A.3 Sheet 3: Continuous-time Markov chains

This assignment sheet is due for your class in week 6 or 7.

1. Suppose that we have a Q-matrix Q on a finite state space. Show that 0 is always an eigenvalue of Q.

2. Let $X$ be a simple birth-death process where individuals have independent Exp$(\mu)$ lifetimes and, during their lifetime give birth at rate $\lambda$ independently of other individuals as in Exercise A.2.4. It is clearly possible that the population dies out. Let

$$T = \inf\{t \geq 0: X_t = 0\}$$

be the extinction time for the population.

(a) Write down the forward equations for this chain.

(b) Suppose now that $X_0 = 1$. Let $G(s, t) = \mathbb{E}(s^{X_t})$ be the probability generating function of $X_t$. Show that $G$ satisfies

$$\frac{\partial}{\partial t} G(s, t) = (\lambda s - \mu)(s - 1) \frac{\partial}{\partial s} G(s, t).$$

(c) It can be shown that the solution to this equation is

$$G(s, t) = \begin{cases} 
\frac{\mu(s - 1) - (\lambda s - \mu)e^{-(\lambda-\mu)t}}{\lambda(s - 1) - \mu} & \text{if } \lambda \neq \mu \\
\frac{\lambda s - \mu}{\lambda(s - 1) - 1} & \text{if } \lambda = \mu.
\end{cases}$$

Using the fact that $G(0, t) = \mathbb{P}(X_t = 0)$, find the distribution function and density of $T$. Find $\mathbb{E}(T)$ in the case $\lambda \leq \mu$. Hint: $\mathbb{E}(T) = \int_0^\infty \mathbb{P}(T > t)dt$.

(d) Using the expressions for $\mathbb{E}(s^{X_t})$ and the continuity theorem for probability generating functions, identify the limiting distribution of $X_t$ as $t \to \infty$.

Hint: $\mathbb{P}(X_\infty \in \{0, \infty\}) = 1$.

3. Two-state Markov chain. Consider the Markov chain with Q-matrix $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$.

(a) Write down the backward and forward equations. Solve either the backward or the forward equations for the transition probabilities $p_{ij}(t)$, $i, j = 1, 2$. Check that your solution also satisfies the other equations.

Hint: one set is easier to solve than the other. The solutions of $y' = a + by$ are $y(x) = ce^{bx} - a/b$, $c \in \mathbb{R}$.

(b) Solve the equation $\xi Q = 0$ for $\xi$ and verify that $p_{ij}(t) \to \xi_j$ as $t \to \infty$. This proves convergence to equilibrium by bare-hands methods, not requiring the general theory presented in the lectures.
4. Consider a continuous-time Markov chain $X$ on $S = \mathbb{N}$ with Q-matrix $Q$, where the only non-zero off-diagonal entries of $Q$ are

$$q_{n,n+1} = 2^n, \quad n \geq 0, \quad q_{n,n-1} = 2^n, \quad n \geq 1.$$ 

(a) Determine the transition probabilities of the underlying jump chain $M$ and show that $M$ is null recurrent.

(b) Show that $X$ has a unique stationary distribution and deduce that $X$ is positive recurrent.

(c) (optional) Find a null recurrent continuous-time Markov chain whose underlying jump chain $M$ is positive recurrent.

5. Consider Markov chains $X$ and $Y$ with Q-matrices on $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 4, 5\}$

$$Q_X = \begin{pmatrix}
-1 & 1/2 & 1/2 & 0 \\
1/4 & -1/2 & 0 & 1/4 \\
1/6 & 0 & -1/3 & 1/6 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad Q_Y = \begin{pmatrix}
-3 & 2 & 0 & 0 & 1 \\
0 & -3 & 3 & 0 & 0 \\
0 & 5 & -5 & 0 & 0 \\
0 & 0 & 0 & -2 & 2 \\
0 & 0 & 0 & 1 & -1
\end{pmatrix}.$$

(a) What are the communicating classes. For each class, say if it is open or closed, and recurrent or transient.

(b) For $X$, calculate the expected time to hit 4 starting from 1

(c) For $X$, calculate the probability of hitting 3 starting from 1.

(d) For $Y$, determine all stationary distributions.

(e) (optional) For $Y$, determine the limit distribution when starting from 1.

*Hint:* For (b) and (c) you may wish to consider the quantities for arbitrary starting points and derive linear equations by conditioning on the first transition (time and state).

6. Consider the M/M/1 queue, that is a single-server queue in which customers arrive in a Poisson process of rate $\lambda$ and service times are independent identically exponentially distributed with parameter $\mu$. Let $X_t$ denote the length of the queue at time $t$ including any customer being served, where $X_0 = 0$.

(a) If the queue length is $k \geq 1$, what is the probability that the next customer arrives before the current customer’s service time ends?

(b) Describe the ‘jump chain’ $M$ of $X$.

(c) Determine the distribution of the number of arrivals during an Exp($\mu$) service period.

7. Let $X$ be the length of an M/M/1 queue, as in the previous question, but now assume that $\lambda < \mu$.

(a) Find the invariant distribution of $X$

(b) Find the invariant distribution of the jump chain $M$.

(c) Formulate the ergodic theorems for $X$ and $M$. Use this to explain why the invariant distributions in (a) and (b) are different.