Let $X_1, X_2, X_3, \ldots$ be i.i.d. with mean μ and variance σ^2.

Weak law of large numbers: for all $\epsilon > 0$,

$$\mathbb{P}\left(\left|\bar{X}_n - \mu\right| \leq \epsilon\right) \to 1 \text{ as } n \to \infty.$$

Central limit theorem:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \to \mathcal{N}(0,1) \text{ as } n \to \infty.$$

For example, for all $z > 0$,

$$\mathbb{P}\left(\left|\bar{X}_n - \mu\right| \leq \frac{\sigma}{\sqrt{n}} z\right) \to \Phi(z) - \Phi(-z) \text{ as } n \to \infty.$$

Simple random walk

Let $p \in (0, 1)$. Let X_1, X_2, $X_3, \ldots$ be i.i.d. with

$$\mathbb{P}(X_i = 1) = p,$$
$$\mathbb{P}(X_i = -1) = 1 - p.$$

Then $\mathbb{E}\, X_i = \mu = 2p - 1$, $\mathrm{Var}\, X_i = \sigma^2 = 4p(1 - p)$.

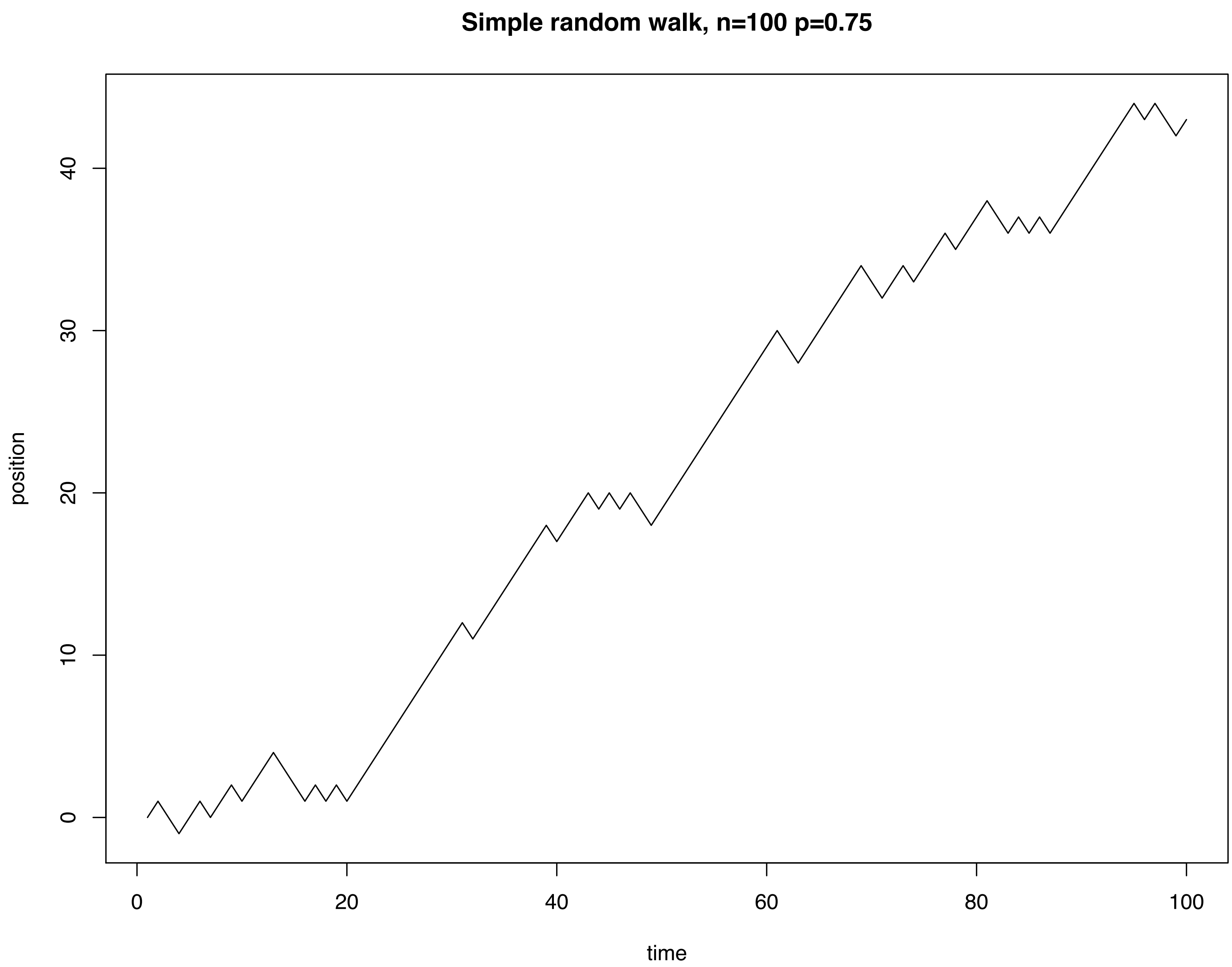
Let $S_n = X_1 + X_2 + \cdots + X_n$ (position of random walk at step n).

