Groups and Group Actions, Sheet 7, TT17

Orbits. Stabilizers. Orbit-Stabilizer Theorem.

1. Consider the following actions. [You are not asked to show they are actions.] In each case describe the orbits of the action and determine the stabiliser of the given s.

(i) $(0,\infty)$ acts on $\mathbb{C}$ by multiplication, that is, $r \cdot z = rz$; $s = i$.

(ii) $\mathbb{Z}$ acts on $\mathbb{Z}_6$ by addition, that is, $n \cdot \overline{m} = \overline{n+m}$ where the line denotes mod 6 congruence; $s = 0$.

(iii) S_3 acts on S_3 by conjugation, that is, $\tau \cdot \sigma = \tau\sigma\tau^{-1}$; $s = (1\,2)$.

(iv) $O(2)$ acts on $\mathbb{R}^2$ by $A \cdot \mathbf{v} = A\mathbf{v}$; $s = \mathbf{i}$.

2. Let f be a polynomial in the (commuting) variables $x_1, x_2, \ldots, x_n$ and let N be the number of distinct polynomials, including f itself, that can be obtained from f by permuting the variables. Prove that N divides $n!$

Give examples to show that every divisor of $n!$ occurs when $n = 3$. Verify the Orbit-Stabilizer Theorem for each of your examples.

3. Let G be a group and let S denote the set of subgroups of G. Show that

$$g \cdot H = gHg^{-1}, \qquad \text{where } g \in G,\ H \leqslant G,$$

defines a left action of G on S.

Now let $G = S_4$. What is $\mathrm{Orb}(H)$ and $\mathrm{Stab}(H)$ in each of the following cases?

$$H = V_4, \qquad H = \mathrm{Sym}\{1,2,3\}, \qquad H = \langle (1234) \rangle.$$

4. Show that $GL_3(\mathbb{R})$ (the group of invertible 3×3 real matrices) acts on $M_{3\times 3}(\mathbb{R})$ (the set of 3×3 real matrices) by $A \cdot M = AM$. Let

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

Show that M_2 and M_3 lie in the same orbit; determine a matrix A such that

$$\mathrm{Stab}(M_2) = A\,\mathrm{Stab}(M_3)\,A^{-1}.$$

Show that M_1 and M_2 lie in different orbits, but that nonetheless $\mathrm{Stab}(M_1) = \mathrm{Stab}(M_2)$.

5. *Cayley's Theorem* states that every finite group is isomorphic to a subgroup of some S_n. For each of the following groups, what is the smallest n such that S_n contains a subgroup isomorphic to that group? Justify your answers and describe such a subgroup.

$$C_5, \qquad D_{10}, \qquad C_2 \times C_2 \times C_2, \qquad S_3 \times S_3.$$