D'Alembert's solution and the characteristic diagram

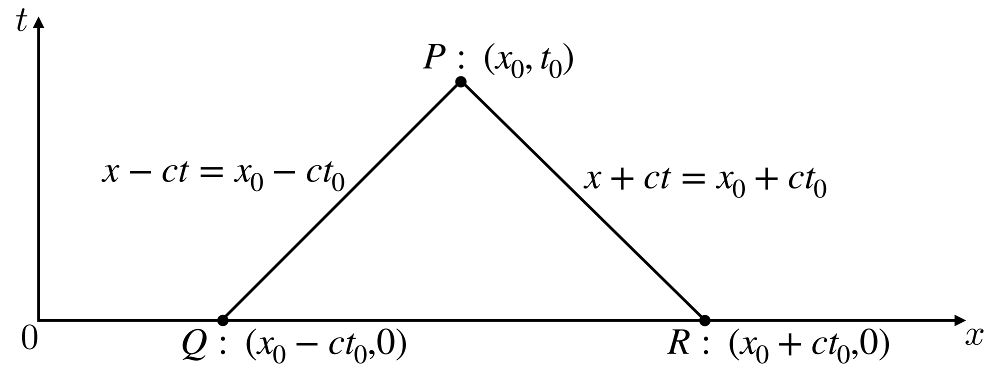
- D'Alembert's solution

$$y(x_0,t_0)=\frac{1}{2}\big(f(x_0-ct_0)+f(x_0+ct_0)\big)+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s \tag{1}$$

  implies

$$y(P)=\frac{1}{2}\big(f(Q)+f(R)\big)+\frac{1}{2c}\int_{Q}^{R} g(s)\,\mathrm{d}s, \tag{2}$$

  where P, Q and R are the points shown in the diagram.



- Note the deliberate abuse of notation in (2) to aid the geometric interpretation of (1).
- **Definition:** The curves $x \pm ct = x_0 \pm ct_0$ are the *characteristic lines* through $P:\ (x_0,t_0)$.
- It follows from (2) that $y(P)$ depends only on

  (i) f though the values f takes at Q and R;

  (ii) g though the values g takes on the x-axis between Q and R.

  This motivates the following definition.
- **Definition:** The interval $[x_0-ct_0, x_0+ct_0]$ of the x-axis between Q and R is called the *domain of dependence* of $P:\ (x_0,t_0)$
- If f or g are modified outside the domain of dependence of P, then $y(P)$ is unchanged.
- We can exploit the geometric interpretation (2) to construct explicit formulae for the solution: the contribution to $y(P)$ from f and g changes at points on the x-axis where f and g change their analytic bevaviour.
- Hence, given a particular f and g, the first task is to identify these points on the x-axis and sketch the characteristic lines $x \pm ct =$ constant through each of them — this is the *characterisrtic diagram.*
- The characteristic diagram divides the (x,t)-plane into regions in which the contributions from f and g may be different: the second task is to evaluate $y(P)$ for P in each of these regions.