y) = T(a, y) = 0, \quad 0 < y < b, \tag{5.4}$$

$$T(x, 0) = 0 \text{ and } T(x, b) = f(x), \quad 0 < x < a. \tag{5.5}$$

Figure 19

To begin we look for special solutions of Laplace's equation of the separable form $T(x, y) = F(x)G(y)$. Substituting into (5.1) and dividing through by $F(x)G(y)$ gives

$$\frac{1}{F(x)}F^{''}(x) + \frac{1}{G(y)}G^{''}(y) = 0. \tag{5.6}$$

Hence there is a constant λ such that

$$F^{''}(x) = -\lambda F(x), \quad G^{''}(y) = \lambda G(y). \tag{5.7}$$

The choices $\lambda = n^2\pi^2/a^2$, $F(x) = \sin(n\pi x/a)$ $(n = 1, 2, 3...)$ give the solutions

$$T(x, y) = \sin\left(\frac{n\pi x}{a}\right) G(y), \tag{5.8}$$

which satisfy the boundary conditions (5.3). Furthermore $G(y)$ is a solution of the ODE

$$G''(y) = \frac{n^2 y^2}{a^2} G(y), \tag{5.9}$$

and so

$$G(y) = A\cosh\left(\frac{n\pi y}{a}\right) + B\sinh\left(\frac{n\pi y}{a}\right), \tag{5.10}$$

in which hyperbolic functions, rather than trigonometric functions, occur. The choice $A = 0$ ensures that the boundary condition $T(x, 0) = 0$ $(0 < x < a)$ holds and thus we are led to consider solutions of the BVP of the form

$$T(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{5.11}$$

On setting $y = b$ we see that the coefficients B_n are determined by the condition that

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = f(x), \quad 0 < x < a. \tag{5.12}$$

Hence

$$B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(s)\sin\left(\frac{n\pi s}{a}\right) \mathrm{d}s, \tag{5.13}$$

and the BVP has the solution

$$T(x, y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{1}{\sinh(n\pi b/a)} \left[\int_0^a f(s)\sin\left(\frac{n\pi s}{a}\right) \mathrm{d}s\right] \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{5.14}$$

5.2 BVP in polar coordinates

Next, let us consider separable solutions of Laplace's equation of the form $T(r, \theta) = F(r)G(\theta)$. Substitution into (5.3) gives

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) + \frac{1}{r^2}F(r)G''(\theta) = 0. \tag{5.15}$$

Hence

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} + \frac{G''(\theta)}{G(\theta)} = 0, \tag{5.16}$$

and there is a constant λ such that

$$r^2 F''(r) + rF'(r) = \lambda F(r), \tag{5.17}$$

$$G''(\theta) = -\lambda G(\theta). \tag{5.18}$$

The function $G(\theta)$ must be periodic with period 2π so that $G(\theta + 2\pi) = G(\theta)$, and this is possible only if $\lambda = n^2$, where n is an integer. If $n = 0$, the only solutions of (5.18) which are periodic are $G(\theta) \equiv$ constant. If $n \neq 0$ the periodic solutions are arbitrary linear combinations of $\cos(n\theta)$ and $\sin(n\theta)$. When $n = 0$ the solutions of (5.17) are of the form $F(r) = A + B\log r$. When $n \neq 0$, equation (5.17) is of Euler's type and $F(r)$ is a linear combination of r^n and r^{-n}. Thus there are separable solutions of Laplace's equation of the forms

$$T(r) = A + B\log r, \tag{5.19}$$

and

$$T(r,\theta) = \left(Ar^n + Br^{-n}\right)\left[C\cos(n\theta) + D\sin(n\theta)\right], \tag{5.20}$$

where n is a positive integer and A, B, C, D are arbitrary constants. The solutions $\log r$, $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$ are not defined at $r = 0$ and so are not admissible if the origin belongs to the domain in which T is defined.

