Fourier Series and PDEs Problem Sheet 6

1. Consider the initial boundary value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the forced wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + F(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the boundary conditions $y(0,t) = \phi(t)$ and $y(L,t) = \psi(t)$ for $t > 0$ and the initial conditions $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$ for $0 < x < L$, where the wave speed c is a positive constant and the functions F, ϕ, ψ, f and g are given.

(a) Let

$$y(x,t) = \phi(t)\left(1 - \frac{x}{L}\right) + \psi(t)\frac{x}{L} + Y(x,t).$$

Determine the functions $\widetilde{F}$, $\widetilde{f}$ and $\widetilde{g}$ for which Y satisfies the initial boundary value problem given by

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial^2 Y}{\partial x^2} + \widetilde{F}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with $Y(0,t) = Y(L,t) = 0$ for $t > 0$ and $Y(x,0) = \widetilde{f}(x)$, $Y_t(x,0) = \widetilde{g}(x)$ for $0 < x < L$.

(b) By considering your answer to question 3 of sheet 5, write down the solution for $y(x,t)$ in the special case in which $\widetilde{F}(x,t) = 0$ for $0 < x < L$, $t > 0$.

(c) Consider now the case in which $\widetilde{F}$ is not identically zero. Suppose that $Y(x,t)$ and $\widetilde{F}(x,t)$ may be expanded as Fourier sine series on $[0,L]$ with Fourier coefficients

$$Y_n(t) = \frac{2}{L}\int_0^L Y(x,t)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \quad \widetilde{F}_n(t) = \frac{2}{L}\int_0^L \widetilde{F}(x,t)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x,$$

where n is a positive integer. By differentiating $Y_n(t)$ under the integral sign, using the wave equation and integrating by parts, show that

$$\frac{\mathrm{d}^2 Y_n}{\mathrm{d}t^2} + \omega_n^2\, Y_n = \widetilde{F}_n \quad \text{for} \quad t > 0,$$

where $\omega_n = n\pi c/L$. What are the initial conditions for Y_n?

(d) Consider now the case in which

$$F(x,t) = 0, \qquad \phi(t) = h\sin(\omega t), \qquad \psi(t) = 0, \qquad f(x) = 0, \qquad g(x) = 0,$$

where h and ω are positive constants. What is the initial value problem for $Y_n(t)$? Given that the solution for $Y_n(t)$ is

$$Y_n(t) = \begin{cases} \dfrac{2h\omega(\omega\sin(\omega t) - \omega_n\sin(\omega_n t))}{n\pi(\omega_n^2 - \omega^2)} & \text{for } \omega \neq \omega_n, \\[2ex] -\dfrac{h}{n\pi}\Big(\omega_n t\cos(\omega_n t) + \sin(\omega_n t)\Big) & \text{for } \omega = \omega_n, \end{cases}$$

write down the solution for $y(x,t)$ when

(i) $\omega \neq \omega_n$ for all positive integers n;

(ii) $\omega = \omega_p$ for some positive integer p.

What is the essential difference between the two cases?

[*You may quote the identity*

$$\int_0^L \left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{L}{n\pi},$$

where n is a positive integer.]

2. At time $t = 0$ an elastic string is stretched to a line density ρ and a tension T between the lines at $x = 0$ and $x = L$ in the (x, y)-plane, where ρ, T and L are positive constants. The small transverse displacement $y(x, t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

where the wave speed $c = \sqrt{T/\rho}$. The end of the string at $x = L$ is fixed so that $y(L, t) = 0$ for $t > 0$. The other end is attached to a small ring of mass M which can move freely on a smooth wire lying along the y-axis.

(a) Assuming that the effects of gravity and air resistance are negligible, write down the y-component of Newton's Second Law for the ring and deduce that, to a first approximation for $|y_x| \ll 1$,

$$M \frac{\partial^2 y}{\partial t^2}(0, t) = T y_x(0, t) \quad \text{for } t > 0.$$

(b) Let ω be a positive constant and ε a constant. Show that there is a non-trivial separable solution of the form $y(x, t) = F(x) \sin(\omega c t + \varepsilon)$ only if ω is a root of the equation

$$\tan(\omega L) = \frac{\alpha}{\omega L},$$

where α is a dimensionless parameter that you should determine.

(c) The energy of the system is given by

$$E(t) = \frac{\rho}{2} \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x + \frac{T}{2} \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x + \frac{M}{2} \left(\frac{\partial y}{\partial t}(0, t)\right)^2.$$

(i) State the physical significance of each of the three terms on the right-hand side of this equality and show that E is constant.

(ii) Deduce that there is at most one solution of the initial boundary value problem for $y(x, t)$ given by the wave equation and boundary conditions above, subject to the given initial conditions $y(x, 0) = f(x)$ and $y_t(x, 0) = g(x)$ for $0 < x < L$.