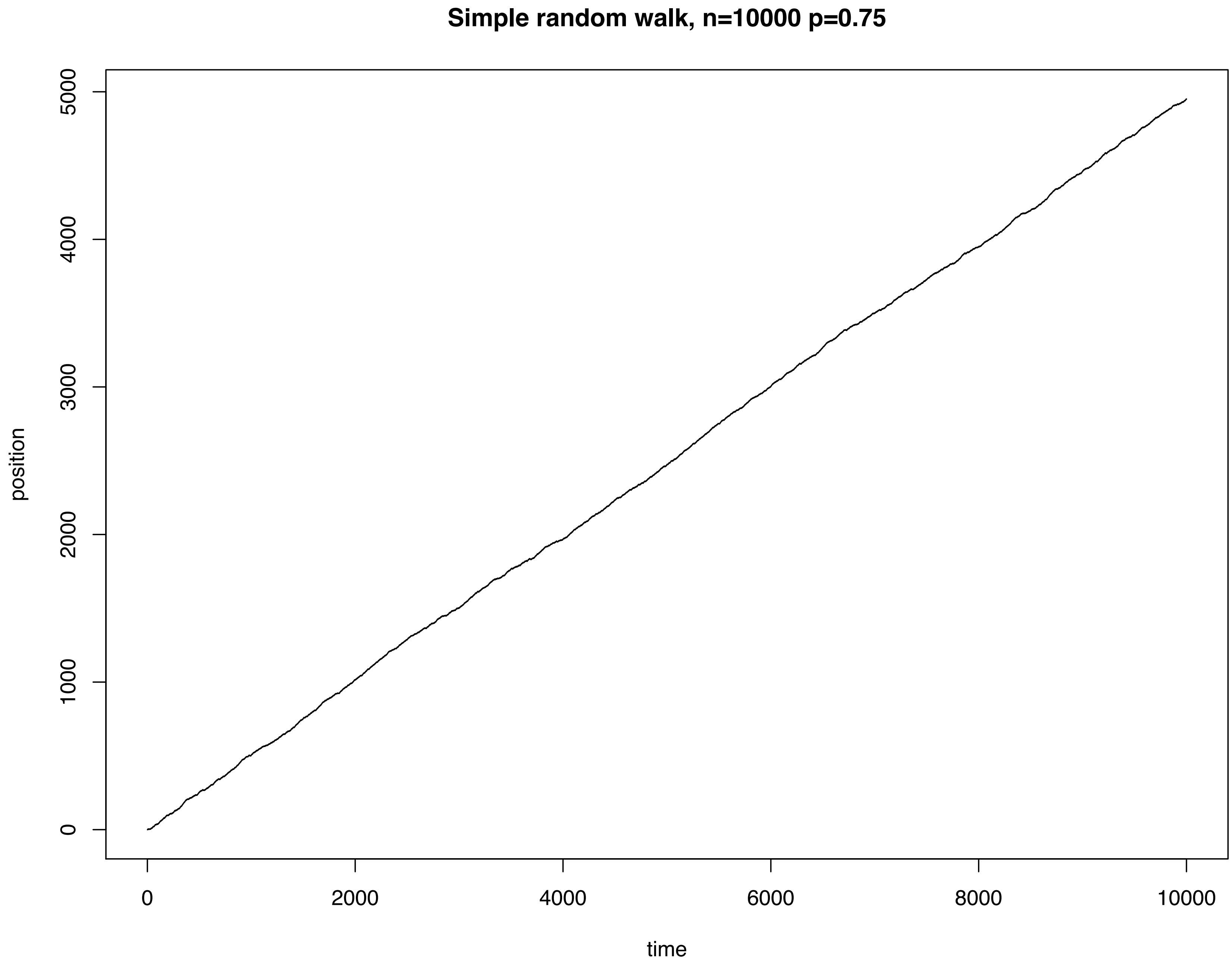
Law of large numbers:

