**B8.1 Probability, Measure and Martingales**

**Problem Sheet 1, 2018 MT**

**Q1.** *(Monotone class theorem for functions)* Let $\mathcal{C}$ be a $\pi$-system on $\Omega$, and let $\mathcal{H}$ be a vector space of real-valued functions on $\Omega$. Suppose

1) the constant functions belong to $\mathcal{H}$, and $1_E \in \mathcal{H}$ for every $E \in \mathcal{C}$, and

2) if $\{f_n\} \subseteq \mathcal{H}$ is an increasing sequence such that $f = \lim_{n \to \infty} f_n$ is finite, then $f \in \mathcal{H}$.

Prove that $\mathcal{H}$ contains all finite real-valued functions which are $\sigma(\mathcal{C})$-measurable, where $\sigma(\mathcal{C})$ is the smallest $\sigma$-algebra containing $\mathcal{C}$.

*[Hint. Show that $\mathcal{D} = \{E \subseteq \Omega : 1_E \in \mathcal{H}\}$ is a monotone class and conclude that $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Then show that any simple $\sigma(\mathcal{C})$-measurable function belongs to $\mathcal{H}$, and conclude your argument by using the structure theorem for measurable functions in terms of simple functions, see item 7, on page 4, Lecture Notes].

**Q2.** 1) We say $f : \mathbb{R}^n \to \mathbb{R}^m$ is Borel measurable, if $f^{-1}(G) \in \mathcal{B}(\mathbb{R}^n)$ whenever $G \in \mathcal{B}(\mathbb{R}^m)$, where $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel $\sigma$-algebra, i.e. the smallest $\sigma$-algebra containing all open subsets of $\mathbb{R}^n$. Prove that, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then $f$ is Borel measurable.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x - y)$.

Show that $F$ is Borel measurable on $\mathbb{R}^2$.

3) Let $m$ be the Lebesgue measure on $\mathbb{R}$. Suppose $f$ and $g$ are Borel measurable and integrable on $\mathbb{R}$ with respect to $m$. Show that $f(x - y)g(y)$ is Borel measurable on $\mathbb{R}^2$ and show that

$$
\int_{\mathbb{R}^2} |f(x - y)g(y)| \, dx \, dy = \|f\|_{L^1} \|g\|_{L^1}
$$

where $dx \, dy$ is the Lebesgue measure on $\mathbb{R}^2$ and

$$
\|f\|_{L^1} = \int_{\mathbb{R}} |f| \, dm
$$

is the $L^1$-norm on the measure space $(\mathbb{R}, \mathcal{M}_{\text{Leb}}, m)$. Hence deduce that if $f, g$ are Borel measurable and Lebesgue integrable, then the convolution $(f \ast g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dy$ is Lebesgue integrable and

$$
\|f \ast g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.
$$

*[You may use the Fubini Theorem in an appropriate setting].

**Q3.** Let $(S, \Sigma)$ be a measurable space and $X_k : \Omega \to S$ be mappings, where $k = 1, 2, \ldots, n$. By definition, $\mathcal{G} = \sigma\{X_k : 1 \leq k \leq n\}$ is the smallest $\sigma$-algebra on $\Omega$ such that $X_k$ are measurable mappings from $(\Omega, \mathcal{G})$ to $(S, \Sigma)$.

1) Show that

$$
\mathcal{G} = \sigma\left\{X_1^{-1}(A_1) \cap \cdots \cap X_n^{-1}(A_n) : A_k \in \Sigma \text{ for } 1 \leq k \leq n\right\}.
$$

2) Suppose $Y : \Omega \to \mathbb{R}$. Then $Y$ is $\mathcal{G}$-measurable if and only if $Y = F(X_1, \ldots, X_n)$ where $F : \prod_{i=1}^n S \to \mathbb{R}$ is $\prod_{i=1}^n \Sigma$-measurable. *[Here for product $\sigma$-algebra $\prod_{i=1}^n \Sigma$, see item 1, page 24 in Lecture Notes].]
Q4. Let \((\Omega, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space. Let \(\mu^*\) be the outer measure

\[
\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : \text{where } A_i \in \mathcal{F} \text{ s.t. } \bigcup_{i=1}^{\infty} A_i \supset E \right\}
\]

where \(E \subset \Omega\). Let \(\mathcal{F}^*\) be the \(\sigma\)-algebra of all \(\mu^*\)-measurable subsets. Thus \((\Omega, \mathcal{F}^*, \mu^*)\) is a measure space, \(\mathcal{F} \subset \mathcal{F}^*\) and \(\mu^* = \mu\) on \(\mathcal{F}\), so that \(\mu^*\) will be denoted by \(\mu\) for simplicity. [Theorem 2.4, page 11 in Lecture Notes]

1) If \(E \in \mathcal{F}^*\), then there is a subset \(B \in \mathcal{F}\) such that \(E \subset B\) and \(\mu^*(B \setminus E) = 0\). Hence conclude that \(\mathcal{F}^* = \mathcal{F}^\mu\), where \(\mathcal{F}^\mu = \sigma \{ \mathcal{F}, \mathcal{N} \}\), \(\mathcal{N}\) is the collection of all \(\mu^*\)-null subsets.