Example 5.1 Find T so as to satisfy Laplace's equation in the annulus $a < r < b$ and the boundary conditions

$$T = T_0 \text{ on } r = a, \quad T = T_1 \text{ on } r = b, \tag{5.21}$$

where T_0 and T_1 are constants. By inspection, $T = A + B\log r$, where A and B must satisfy

$$A + B\log a = T_0, \quad A + B\log b = T_1. \tag{5.22}$$

Then

$$A = \frac{T_0\log b - T_1\log a}{\log(b/a)}, \quad B = \frac{T_1 - T_0}{\log(b/a)}, \tag{5.23}$$

and

$$T(r,\theta) = \frac{T_0\log b - T_1\log a}{\log(b/a)} + \frac{T_1 - T_0}{\log(b/a)}\log r. \tag{5.24}$$

Example 5.2 A conductor occupies the region $r \leq a$ and the temperature satisfies the boundary condition $T(a,\theta) = \sin^3\theta$. Find $T(r,\theta)$ in $r < a$.

Note that

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta). \tag{5.25}$$

Hence

$$T = \frac{3}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{1}{4}\left(\frac{r}{a}\right)^3\sin(3\theta). \tag{5.26}$$

5.2.1 Application of Fourier series

We wish to find T so as to satisfy Laplace's equation in the disc $0 \leq r < a$ and the boundary condition $T = f(\theta)$ on $r = a$, $(0 \leq \theta \leq 2\pi)$, where f is a prescribed function. Here the solution is of the form

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n\left[C_n\cos(n\theta) + D_n\sin(n\theta)\right], \tag{5.27}$$

and the boundary condition gives

$$A + \sum_{n=1}^{\infty} a^n\left[C_n\cos(n\theta) + D_n\sin(n\theta)\right] = f(\theta), \quad 0 \leq \theta \leq 2\pi. \tag{5.28}$$

Thus

$$A = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta, \tag{5.29}$$

$$C_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)\,\mathrm{d}\theta, \tag{5.30}$$

$$D_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)\,\mathrm{d}\theta. \tag{5.31}$$

Example 5.3 Find $T(r,\theta)$ so as to satisfy Laplace's equation in the disc $r < a$ and the boundary condition

$$T(a,\theta) = |\sin\theta|, \quad 0 \le \theta \le 2\pi. \tag{5.32}$$

The solution is

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n \left[C_n\cos(n\theta) + D_n\sin(n\theta)\right], \tag{5.33}$$

where

$$A = \frac{1}{2\pi}\int_0^{2\pi} |\sin\theta|\,\mathrm{d}\theta = \frac{1}{2\pi}\left[\int_0^{\pi}\sin\theta\,\mathrm{d}\theta - \int_{\pi}^{2\pi}\sin\theta\,\mathrm{d}\theta\right] = \frac{2}{\pi}, \tag{5.34}$$

$$C_n = \frac{1}{\pi a^n}\int_0^{2\pi} |\sin\theta|\cos(n\theta)\,\mathrm{d}\theta = \begin{cases} 0 & n \text{ odd},\\ -4/(\pi a^n(n^2-1)) & n \text{ even},\end{cases} \tag{5.35}$$

$$D_n = \frac{1}{\pi a^n}\int_0^{2\pi} |\sin\theta|\sin(n\theta)\,\mathrm{d}\theta = 0, \tag{5.36}$$

and so

$$T(r,\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{2n}\frac{\cos(2n\theta)}{4n^2-1}. \tag{5.37}$$

5.2.2 Poisson's formula

Consider again the problem from Section 5.2.1: find T so as to satisfy Laplace's equation in the disc $0 \le r < a$ and the boundary condition $T = f(\theta)$ on $r = a$, $(0 \le \theta \le 2\pi)$, where f is a prescribed function. Poisson's formula states that the solution to this problem can be written

$$T(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_0^{2\pi}\frac{f(\phi)}{a^2+r^2-2ar\cos(\theta-\phi)}\,\mathrm{d}\phi. \tag{5.38}$$

Lemma 5.1 If λ and α are real and $|\lambda| < 1$ then

$$\frac{1}{2} + \sum_{n=1}^{\infty}\lambda^n\cos n\alpha = \frac{1-\lambda^2}{2(1+\lambda^2-2\lambda\cos\alpha)}. \tag{5.39}$$

Proof.