$$\text{“}\boxed{\frac{S_n}{n} \approx \mu}\text{”}$$

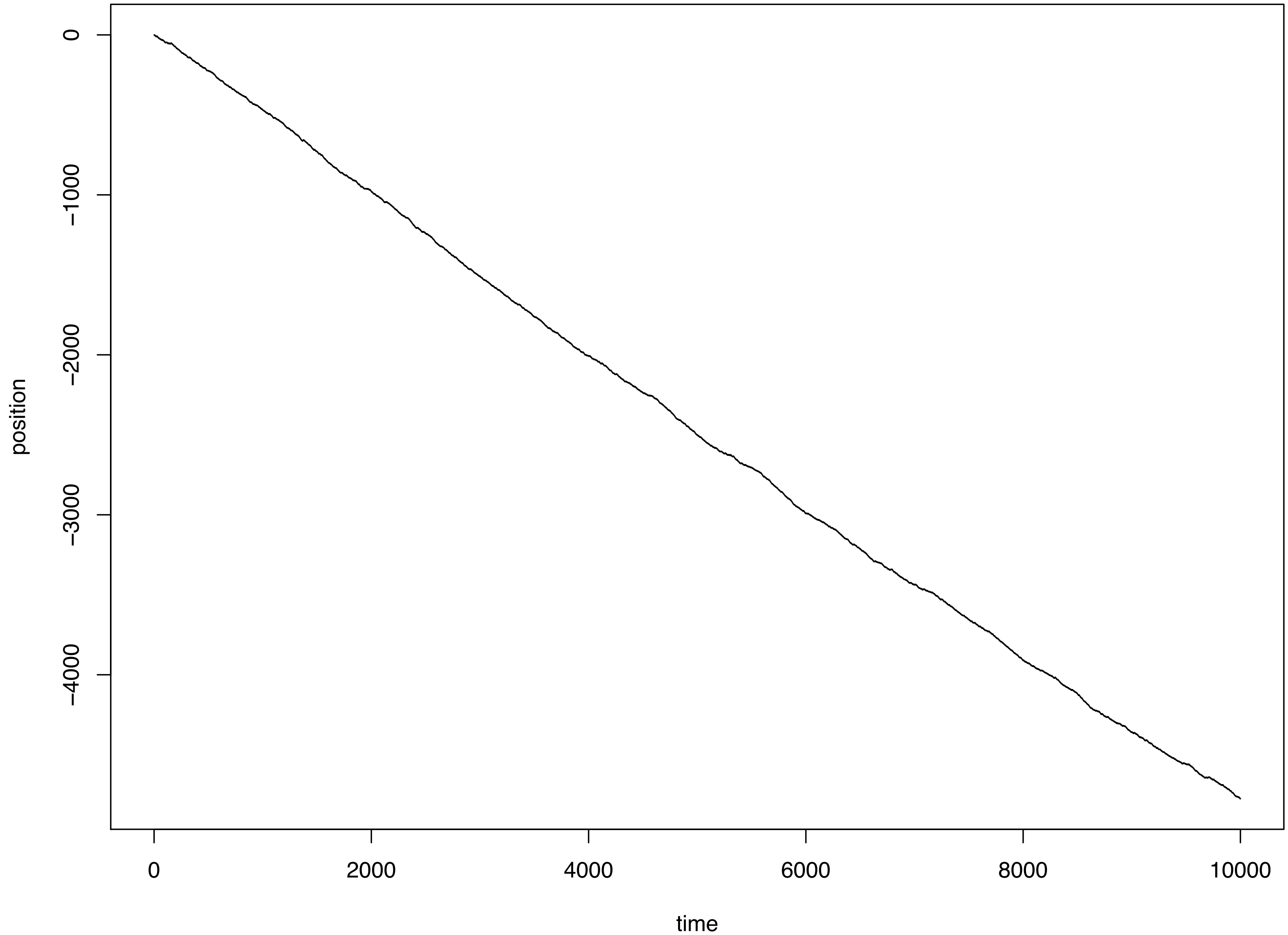
Central limit theorem:

