Nonlinear Systems HT2020 — Sheet 1

Questions marked [$*$] are harder and optional.

1. Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^3$ and

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}.$$

   Without solving the system, find the stable, unstable and center subspaces and sketch the phase portrait.

2. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a semi-simple matrix (i.e. a $n \times n$ matrix with real coefficients that can be diagonalised) and let $\mathbf{x} = \mathbf{x}(t)$ be a solution of

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}.$$

   Show that:

   (i) If $\mathbf{x}_0 \in E^s$, then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ and $\lim_{t\to-\infty} |\mathbf{x}(t)| = \infty$

   (ii) If $\mathbf{x}_0 \in E^u$, then $\lim_{t\to\infty} |\mathbf{x}(t)| = \infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

   (iii) If $\mathbf{x}_0 \in E^c$, then $\exists M \in \mathbb{R}$ such that $\forall\ t \in \mathbb{R}$,

$$|\mathbf{x}(t)| \leq M.$$

   (iv) [$*$] Which of these properties hold if A is not semi-simple? (prove or give a counter-example)

3. A **heteroclinic orbit** is an orbit that connects two fixed points. Find the value of α such that the system

$$\begin{aligned} \dot{x} &= x - y, \\ \dot{y} &= -\alpha x + \alpha x y, \end{aligned}$$

   admits the first integral $I = (y - 2x + x^2)\mathrm{e}^{-2t}$. (A scalar function $I = I(\mathbf{x}, t)$ is a *first integral* if $\dot{I} = 0$ on all trajectories.) Compute the fixed points and show that a branch of the level set of this first integral is a heteroclinic orbit. Can you find a closed form solution of this orbit?

4. The system

$$\dot{x} = -x \tag{1}$$
$$\dot{y} = -y + x^2 \tag{2}$$
$$\dot{z} = z + x^2 \tag{3}$$

   defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 | z = -x^2/3\}$ is an **invariant set** of this flow. Sketch this set in phase space and identify other interesting orbits (such as fixed points).

5. The system

$$\begin{aligned} \dot{x} &= -y + x(1 - z^2 - x^2 - y^2) \\ \dot{y} &= x + y(1 - z^2 - x^2 - y^2) \\ \dot{z} &= 0 \end{aligned}$$

   defines a flow $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$. Show that the union of the unit sphere with the portions of the z-axis outside the sphere is an **attracting set** for this flow. Find its domain of attraction. (Hint: rewrite the system in cylindrical coordinates).

6. Consider the system

$$\dot{r} = r(1-r)$$
$$\dot{\theta} = \sin^2\frac{\theta}{2},$$

where (r, θ) are the usual polar coordinates of a point in the plane. Show that this system has two fixed points. Show that the fixed point $(x = 1, y = 0)$ is the ω-limit set of almost all initial conditions. That is $\varphi_t(\mathbf{x}_0) \to (1, 0))$ for all initial conditions $\mathbf{x}_0 \neq (0, 0)$. Despite that, show that $(1, 0)$ is not stable. Is it an attracting set? Is the unit circle an attracting set? Find the domain of attraction (if any).

7. The equation for the simple pendulum is

$$\ddot{x} + \sin x = 0, \quad x \in \mathbb{R}$$

Find the potential for this system and use it to identify important orbits. In particular identify the fixed points and show that there exist heteroclinic orbits for this system. Sketch the phase portrait. Show that the orbits contained within a symmetric pair of heteroclinic orbits (called a *heteroclinic cycle*) form an invariant set. Is this an attracting set?

(This equation can be solved in terms of elliptic integrals of the first kind, but you should answer all these questions without solving the equation explicitly).

One of the main tools of dynamical system is the linearisation of a nonlinear system close to its fixed points. From Part A, you should be familiar with the basic ideas. The following two **optional** exercises are meant as a refresher from last year.

8. Consider the systems below. Find the fixed points and determine their stability through linearisation whenever possible. For systems with parameters, discuss stability with respect to the parameters

   (i)

$$\dot{x} = 2x - 2xy$$
$$\dot{y} = 2y - x^2 + y^2$$

   (ii)

$$\dot{x} = -4y + 2xy - 8$$
$$\dot{y} = -x^2 + 4y^2$$

   (iii)

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0$$

9. Consider the two systems

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

and

$$\dot{x} = xf(x, y)$$
$$\dot{y} = y\,g(x, y),$$

where f and g are both C^1 functions.

(i) Show that for the second system, the first positive quadrant (defined as the set $S = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$) is an invariant set.

(ii) Show that if the first system has an exponentially stable or unstable fixed point in S of the form $(x, y) = (c, c)$ with $c > 0$, then the second one admits the same fixed point with the same stability properties (a fixed point is *exponentially stable* (resp. unstable) if the real parts of all its eigenvalues are strictly negative (resp. positive)). Is this property true for fixed points in the other quadrants (i.e. with fixed points $(\pm c, \pm c)$)? (Prove or give a counter-example)

(iii) [*] Are these properties true for the similar problem in n dimensions. Formulate the corresponding results.