[Hint. First consider the case that \(\mu(E) < \infty\), so by definition, for every \(N = 1, 2, \cdots\), there is a countable cover \(A_i^{(N)}\) of \(E\), \(A_i^{(N)} \in \mathcal{F}\), such that

\[
\mu(E) \leq \sum_{i=1}^{\infty} \mu(A_i^{(N)}) < \mu(E) + \frac{1}{2^N}.
\]

Prove that \(B = \bigcap_{N=1}^{\infty} \bigcup A_i^{(N)}\) is what you want].

2) Let \(\rho\) be an increasing function on \(\mathbb{R}\), \(m_\rho\) be the Lebesgue-Stieltjes measure on the \(\sigma\)-algebra \(\mathcal{M}_\rho\). We know that \(\mathcal{B}(\mathbb{R}) \subset \mathcal{M}_\rho\) [Section 3, Lecture Notes], so that \((\mathbb{R}, \mathcal{B}(\mathbb{R}), m_\rho)\) is a measure space. Show that for every \(E \in \mathcal{M}_\rho\) there is a Borel subset \(B \in \mathcal{B}(\mathbb{R})\) such that \(E \subset B\), and \(m_\rho(B \setminus E) = 0\).

[Hint. Show that \(m_\rho\) is \(\sigma\)-finite, and apply 1) to \(\mathcal{F} = \mathcal{B}(\mathbb{R})\).]

Q5. 1) Let \(\rho(t) = t + 1\) for \(t \geq 1\) and \(\rho(t) = 0\) for \(t < 1\). Prove that \(\rho\) is increasing, right continuous on \(( -\infty, \infty)\). Calculate \(m_\rho(A)\) where \(A \subset ( -\infty, 1)\), \(m_\rho(\{1\})\) and \(m_\rho(A)\) for \(A \subset (1, \infty)\), where \(A\) is Borel measurable. Hence describe the Lebesgue-Stieltjes measure \(m_\rho\) in terms of Lebesgue measure (and integrals).

2) \(\rho\) as in 1). Show that the right derivative \(\rho'(t+)^\) exists for all \(t\), and is non-negative, hence it is Borel measurable. Define \(\mu(A) = \int_A \rho'(t+)dt\) [where \(dt\) denotes the Lebesgue measure] for \(A \in \mathcal{B}(\mathbb{R})\). Prove \(\mu\) is a measure on \(\mathcal{B}(\mathbb{R})\). Calculate \(\mu(A)\) for \(A \subset ( -\infty, 1)\) and \(A \subset (1, \infty)\), and calculate \(\mu(\{1\})\) in terms of Lebesgue measure \(m\). Conclude that \(\mu \neq m_\rho\) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\).

3) Suppose \(\rho\) is a continuous increasing function on \(( -\infty, \infty)\), which is piece-wise differentiable in the sense that there are finite many \(a_1 < a_2 < \cdots < a_n\) such that \(\rho\) has continuous derivative on \((a_i, a_{i+1})\) for \(i = 0, \cdots, n\) (with \(a_0 = -\infty\) and \(a_{n+1} = \infty\)). [Examples including such as \(1)\) \(\rho(t) = t - 1\) for \(t \geq 1\) and \(\rho(t) = 0\) for \(t < 0\); \(2)\) \(\rho(t) = t^p\) for \(t > 0\) \(\rho(t) = 0\) for \(t \leq 0\) where \(p > 1\) a constant].

In particular the derivative \(\rho'\) is non-negative, continuous except for finite many points, thus must be Borel measurable. Let \(\mu(A) = \int_A \rho'(t)dt\) and \(m_\rho\) denote the associated Lebesgue-Stieltjes measure. Prove that \(\mu = m_\rho\) on \(\mathcal{B}(\mathbb{R})\).

[Hint. Show that \(\mu = m_\rho\) on the \(\pi\)-system \(\mathcal{G}\) of all \((s, t]\), by using the Fundamental Theorem in Calculus].