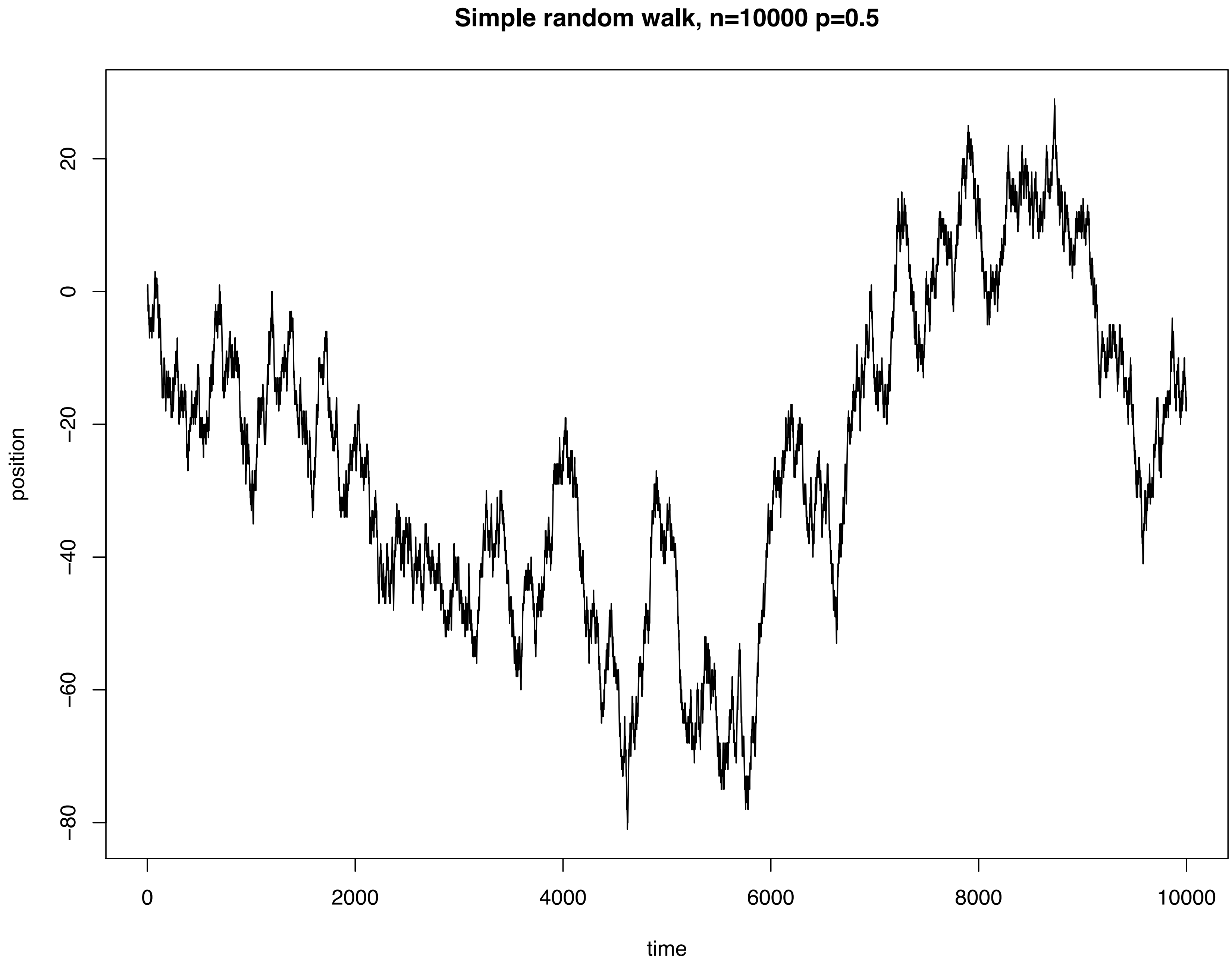
$$\text{“}\boxed{\frac{S_n/n - \mu}{\sigma/\sqrt{n}} \approx \mathcal{N}(0, 1)}\text{”}$$

