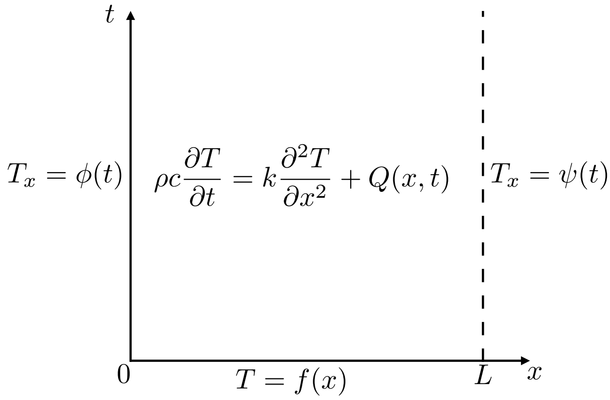
Inhomogeneous heat equation and boundary conditions

Consider the IBVP for the temperature $T(x,t)$ in a rod of length L given by the inhomogeneous heat equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the inhomogeneous boundary conditions $T_x(0,t) = \phi(t)$ and $T_x(L,t) = \psi(t)$ for $t > 0$ and the initial condition $T(x,0) = f(x)$ for $0 < x < L$, where ρ, c and k are positive constants and the functions $Q(x,t)$, $\phi(t)$, ψ(t) and $f(x)$ are given.



Note that Q is the volumetric heat source (due to *e.g.* radiation or chemical reactions) and the heat flux in the positive direction $q = -kT_x$ according to Fourier's law, so that the boundary conditions prescribe q at each end of the rod.

Generalizing Fourier's method

- In general Fourier's method cannot be used to solve the IBVP for T because the heat equation and boundary conditions are inhomogeneous (*i.e.* Q, ϕ and ψ are non-zero). We now describe a generalization of Fourier's method that works.
- We deal first with the boundary conditions: if we let $T(x,t) = S(x,t) + U(x,t)$, where

$$S(x,t) = -\phi(t)\frac{(x-L)^2}{2L} + \psi(t)\frac{x^2}{2L},$$

then the IBVP for T implies that the IBVP for U is given by

$$\rho c \frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2} + \widetilde{Q}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with $U_x(0,t) = 0$ and $U_x(L,t) = 0$ for $t > 0$ and $U(x,0) = \widetilde{f}(x)$ for $0 < x < L$; here

$$\widetilde{Q}(x,t) = Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c \frac{\partial S}{\partial t} \quad \text{and} \quad \widetilde{f}(x) = f(x) - S(x,0)$$

are functions that are known in terms of Q, ϕ, ψ and f. Thus, the boundary conditions have been rendered homogeneous using a technique called 'shifting the data' (because ϕ and ψ have moved from the boundary conditions in the IBVP for T to the PDE in the IBVP for U).

- If $\widetilde{Q} = 0$, then we can solve the IBVP for U using Fourier's method as in Example 3.4 to obtain

$$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 kt}{\rho c L^2}\right),$$

where the Fourier coefficients a_n are chosen to satisfy the initial condition so that

$$a_n = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- This series solution for $U(x,t)$ suggests that if $\widetilde{Q}$ is not identically zero, then we should seek a solution for U in the form of the Fourier cosine series

$$U(x,t) = \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t)\cos\left(\frac{n\pi x}{L}\right),$$

where the Fourier coefficients $U_n(t)$ depend on time and are to be determined. From the formulae for the Fourier coefficients of a cosine series, we deduce that $U_n(t)$ are given in terms of $U(x,t)$ by the integral expressions

$$U_n(t) = \frac{2}{L}\int_0^L U(x,t)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- We now use Leibniz's Integral Rule and the heat equation for U to deduce that

$$\rho c\frac{\mathrm{d}U_n}{\mathrm{d}t} = \frac{2}{L}\int_0^L \rho c\frac{\partial U}{\partial t}\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{2}{L}\int_0^L \left(k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}\right)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Integration by parts using the boundary conditions for U reveals that

$$\int_0^L \frac{\partial^2 U}{\partial x^2}\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = -\left(\frac{n\pi}{L}\right)^2\int_0^L U\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x,$$

while we recognize the functions

$$\widetilde{Q}_n(t) = \frac{2}{L}\int_0^L \widetilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x$$

as the Fourier coefficients of the cosine series for $\widetilde{Q}(x,t)$. Combining these equations, we find that U_n is governed by the ODE

$$\rho c\frac{\mathrm{d}U_n}{\mathrm{d}t} + \frac{kn^2\pi^2}{L^2}U_n = \widetilde{Q}_n(t) \quad \text{for } t > 0,$$

with the initial condition for U giving the initial condition

$$U_n(0) = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Remarks

- We have reduced the problem to a countably infinite set of IVPs for $U_0(t)$, $U_1(t)$, ...
- The IVP for $U_n(t)$ can be solved explicitly using an integrating factor.
- If $\widetilde{Q} = 0$, then $\widetilde{Q}_n = 0$ and we recover the solution for U obtained by Fourier's method.