Problem Sheet 6

Please send any comments or corrections to `martin@stats.ox.ac.uk`.

1. (a) If X is a constant random variable, say $\mathbb{P}(X = a) = 1$ for some $a \in \mathbb{N}$, what is its probability generating function?

   (b) If Y has probability generating function $G_Y(s)$, and m, n are positive integers, what is the probability generating function of $Z = mY + n$?

2. (a) Suppose that we perform a sequence of independent trials, each of which has probability p of success. Let Y be the number of trials up to and including the mth success, where $m \geq 1$ is fixed. Explain why

$$\mathbb{P}(Y = k) = \binom{k-1}{m-1} p^m (1-p)^{k-m}, \quad k = m, m+1, \ldots.$$

   (This is called the *negative binomial* distribution.)

   (b) By expressing Y as a sum of m independent random variables, find its probability generating function.

3. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed non-negative integer valued random variables, and let N be a non-negative integer valued random variable which is independent of the sequence $X_1, X_2, \ldots$.

   Let $Z = X_1 + \ldots + X_N$ (where we take $Z = 0$ if $N = 0$).

   (a) Show that

$$\mathbb{E}[Z] = \mathbb{E}[N]\,\mathbb{E}[X_1]$$

   and

$$\operatorname{var}(Z) = \operatorname{var}(N)\,(\mathbb{E}[X_1])^2 + \mathbb{E}[N] \operatorname{var}(X_1).$$

   (b) If $N \sim \text{Po}(\lambda)$ and $X_1 \sim \text{Ber}(p)$, find $\operatorname{var}(Z)$.

   (c) [*Optional*] Suppose we remove the condition that N is independent of the sequence (X_i). Is it still necessarily the case that $\mathbb{E}[Z] = \mathbb{E}[N]\,\mathbb{E}[X_1]$? Find a proof or a counterexample.

4. A random variable X has probability generating function G_X. Find a simple expression using G_X for the probability that X is even. [*Hint: consider the value of* $G_X(-1)$. *Possible extension: suggest a similar expression for the probability that* X *is divisible by* 4 *– be creative about what values of the generating function you might evaluate!*]

5. A population of cells is grown on a petri dish. Once a minute, each cell tries to reproduce by splitting in two. This is successful with probability 1/4; with probability 1/12, the cell dies instead; and with the remaining probability 2/3, nothing happens. Assume that different cells behave independently and that we begin with a single cell. What is the probability generating function $G(s)$ of the number of cells on the dish after 1 minute? How about after 2 minutes? What is the probability that after 2 minutes the population has died out?

6. Consider a branching process in which each individual has 2 offspring with probability p, and 0 offspring with probability $1-p$. Let X_n be the size of the nth generation, with $X_0 = 1$.

   (a) Write down the mean μ of the offspring distribution, and its probability generating function $G(s)$.

   (b) Find the probability that the process eventually dies out. [*Recall that this probability is the smallest non-negative solution of the equation* $s = G(s)$.] Verify that the probability that the process survives for ever is positive if and only if $\mu > 1$.

   (c) Let $\beta_n = \mathbb{P}(X_n > 0)$, the probability the process survives for at least n generations. Write down $G(s)$ in the case $p = 1/2$. Deduce that in that case,

   $$\beta_n = \beta_{n-1} - \beta_{n-1}^2/2,$$

   and use induction to prove that, for all n,

   $$\frac{1}{n+1} \leq \beta_n \leq \frac{2}{n+2}.$$

   (d) [*For further exploration!*] In lectures we considered a simple random walk, which at each step goes up with probability p and down with probability $1-p$. Suppose the walk starts from site 1. By taking limits in the gambler's ruin model, we showed that the probability that the walk ever hits site 0 equals 1 for $p \leq 1/2$, and $(1-p)/p$ for $p > 1/2$.

   Compare this probability to your answer in part (b). Can you find a link between the branching process and the random walk? [*Hint: if I take an individual in the branching process and replace it by its children (if any), what happens to the size of the population?*]