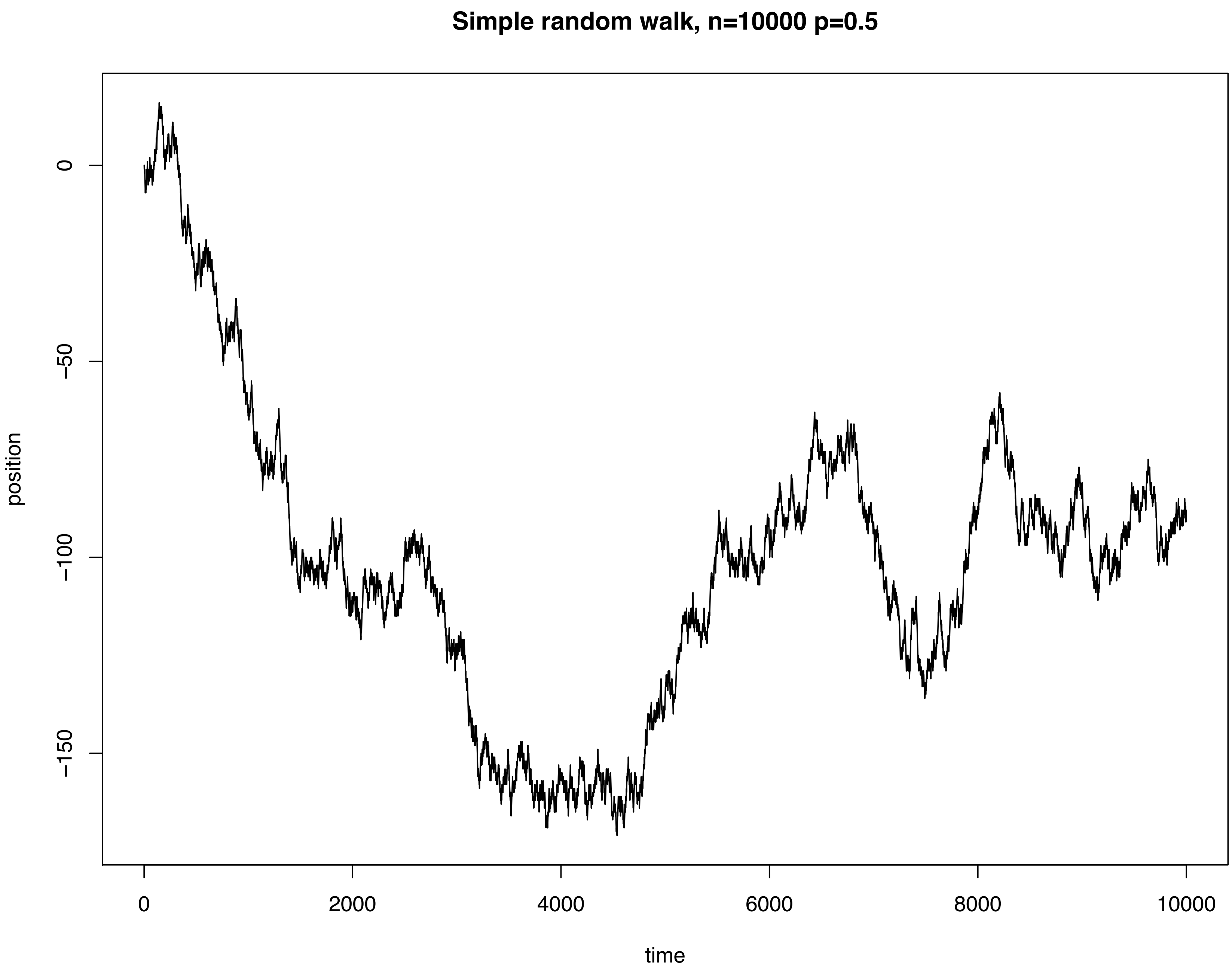

Simple random walk, n=100 p=0.75
position
0
10
20
30
40
0
20
40
60
80
100
time


Simple random walk, n=10000 p=0.75
position
time
0
1000
2000
3000
4000
5000
0
2000
4000
6000
8000
10000

