Nonlinear Systems HT2020 — Sheet 3

1. A bead is free to slide without friction on a circular wire hoop of radius L. The hoop spins about its vertical axis with angular velocity ω. After nondimensionalisation, the equation governing the position $\theta(t)$ (measured from the bottom of the hoop) is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \sin\theta - \alpha\sin\theta\cos\theta = 0,$$

where $\alpha = \omega^2 L/g$.

   (i) Discuss the behaviour of this system, as α increases from zero, from the point of view of bifurcation theory.

   (ii) Write down an energy integral for the system. Find the smallest constant $v > 0$ (in terms of the parameters) such that, if initially $\theta = \pi/2, |\dot{\theta}| > v$, then the bead will continually encircle the hoop in one direction.

   (iii) [*] What happens if linear damping is added to the system (that is, $-\mu\dot{\theta}$ is added on the equation's RHS with $\mu > 0$)? (NB: There is a real zoo of possible bifurcations in this system. A simple and good starting point is to find the critical rotation speed at which $\theta = 0$ becomes unstable. Describe this bifurcation).

2. Consider the system

$$\dot{x} = y - x - x^2,$$
$$\dot{y} = \mu x - y - y^2.$$

Find the value of μ for which there is a bifurcation at the origin. Find the evolution equation on the extended centre manifold correct to quadratic terms in the Taylor expansion and determine the type of bifurcation.

3. Consider the map

$$x_{n+1} = (1+\mu)x_n - \mu x_n^2,$$

for $\mu \geq 0$.

   (i) Find the fixed points and analyse their stabilities.

   (ii) Find the period-2 cycles and analyse their stabilities.

   (iii) Draw the bifurcation diagram in the (μ, x)-plane.

   (iv) [*] Verify your results by computing (numerically) the full bifurcation diagram of x_n for $0 < \mu \leq 3$.

4. Consider the 1D map

$$x_{n+1} = f(x_n),$$

and assume that it supports a p-periodic orbit $\{x_1, x_2, \ldots, x_p\}$ such that $x_i \neq x_j\ \forall i, j \in \{1, \ldots, p\}$ with $i \neq j$, and $x_{p+1} = x_1$. Show that the stability of this orbit is determined by the multiplier

$$\lambda = \prod_{i=1}^{p} f'(x_i),$$

whenever $|\lambda| \neq 1$.

5. Consider the map T defined by

$$x_{n+1} = y_n,$$
$$y_{n+1} = -x_n + 7y_n - y_n^3.$$

(i) Find the fixed points, determine their stability and compute their local stable and unstable subspaces.

(ii) Show that T admits 3 orbits of period 2, all within the square $S = \{(x, y) \mid |x| \leq 3, |y| \leq 3\}$.

(iii) Show that every orbit starting outside S tends to infinity for either $n \to \infty$ or $n \to -\infty$ and hence there are no periodic orbits outside S.

[Hint: Divide the complement of S into the regions

$$R_1 = \{(x, y) \mid x + y \geq 0, y > 3\}, \qquad R_2 = \{(x, y) \mid x + y < 0, y \leq -3\},$$

$$R_3 = \{(x, y) \mid x + y \geq 0, x > 3\}, \qquad R_4 = \{(x, y) \mid x + y < 0, x \leq -3\}.$$

Show a point in R_1 is mapped to a point in R_2 and vice versa, and that for such points $|y_{n+1}| > |y_n|$. For R_3 and R_4 consider the inverse map.]

(The map T admits infinitely many periodic orbits but the proof is somewhat more involved.)

6. Consider the system

$$\begin{aligned} \dot{x} &= \mu x + y + \sin x, \\ \dot{y} &= x - y. \end{aligned} \tag{1}$$

Show that a bifurcation occurs at the origin of this system. Classify it and draw the local (close to the origin) phase portraits before and after the bifurcation.