$$\sum_{n=1}^{\infty} \lambda^n \cos n\alpha = R_e \sum_{n=1}^{\infty} \lambda^n e^{in\alpha}, \quad (5.40)$$

$$= R_e \sum_{n=1}^{\infty} (\lambda e^{i\alpha})^n, \quad (5.41)$$

$$= R_e \left[\frac{\lambda e^{i\alpha}}{1 - \lambda e^{i\alpha}} \right], \quad (5.42)$$

$$= \frac{\lambda \cos\alpha - \lambda^2}{1 + \lambda^2 - 2\lambda\cos\alpha}. \quad (5.43)$$

Hence

$$\frac{1}{2} + \sum_{n=1}^{\infty} \lambda^n \cos n\alpha = \frac{1 - \lambda^2}{2(1 + \lambda^2 - 2\lambda\cos\alpha)}. \quad (5.44)$$

□

Proof of Poisson's formula. Taking care over the dummy variable of integration we get

$$T(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, \mathrm{d}\phi \quad (5.45)$$

$$+\frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\cos(n\theta) \int_0^{2\pi} f(\phi) \cos(n\phi)\, \mathrm{d}\phi \right.$$

$$\left. + \sin(n\theta) \int_0^{2\pi} f(\phi) \sin(n\phi)\, \mathrm{d}\phi \right], \quad (5.46)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta - \phi)] \right\} f(\phi)\, \mathrm{d}\phi, \quad (5.47)$$

and, on appealing to Lemma 5.1, with $\lambda = r/a$ and $\alpha = \theta - \phi$, the result follows. □

Corollary 5.2 (Mean value property of solution of Laplace's equation) The value of T at the centre of the disc is

$$T(0,\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, \mathrm{d}\phi = \text{mean value over boundary}. \quad (5.48)$$

5.3 Uniqueness

Uniqueness is established with the aid of *Green's Theorem*:

Theorem 5.3 (Green's theorem) If R is a bounded and connected plane region whose boundary ∂R is the union $C_1 \cup \ldots \cup C_n$ of a finite number of simple closed curves, oriented so that R is on the left, and if $p(x,y)$ and $q(x,y)$ are continuous and have continuous first derivatives on $R \cup \partial R$, then

$$\int_{\partial R} p\, \mathrm{d}x + q\, \mathrm{d}y = \int\int_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y. \quad (5.49)$$

Figure 20

In the figure, R is the shadowed region. It has two 'holes' and ∂R is the union of three simple closed curves oriented as shown.

5.3.1 Uniqueness for the Dirichlet problem

We consider uniqueness of solutions to the Dirichlet problem, working in Cartesian coordinates.

Theorem 5.4 Consider the BVP

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ in } R, \quad T = f \text{ on } \partial R, \tag{5.50}$$

where f is a prescribed function and R is a bounded and connected region as in the statement of Green's theorem. Then the BVP has at most one solution.

Proof. Let S also be a solution, so that

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0 \text{ in } R, \quad S = f \text{ on } \partial R. \tag{5.51}$$

Then the difference $W := T - S$ in a solution of the BVP

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \text{ in } R, \quad W = 0 \text{ on } \partial R. \tag{5.52}$$

Consider the identity

$$W\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) + \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = \frac{\partial}{\partial x}\left(W\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right). \tag{5.53}$$

Integrate both sides over R and appeal to Laplace's equation and Green's theorem to find that

$$\begin{aligned}\int\!\!\int_R \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y &= \int\!\!\int_R \frac{\partial}{\partial x}\left[\left(W\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right)\right] \mathrm{d}x\,\mathrm{d}y, \\ &= \int_{\partial R}\left[-W\frac{\partial W}{\partial y}\,\mathrm{d}x + W\frac{\partial W}{\partial x}\,\mathrm{d}y\right]. \end{aligned} \tag{5.54}$$

Since $W = 0$ on ∂R the line integral must vanish and so

$$\int\!\!\int_R \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y = 0. \tag{5.55}$$

This is possible only if $\partial W/\partial x = 0$, $\partial W/\partial y = 0$ in R. Hence W is constant and since $W = 0$ on ∂R the constant can only equal zero. Hence $T = S$ and the solution is unique. □

Example 5.4 We now consider a BVP in an unbounded region, for which our uniqueness proof does not apply. A conductor occupies the region $r \geq a$, *i.e.* that exterior to the circle $r = a$, and T satisfies the boundary condition $T = x$ on $r = a$. Find T in $r > a$, given that T remains bounded as $r \to \infty$.