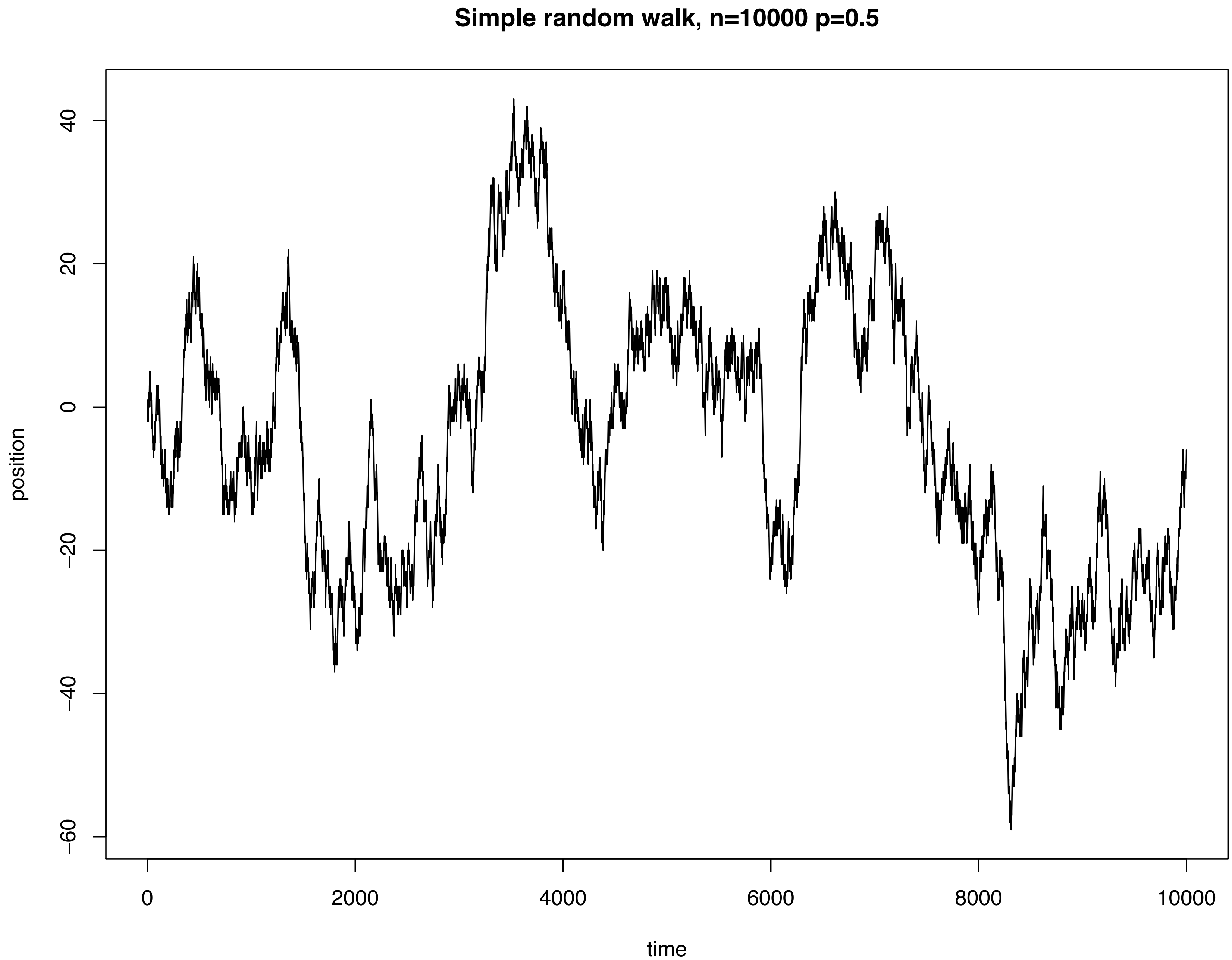
Simple random walk, n=10000 p=0.25



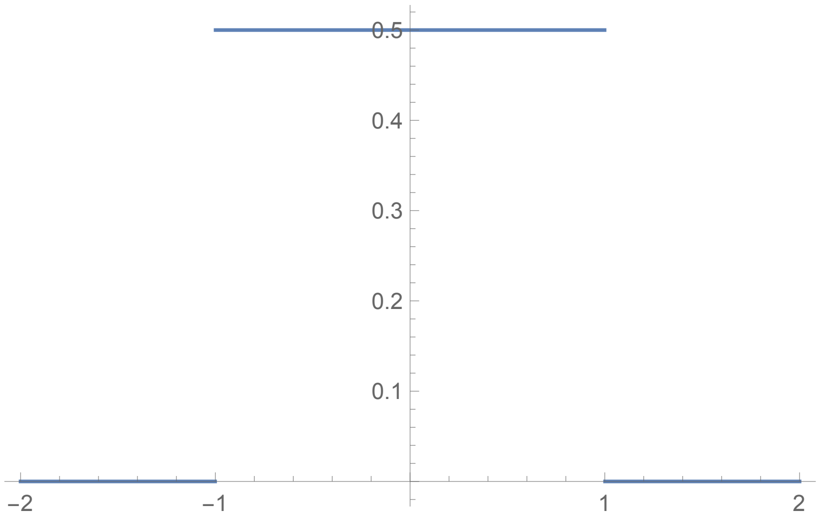

Simple random walk, n=10000 p=0.5
position
time
20
0
−20
−40
−60
−80
0
2000
4000
6000
8000
10000



