Heat conduction in a finite rod

Consider the initial boundary value problem (IBVP) for the temperature $T(x,t)$ in a rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$

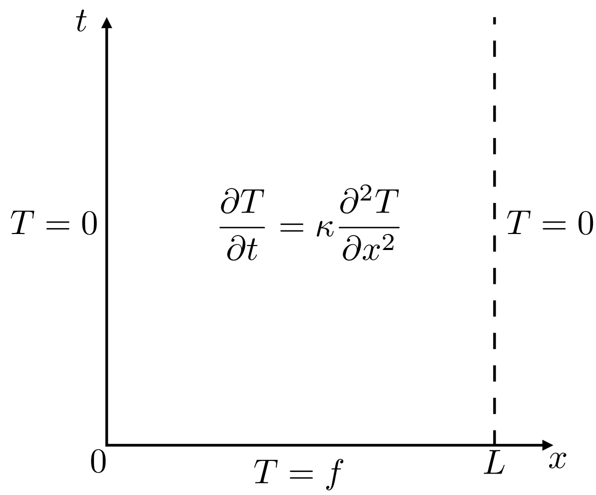
with the boundary conditions

$$T(0,t) = 0,\ T(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2}$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3}$$

where κ is the thermal diffusivity and the initial temperature profile $f(x)$ is given.



Fourier's method

We will solve the IBVP (1)–(3) using Fourier's method, which consists of the following three steps.

(I) Use the method of separation of variables to find the countably infinite set of nontrivial separable solutions satisfying the heat equation (1) and boundary conditions (2), each of them containing an arbitrary constant.

(II) Use the principle of superposition — that the sum of any number of solutions of a linear problem is also a solution (assuming convergence) — to form the general series solution that is the infinite sum of the nontrivial separable solutions.

(III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the initial conditions (3).

Remarks

- The heat equation (1) and boundary conditions (2) are linear: if T_1 and T_2 satisfy them, then so too does $\alpha T_1 + \beta T_2$ for all real constants α and β.
- To verify that the resulting series is actually a solution of the heat equation, it is necessary that is converges sufficiently rapidly that the partial derivatives T_t and T_{xx} may be computed by term-wise differentiation — we largely gloss over such technical issues (though comparison and summation methods are effective), *i.e.* we proceed formally.