Problem Sheet 8

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1. Continuous random variables X and Y have joint probability density function

   (a) $f_{X,Y}(x,y) = C_1\left(x^2 + \frac{1}{3}xy\right)$, $x \in (0,1), y \in (0,2)$.

   (b) $f_{X,Y}(x,y) = C_2 e^{-x-y}$, $0 < x < y < \infty$.

   Find the values of the constants C_1 and C_2. For each of the joint densities above:

   - are X and Y independent?
   - find the marginal probability density functions of X and of Y.
   - find $\mathbb{P}(X \le 1/2, Y \le 1)$.

   In case (b), if the region had been $0 < x, y < \infty$, how would this affect your answer to the question about independence?

2. In the game of Oxémon Ko, you wander the streets of an old university town in search of a set of n different small furry creatures.

   Let T_i be the time (in hours) at which you first see a creature of type i, for $1 \le i \le n$. Suppose that $(T_i, 1 \le i \le n)$ are independent, and that T_i has exponential distribution with parameter λ_i.

   (a) Let $X = \min\{T_1, T_2, \ldots, T_n\}$ be the time at which you see your first creature. Show that X has an exponential distribution and give its parameter. [*Hint: consider* $\mathbb{P}(X > t)$ *and use independence.*]

   (b) What is the expected number of types of creature that you have not met by time 1?

   (c) Let $M = \max\{T_1, T_2, \ldots, T_n\}$ be the time until you have met all n different types of creature. Suppose now they are all equally common, with $\lambda_i = 1$ for all i. Find the median of the distribution of M. (As well as giving an exact expression, try to describe how quickly it grows as n becomes large.) [*Here you may wish to consider instead* $\mathbb{P}(M \le t)$. *You may find useful an estimate like* $\alpha^{1/n} - 1 = e^{\frac{1}{n}\log\alpha} - 1 \approx \frac{1}{n}\log\alpha$ *for large* n.]

3. Let U and V be independent random variables, both uniformly distributed on $[0,1]$. Find the probability that the quadratic equation $x^2 + 2Ux + V = 0$ has two real solutions.

4. A fair die is thrown n times. Using Chebyshev's inequality, show that with probability at least $31/36$, the number of sixes obtained is between $n/6 - \sqrt{n}$ and $n/6 + \sqrt{n}$.

5. Suppose that you take a random sample of size n from a distribution with mean μ and variance σ^2. Using Chebyshev's inequality, determine how large n needs to be to ensure that the difference between the sample mean and μ is less than two standard deviations with probability exceeding 0.99.

6. A fair coin is tossed $n+1$ times. For $1 \leq i \leq n$, let A_i be 1 if the ith and $(i+1)$st outcomes are both heads, and 0 otherwise.

   (a) Find the mean and the variance of A_i.

   (b) Find the covariance of A_i and A_j for $i \neq j$. (Consider the cases $|i-j| = 1$ and $|i-j| > 1$.)

   (c) Define $M = A_1 + \cdots + A_n$, the number of occurrences of the motif HH in the sequence. Find the mean and variance of M. [*Recall the formula for the variance of a sum of random variables, in terms of their variances and pairwise covariances.*]

   (d) Use a similar method to find the mean and variance of the number of occurrences of the motif TH in the sequence.

7. Let $a, b, p \in (0,1)$. What is the distribution of the sum of n independent Bernoulli random variables with parameter p? By considering this sum and applying the weak law of large numbers, identify the limit

$$\lim_{n\to\infty} \sum_{\substack{r\in\mathbb{N}: \\ an<r<bn}} \binom{n}{r} p^r (1-p)^{n-r}$$

in the cases (i) $p < a$; (ii) $a < p < b$; (iii) $b < p$.