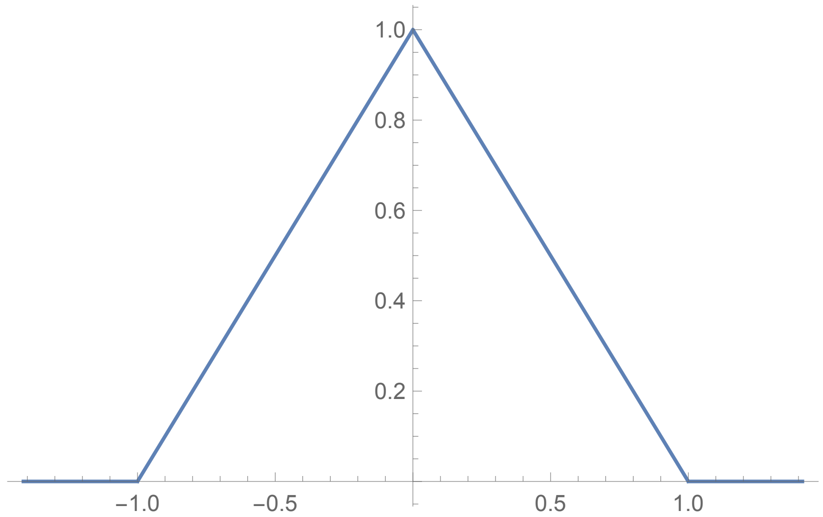
Simple random walk, n=10000 p=0.5
position
0
−50
−100
−150
0
2000
4000
6000
8000
10000
time


Simple random walk, n=10000 p=0.5
position
time
40
20
0
−20
−40
−60
0
2000
4000
6000
8000
10000

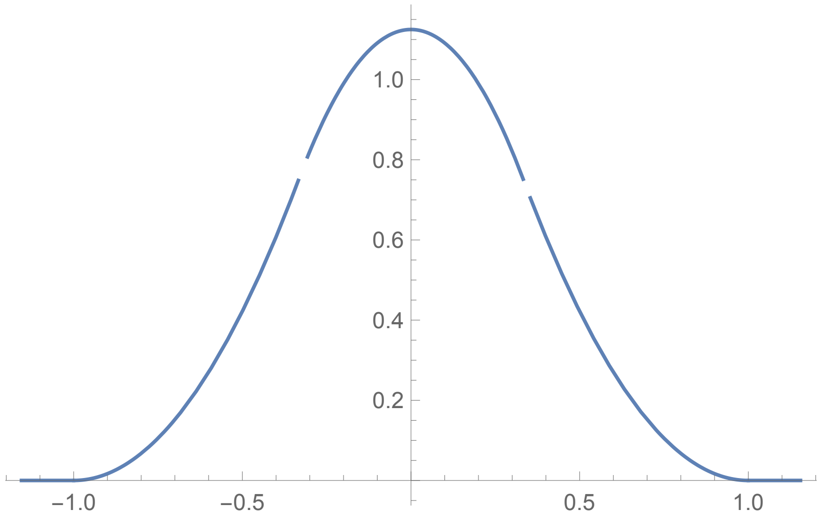
Let X_i be i.i.d. with uniform distribution on the interval $[-1, 1]$. We can plot the probability density function of $\bar{X}_n$.