In terms of polar coordinates the boundary condition is $T = a\cos\theta$ for $r = a$ and $0 \leq \theta \leq 2\pi$. This suggests that we seek a solution of the form

$$T = \left(Ar + \frac{B}{r}\right)\cos\theta. \tag{5.56}$$

If T is to remain bounded as $r \to \infty$ we must have $A = 0$, and to match the boundary condition on $r = a$ we must have $B = a^2$. Hence the solution is

$$T(r, \theta) = \frac{a^2\cos\theta}{r}, \tag{5.57}$$

or, in Cartesian coordinates,

$$T(x, y) = \frac{a^2 x}{x^2 + y^2}. \tag{5.58}$$

Example 5.5 (Poisson's equation) The same argument establishes uniqueness for the BVP:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = F(x, y) \text{ in } R, \quad T = f(x, y) \text{ on } \partial R, \tag{5.59}$$

where F and f are prescribed functions.

5.3.2 Uniqueness for the Neumann problem

Firstly, we consider the Neumann problem in Cartesian coordinates.

Theorem 5.5 Consider the BVP

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = F(x, y) \text{ in } R, \quad \frac{\partial T}{\partial n} = g(x, y) \text{ on } \partial R, \tag{5.60}$$

where F and g are prescribed functions. Then the BVP has no solution unless

$$\int\!\!\int_R F \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} g \,\mathrm{d}s. \tag{5.61}$$

When a solution exists it is not unique; all solutions differ by a constant.

Proof. For the first part we use Green's theorem

$$\int\!\!\int_R \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{F} \mathrm{d}x\,\mathrm{d}y = \int_{\partial R} \frac{\partial T}{\partial x}\,\mathrm{d}y - \frac{\partial T}{\partial y}\,\mathrm{d}x = \int_{\partial R} \underbrace{\frac{\partial T}{\partial n}}_{g} \mathrm{d}s. \tag{5.62}$$

For the second part, let U be a solution of the same BVP *i.e.*

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x, y) \text{ in } R, \quad \frac{\partial U}{\partial n} = g(x, y) \text{ on } \partial R, \tag{5.63}$$

and consider $W := U - T$. Then W is a solution of the problem

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = 0 \text{ in } R, \quad \frac{\partial W}{\partial n} = 0 \text{ on } \partial R. \tag{5.64}$$

Using the identity

$$W\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2=\frac{\partial}{\partial x}\left(W\frac{\partial W}{\partial x}\right)+\frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right), \tag{5.65}$$

we find that

$$\begin{aligned}\int\int_R\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y &= \int_{\partial R} -W\frac{\partial W}{\partial y}\,\mathrm{d}x+W\frac{\partial W}{\partial x}\,\mathrm{d}y, &(5.66)\\ &= \int_{\partial R} W\frac{\partial W}{\partial n}\,\mathrm{d}s, &(5.67)\\ &= 0. &(5.68)\end{aligned}$$

Thus $\partial W/\partial x = 0$, $\partial W/\partial y = 0$ so that $W = U - T$ is constant. □

Example 5.6 (Polar Neumann problem) Find T so as to satisfy Laplace's equation in the disk $0 \le r < a$ and the boundary condition

$$\frac{\partial T}{\partial n}(a,\theta) = g(\theta), \quad 0 \le \theta \le 2\pi, \tag{5.69}$$

where g is a prescribed function.

