A.5 Modes of convergence and renewal theory (Sheet4 for classes)

Please see MINERVA for hand in times. This sheet will be covered in the final class.

1. (a) Detailed balance equations. Let \((X_t)_{t \geq 0}\) be a continuous-time Markov chain with Q-matrix \(Q = (q_{ij})_{i,j \in \mathbb{S}}\). Suppose that a distribution \(\xi = (\xi_i)_{i \in \mathbb{S}}\) satisfies

\[
\xi_i q_{ij} = \xi_j q_{ji}, \quad \text{for all } i, j \in \mathbb{S}
\]

(detailed balance equations).

Show that this implies that \(\xi Q = 0\), i.e. that \(\xi\) is an invariant distribution of \((X_t)_{t \geq 0}\).

(b) Consider the Q-matrix \(Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}\). What does this example tell you about the converse of (a)?

2. For \(n \geq 1\), let \(X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \end{cases}\)

(a) Show that \(X_n \to 0\) in probability, as \(n \to \infty\), but that \(\mathbb{E}(X_n) \not\to 0\), as \(n \to \infty\).

(b) (optional) Show that \(X_n \to 0\) almost surely.

Now suppose that \((S_n)_{n \geq 0}\) is a simple symmetric random walk on \(\mathbb{Z}\), started from \(S_0 = 0\).

Let \(B_n = 1_{\{S_n = 0\}}\), so \(B_n = 1\) if \(S_n = 0\) and \(B_n = 0\) otherwise.

(c) Show that \(B_n \to 0\) in probability, as \(n \to \infty\).

(d) We know that \((S_n)_{n \geq 0}\) is recurrent. Use this fact to show that \(B_n\) does not converge almost surely, as \(n \to \infty\).

3. Let \(X)\) be a renewal process whose inter-renewal times \((Z_n)_{n \geq 0}\) satisfy \(0 < \sigma^2 = \text{Var}(Z_1) < \infty\) and \(\mu = \mathbb{E}(Z_1)\). Deduce from the Central Limit Theorem for \((Z_n)_{n \geq 0}\) that

\[
\frac{X_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \to Z \sim \text{Normal}(0, 1) \quad \text{in distribution, as } t \to \infty.
\]

Hint: Express probabilities involving \(X_t\) in terms of \(T_n\).

4. Proof of the Ergodic Theorem. Let \(X\) be an irreducible positive recurrent continuous-time Markov chain on a countable state space \(\mathbb{S}\), with holding time parameters \(\lambda_i\) and mean passage times \(m_i\), \(i \in \mathbb{S}\). Denote by \(H^{(m)}_i\), \(m \geq 1\), the successive passage times of \(X\) in \(i\).

(a) Fix \(i \in \mathbb{S}\) and let \(X_0 = i\). Show that the increments \(Z_m = S_{m+1} - S_m\) of \(S_m = H^{(m)}_i\), \(m \geq 0\), form a sequence of independent and identically distributed random variables.

Hint: Use the strong Markov property at \(S_m\), \(m \geq 1\).

(b) Fix \(i \in \mathbb{S}\), as in (a). Let \(X_0 = i\). Show that

\[
\frac{S_m}{m} \to m_i = \mathbb{E}(Z_1) \quad \text{almost surely, as } m \to \infty.
\]

What if \(X_0 = j\) for some \(j \in \mathbb{S}\) with \(j \neq i\)?

Hint: Only the distribution of \(Z_0\) is different now. Consider \(Z_0\) separately.

(c) Prove the following form of the ergodic theorem.

\[
\frac{1}{t} \int_0^t 1_{X_s = i} ds \to \frac{1}{m_i \lambda_i} \quad \text{almost surely, as } t \to \infty.
\]

Hints: Use (b) and also apply the strong law of large numbers to the holding times at \(i\). Consider \(t = H^{(m)}_i\), \(m \to \infty\), first and deduce the general statement.
5. **Inspection paradox.** Suppose that buses arrive at a bus stop as a Poisson process of rate \( \lambda \). Consider the duration \( D_t \) of the inter-arrival time containing \( t \), i.e. \( D_t = A_t + E_t \), where, at time \( t \), \( E_t \) is the time until the next bus arrives, and \( A_t \) is the time since the last one has passed (and \( A_t = t \) if no bus arrived in \([0,t]\)). What is the distribution of \( E_t \)? What is the distribution of \( A_t \)? Show that \( \mathbb{E}(D_t) > 1/\lambda \), so that the inter-arrival time we see ("inspect") has larger mean than a standard inter-arrival time.

6. Let \( p_n = \frac{\lambda^n}{n!} e^{-\lambda}, \ n \geq 0 \), be the probability function of the Poisson distribution. Calculate the associated size-biased distribution. For a random variable \( X^{sb} \) with the size-biased distribution, show that \( X^{sb} - 1 \) is Poisson distributed.

7. **Waiting time paradox.** Consider the \( \text{Gamma}(a, \lambda) \) distribution with density

\[
 f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}, \quad x > 0.
\]

(a) Calculate and identify the associated size-biased distribution.

(b) Suppose that the counting process of buses at a particular bus stop can be modelled by a renewal process \( X \) with stationary increments and \( \text{Gamma}(a, \lambda) \) inter-arrival times. Calculate the average waiting time \( m_{\text{stat}} \) of a customer arriving at time \( t \).

(c) Also calculate the average waiting time \( m_{\text{ren}} \) of a customer arriving just after a bus has passed. Deduce that

\[
m_{\text{stat}} > m_{\text{ren}} \iff a < 1
\]

This is a version of the waiting time paradox. What is paradoxical here?

8. Let \( X \) be an (undelayed) renewal process with finite mean inter-renewal times with density \( f \). Let \( m(t) = \mathbb{E}(X_t) \) be the associated renewal function. Recall that \( m'(t) = \sum_{k=1}^{\infty} f^{*k}(t) \).

(a) Suppose that \( H: [0, \infty) \to \mathbb{R} \) is bounded on bounded intervals. Show that the function \( r = H + H \ast m' \) satisfies the renewal-type equation \( r = H + r \ast f \).

(b) (For the keen) Show that \( r = H + H \ast m' \) is, in fact, the unique solution.

(c) Show that the distribution of the excess lifetime \( E_t = T_{X_{t+1}} - t \) satisfies

\[
 \mathbb{P}(E_t > y) = \bar{F}(t + y) + \int_0^t \bar{F}(t + y - x) m'(x) dx, \quad \text{where} \quad \bar{F}(t) = \int_t^\infty f(s) ds.
\]

(d) Let \( X \) be a renewal process with continuous inter-renewal times of finite mean \( \mu \). Deduce, using the Key Renewal Theorem, that the limit of \( \mathbb{P}(E_t > y) \) as \( t \to \infty \) is

\[
 \frac{1}{\mu} \int_y^\infty \bar{F}(z) dz.
\]

9. Let \( X \) be a renewal process with 1-arithmetic (in particular integer-valued) inter-renewal times \( Z_j \) of finite mean \( \mu \).

(i) Show that \( E_n = T_{X_{n+1}} - n \) is a discrete-time Markov chain.

(ii) Calculate its stationary distribution and deduce that

\[
 \mathbb{P}(E_n = k) \to \mu^{-1} \mathbb{P}(Z_1 \geq k) \quad \text{as} \ n \to \infty.
\]

(iii) Show that \( \mu^{-1} \mathbb{P}(Z_1 \geq k) \) is the probability function of a random variable \( U \) picked uniformly from \( \{1, \ldots, S\} \) conditionally given \( S \), where \( S \) has the size-biased distribution associated with the distribution of \( Z_1 \).