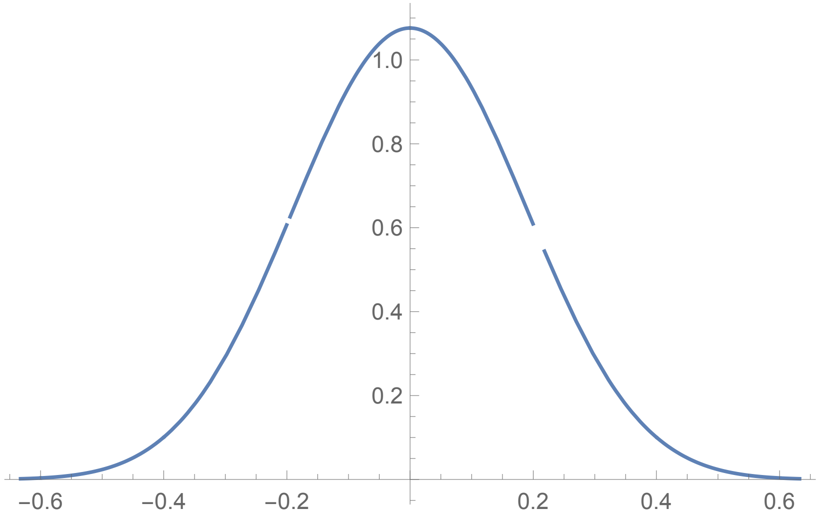
$n = 1$



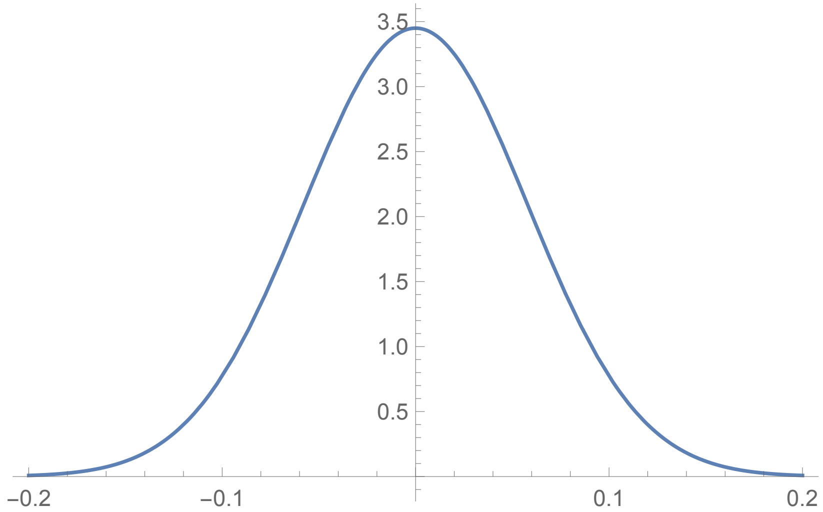
$n = 2$



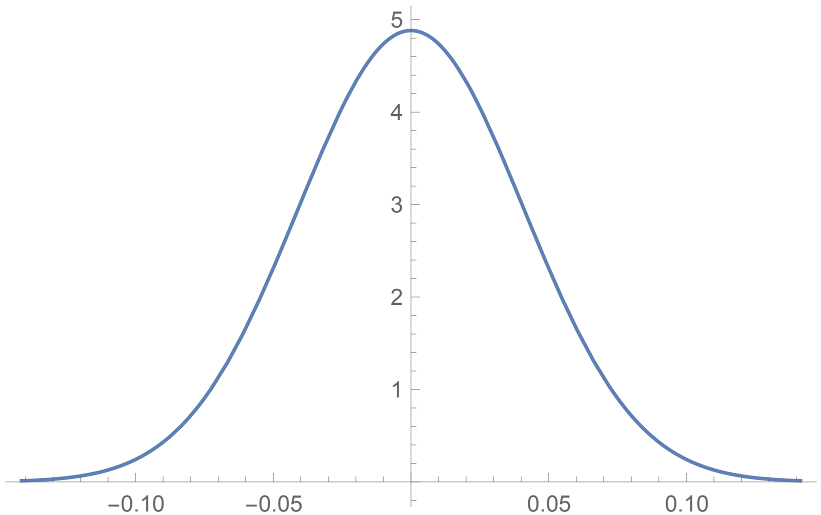
$n = 3$



$n = 10$



$n = 100$



$n = 200$

Topics of the course

- Axioms of probability; sample space, events, ...
- equally likely outcomes, "counting": permutations, combinations.
- conditional probability, Bayes theorem, independence, partition theorems.
- discrete random variables: mass function, expectation, variance, joint distributions, independence...
- recurrence relations, random walks.
- probability generating functions, branching processes.
- continuous random variables: distribution function, density function, expectation, geometrical probability
- sums of random variables: variance & covariance, Markov/Chebyshev inequalities, weak law of large numbers.