First, we define $\boldsymbol{n} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and note that since $x = r\cos\theta$, $y = r\sin\theta$,

$$\begin{aligned}\frac{\partial T}{\partial r} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial T}{\partial y}\frac{\partial y}{\partial r}, &(5.70)\\ &= \cos\theta\frac{\partial T}{\partial x}+\sin\theta\frac{\partial T}{\partial y}, &(5.71)\\ &= (\cos\theta\boldsymbol{i}+\sin\theta\boldsymbol{j})\cdot\left(\frac{\partial T}{\partial x}\boldsymbol{i}+\frac{\partial T}{\partial y}\boldsymbol{j}\right), &(5.72)\\ &= \boldsymbol{n}\cdot\nabla T, &(5.73)\\ &= \frac{\partial T}{\partial n}. &(5.74)\end{aligned}$$

[In this case $\partial T/\partial n$ is a genuine derivative.]

Let $g(\theta)$ have the Fourier expansion

$$g(\theta) = \frac{1}{2}p_0 + \sum_{n=1}^{\infty}\left[p_n\cos(n\theta) + q_n\sin(n\theta)\right], \tag{5.75}$$

where

$$\begin{aligned}p_0 &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\,\mathrm{d}\theta, &(5.76)\\ p_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\,\mathrm{d}\theta, &(5.77)\\ q_n &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\,\mathrm{d}\theta. &(5.78)\end{aligned}$$

Look for solutions in the form

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n\left[C_n\cos(n\theta) + D_n\sin(n\theta)\right]. \tag{5.79}$$

We have

$$\frac{\partial T}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} \left[C_n \cos(n\theta) + D_n \sin(n\theta) \right], \tag{5.80}$$

and the boundary condition gives

$$\sum_{n=1}^{\infty} n a^{n-1} \left[C_n \cos(n\theta) + D_n \sin(n\theta) \right] = g(\theta), \quad 0 \le \theta \le 2\pi. \tag{5.81}$$

We conclude immediately that the condition

$$\int_0^{2\pi} g(\theta)\,\mathrm{d}\theta = 0, \tag{5.82}$$

is necessary for a solution to exist.

If this condition is satisfied then there are solutions

$$T(r,\theta) = A + \sum_{n=1}^{\infty} r^n \left[C_n \cos(n\theta) + D_n \sin(n\theta) \right], \tag{5.83}$$

where

$$C_n = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} g(\theta) \cos(n\theta)\ \mathrm{d}\theta, \tag{5.84}$$

$$D_n = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} g(\theta) \sin(n\theta)\ \mathrm{d}\theta, \tag{5.85}$$

and A is an arbitrary constant, *i.e.* solutions are *non-unique*, if they exist.

Example 5.7 Find T so as to satisfy Laplace's equation in the disc $0 \le r < a$ and the boundary condition

$$\frac{\partial T}{\partial n}(a,\theta) = \sin^3\theta, \quad 0 \le \theta \le 2\pi. \tag{5.86}$$

Here

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta), \tag{5.87}$$

and

$$\int_0^{2\pi} \sin^3\theta\,\mathrm{d}\theta = 0, \tag{5.88}$$

and the solutions are

$$T(r,\theta) = A + \frac{3}{4} r \sin\theta - \frac{1}{12}\frac{r^3}{a^2}\sin(3\theta), \tag{5.89}$$

where A is arbitrary.

5.4 Well-posedness

PDE problems often arise from modelling a particular physical system. In this case we could like to be able to make predictions as to the behaviour of the system based on our analysis of the PDE under consideration.

Definition A problem is said to be well-posed (well-set) if the following three conditions are satisfied:

1. EXISTENCE—there is a solution;
2. UNIQUENESS—there is no more than one solution;
3. CONTINUOUS DEPENDENCE—the solution depends continuously on the data.

Example 5.8 As an illustration consider the IVP

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -\infty < x < \infty,\, t > 0, \tag{5.90}$$

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty, \tag{5.91}$$

where f and g are the initial data. We know that there is exactly one solution, given by D'Alembert's formula:

$$y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{5.92}$$

Thus 1. and 2. hold.

Suppose we are interested in making predictions in the time interval $0 < t < T$ for some time T. Consider a similar problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -\infty < x < \infty,\, t > 0, \tag{5.93}$$

$$y(x,0) = F(x), \quad \frac{\partial y}{\partial t}(x,0) = G(x), \quad -\infty < x < \infty, \tag{5.94}$$

where F and G are different initial data. Again, we know that there is exactly one solution:

$$Y(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,\mathrm{d}s, \tag{5.95}$$

and

$$Y(x,t) - y(x,t) = \frac{1}{2}\left[(F(x-ct) - f(x-ct)) + (F(x+ct) - f(x+ct))\right] \tag{5.96}$$

$$+\frac{1}{2c}\int_{x-ct}^{x+ct} \left[G(s) - g(s)\right]\,\mathrm{d}s. \tag{5.97}$$

Now let $\epsilon > 0$ be arbitrary and suppose that

$$|F(x) - f(x)| < \delta \text{ and } |G(x) - g(x)| < \delta \text{ for } -\infty < x < \infty. \tag{5.98}$$

Then

$$\begin{aligned}|Y(x,t)-y(x,t)| &\leq \frac{1}{2}|F(x-ct)-f(x-ct)| \\ &+\frac{1}{2}|F(x+ct)-f(x+ct)| \\ &+\frac{1}{2c}\int_{x-ct}^{x+ct}|G(s)-g(s)|\,\mathrm{d}s, && (5.99)\\ &< \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2c}\int_{x-ct}^{x+ct}\delta\,\mathrm{d}s, && (5.100)\\ &= \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2c}\cdot 2ct\delta, && (5.101)\\ &= (1+t)\delta && (5.102)\\ &< (1+T)\delta. && (5.103)\end{aligned}$$

Thus if the new data (F, G) are close to the original data (f, g) in the sense that

$$|F(x)-f(x)| < \frac{\epsilon}{1+T} \text{ and } |G(x)-g(x)| < \frac{\epsilon}{1+T} \quad \text{for } -\infty < x < \infty, \qquad (5.104)$$

then the corresponding solutions are close together in the sense that

$$|Y(x,t)-y(x,t)| < \epsilon \text{ for } -\infty < x < \infty \text{ and } 0 < t < T. \qquad (5.105)$$

In this sense 3. holds and the IVP is well-posed.

Example 5.9 By contrast the corresponding IVP for Laplace's equation is *not* well-posed. If $y(x,t)=0$, $f(x)=0$, $g(x)=0$ then y is a solution of the IVP

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial t^2}=0, \quad -\infty < x < \infty,\ t>0, \qquad (5.106)$$

$$y(x,0)=f(x), \quad \frac{\partial y}{\partial t}(x,0)=g(x), \quad -\infty < x < \infty. \qquad (5.107)$$

If

$$Y(x,t)=\delta^2\cos\left(\frac{x}{\delta}\right)\sinh\left(\frac{t}{\delta}\right),\ F(x)=0,\ G(x)=\delta\cos\left(\frac{x}{\delta}\right), \qquad (5.108)$$

Then $Y(x,t)$ is a solution of the IVP

$$\frac{\partial^2 Y}{\partial x^2}+\frac{\partial^2 Y}{\partial t^2}=0, \quad -\infty < x < \infty,\ t>0, \qquad (5.109)$$

$$Y(x,0)=F(x), \quad \frac{\partial Y}{\partial t}(x,0)=G(x), \quad -\infty < x < \infty. \qquad (5.110)$$

Again suppose we want to make predictions in $0<t<T$. Then

$$|F(x)-f(x)|=0<\delta, \quad |G(x)-g(x)|=\delta\left|\cos\left(\frac{x}{\delta}\right)\right|<\delta, \qquad (5.111)$$

and

$$|Y(0,t)-y(0,t)|=\delta^2\sinh\left(\frac{t}{\delta}\right)<\delta^2\sinh\left(\frac{T}{\delta}\right). \qquad (5.112)$$

But

$$\delta^2\sinh\left(\frac{T}{\delta}\right)=\frac{1}{2}\delta^2\left(e^{T/\delta}-e^{-T/\delta}\right)\to\infty \quad \text{as} \quad \delta\to 0, \qquad (5.113)$$

and we cannot make

$$|Y(0,t)-y(0,t)|<\epsilon \quad \text{for} \quad 0<t<T, \qquad (5.114)$$

by making δ suitably small.